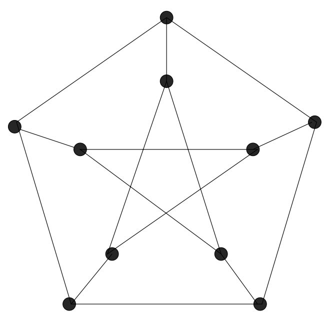
1 Hamiltonian path problem

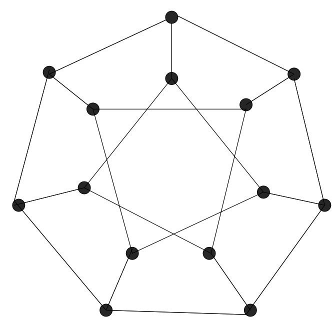
Definition 1 *A* **Hamiltonian path** P *in a directed graph* $G(V, H)$ *is a simple path which contains every vertex of the graph exactly once.*

A **Hamiltonian cycle** C *in a graph* $G(V, E)$ *is a simple cycle which contains every vertex exactly once.*

A graph is called **Hamiltonian** *if it has a Hamiltonian cycle.*

$$HAMPATH = \{\langle G, x, y \in V(G)\rangle : G \text{ contains a Hamiltonian path connecting } x \text{ with } y\}.$$





One of these graphs is Hamiltonian and one is not

The Traveling Salesman Problem

One of the first books on the travelling salesman problem was published in 1832:

"Der Handlungsreisende wie er sein soll und was er zu thun [sic] hat, um Auftrge zu erhalten und eines glcklichen Erfolgs in seinen Geschften gewi zu sein von einem alten Commis-Voyageur"

(The traveling salesman how he must be and what he should do in order to be sure to perform his tasks and have success in his business by a high commis-voyageur)

The travelling salesman problem was defined in the 1800s by the Irish mathematician W. R. Hamilton and by the British mathematician Thomas Kirkman.
Hamiltonian's Icosian Game (Hamilton; 1857) was a recreational puzzle based on finding a Hamiltonian cycle.

The general form of the TSP appears to have been first studied by mathematicians during the 1930s in Vienna and at Harvard, notably by Karl Menger, who defines the problem, considers the obvious brute-force algorithm, and observes the non-optimality of the nearest neighbor heuristic.

TSP *(optimization):* Given an $n \times n$ matrix

$$M = (w(i, j))_{i,j=1}^{n},$$

of non-negative real numbers, find a sequence $i_1, i_2, \ldots, i_n$ of distinct integers within $[1, n]$ (the tour of the salesman) which minimizes the cost function

$$Cost = w(i_1, i_2) + w(i_2, i_3) + \ldots + w(i_{n-1}, i_n) + w(i_n, i_1).$$

TSP *(decision):* Given an $n \times n$ matrix $M = (w(i, j))_{i,j=1}^{n}$, of non-negative real numbers and a number $T \geq 0$, is there a tour of the cost $\leq T$?

TSP *(special case):* Given an undirected graph $G(V, E)$, define $M = (w(i, j))$ as follows

$$m(i, j) = \begin{cases} 1, & \text{if } i, j \in E; \\ 2, & \text{if } i, j \notin E. \end{cases}$$

Is there a tour of the cost n?

Proposition 1 *The TSP originated by a graph (the special case) is equivalent to asking if a given graph G Hamiltonian.*

Consider the following two problems

Directed Hamiltonian Cycle, DHC: given a directed graph $G(V, E)$, is there a (directed) Hamiltonian cycle in G?

Undirected Hamiltonian Cycle, UHC: given an undirected graph $G(V, E)$, is there a Hamiltonian cycle in G?

Theorem 1 *DHC* $\leq_P$ *UHC*

Reduction:
Given a directed graph $G(V, E)$, construct an undirected graph H containing $3n$ vertices, where $n = |V|$, and $2n + |E|$ edges.

- for every vertex $v \in V(G)$, the new graph has three vertices $v_{in}, v, v_{out} \in V(H)$; and two edges $v_{in}v$ and vv_{out};
- for every edge $vu \in E(G)$, the new graph has an edge $v_{out}u_{in}$.

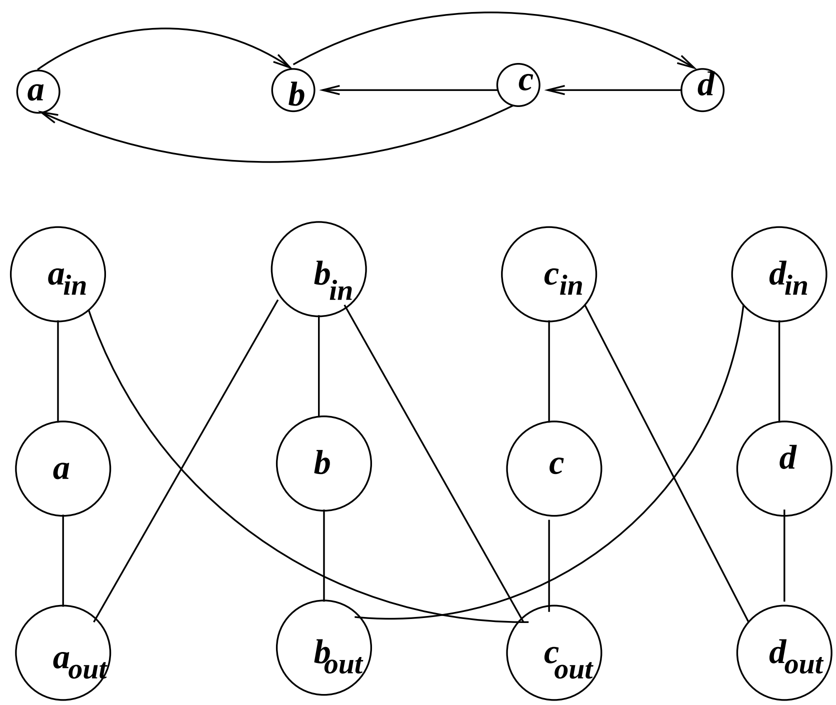
Proof.

First we prove that $G \in DHC$ implies $H \in UHC$.

Let $C = \{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\}$ be the sequence of vertices in the Directed Hamiltonian Cycle in G. Then consider the following sequence in H:

$$C' = \{x_{in}^{(1)}, x^{(1)}, x_{out}^{(1)}, x_{in}^{(2)}, x^{(2)}, x_{out}^{(2)}, \ldots, x_{in}^{(n)}, x^{(n)}, x_{out}^{(n)}.\}$$

From the reduction mapping, it immediately follows that C' is an Undirected Hamiltonian Cycle in H.



Now let C' be a Hamiltonian cycle in H. The vertex set of H consists of the old vertices from G and the new vertices $\{x_{in}, x_{out}\}$.

For every old vertex x, its degree in H is exactly 2, and the two edges incident to x are $x_{in}x$ and xx_{out}, thus C' must contain both of these edges.

We can assume, that up to symmetry, for some vertex x, C' traces the two incident edges from x_{in} to x followed by x_{out}.

Since all edges incident to x_{out} except for xx_{out} are of the type $x_{out}y_{in}$, for some old vertex y, the vertex which follows x_{out} in C' is y_{in} for some old y. Moreover (**the crucial point**), the vertex which follows y_{in} in C' is y, since not using edge $y_{in}y$ would make it impossible to include y into C'.

Hence, C' must be of the following pattern:

$$C' = \{x^1_{in}, x^1, x^1_{out}, x^2_{in}, x^2, x^2_{out}, \ldots, x^n_{in}, x^n, x^n_{out}.\}$$

for some sequence $\{x^1, x^2, \ldots, x^n\}$ of old vertices. This sequence is, obviously, a Hamiltonian Cycle. ∎

Theorem 2 *HAMPATH is NP-complete.*

Proof.
Part 1: HAMPATH is in class NP: a polynomial **verifier** for HAMPATH processes G with a **path** (the string c from the definition) and checks in polynomial time if that path is Hamiltonian and it connects x with y

Part 2: We will prove that 3-SAT is reducible to HAMPATH: thus, we describe a polynomial algorithm which, given a formula $\mathcal{F}$, constructs $\langle G, x, y \in V(G)\rangle$ so that $\mathcal{F}$ is satisfiable iff G has a Hamiltonian path starting at x and ending at y.

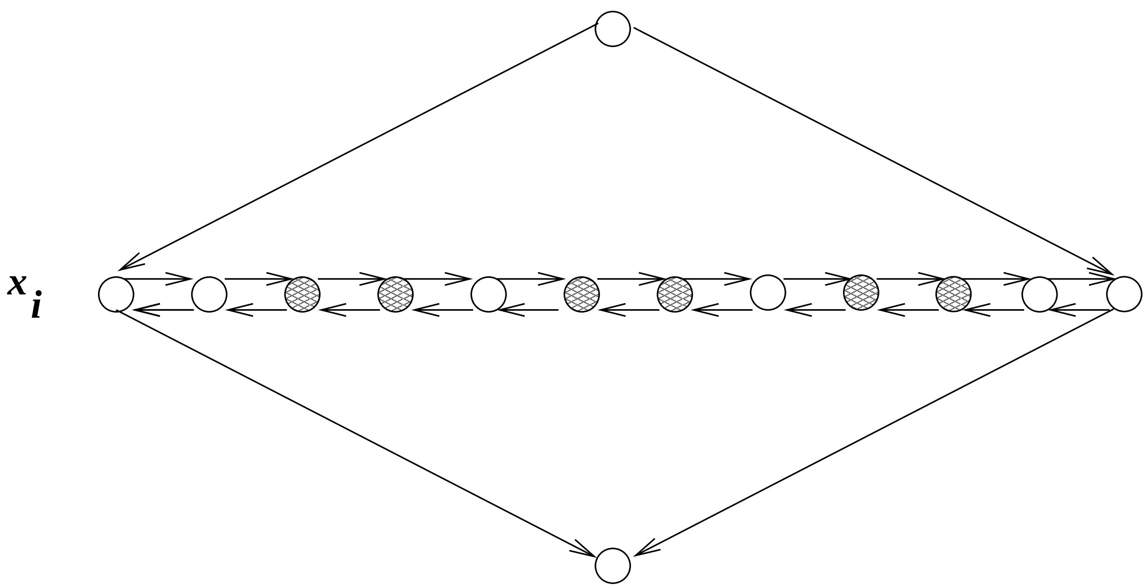
Construction:

Let $\mathcal{F}$ have n variables $x_1, x_2, \ldots, x_n$ and k clauses

$$C_1, C_2, \ldots, C_k,$$

each with 3 literals.

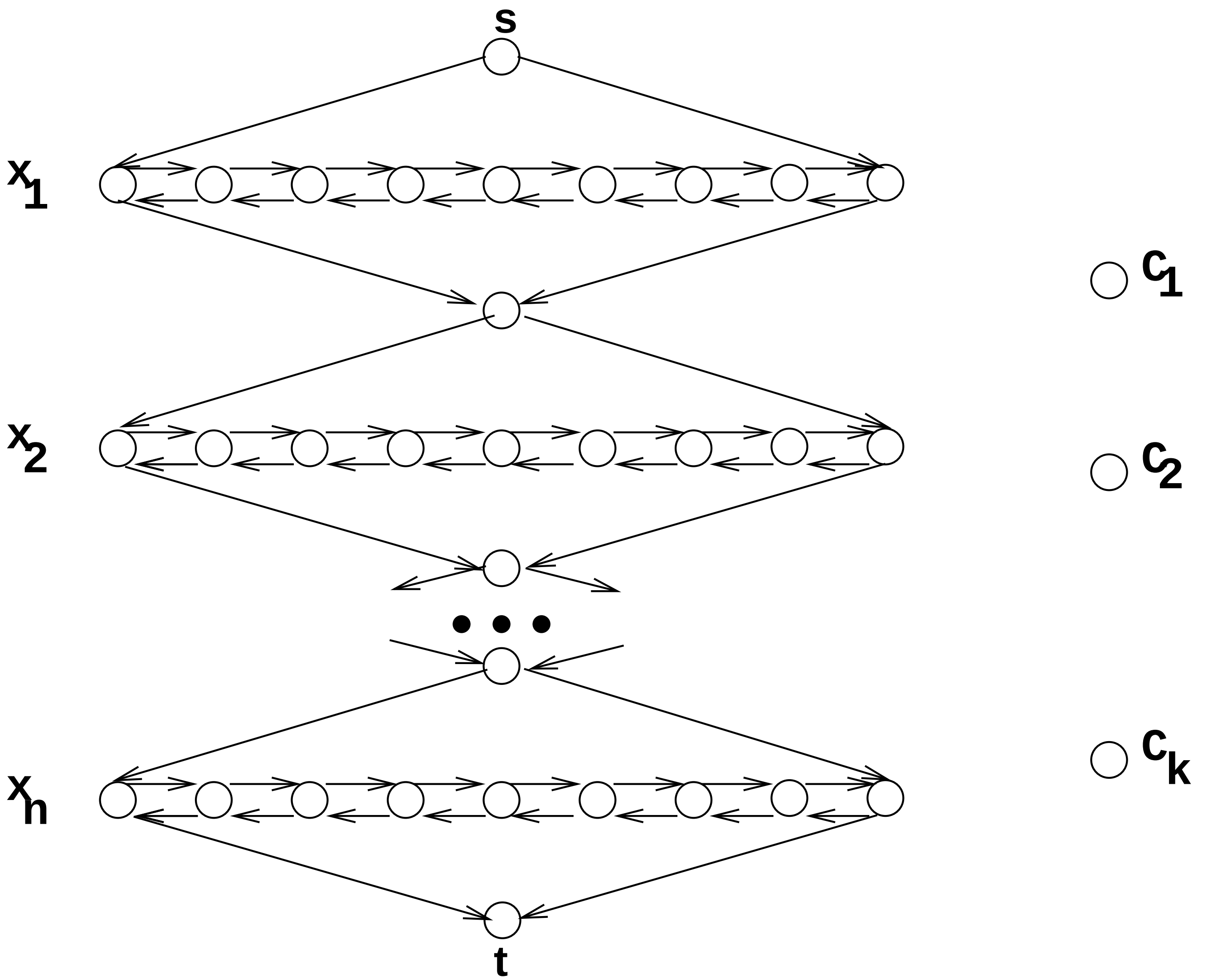
The first stage is to represent the variables and the clauses.



Every variable is represented by a diamond-graph; the horizontal raw of every diamond contains $3k+3$ vertices.

In addition, there are k vertices $\{C_i\}$ representing k clauses $\{C_i\}$.

The $3k+1$ internal vertices of the horizontal raw of every diamond are split into adjacent pairs, one per clause, separated by vertices as illustrated in the Figure.


s
x
1
x
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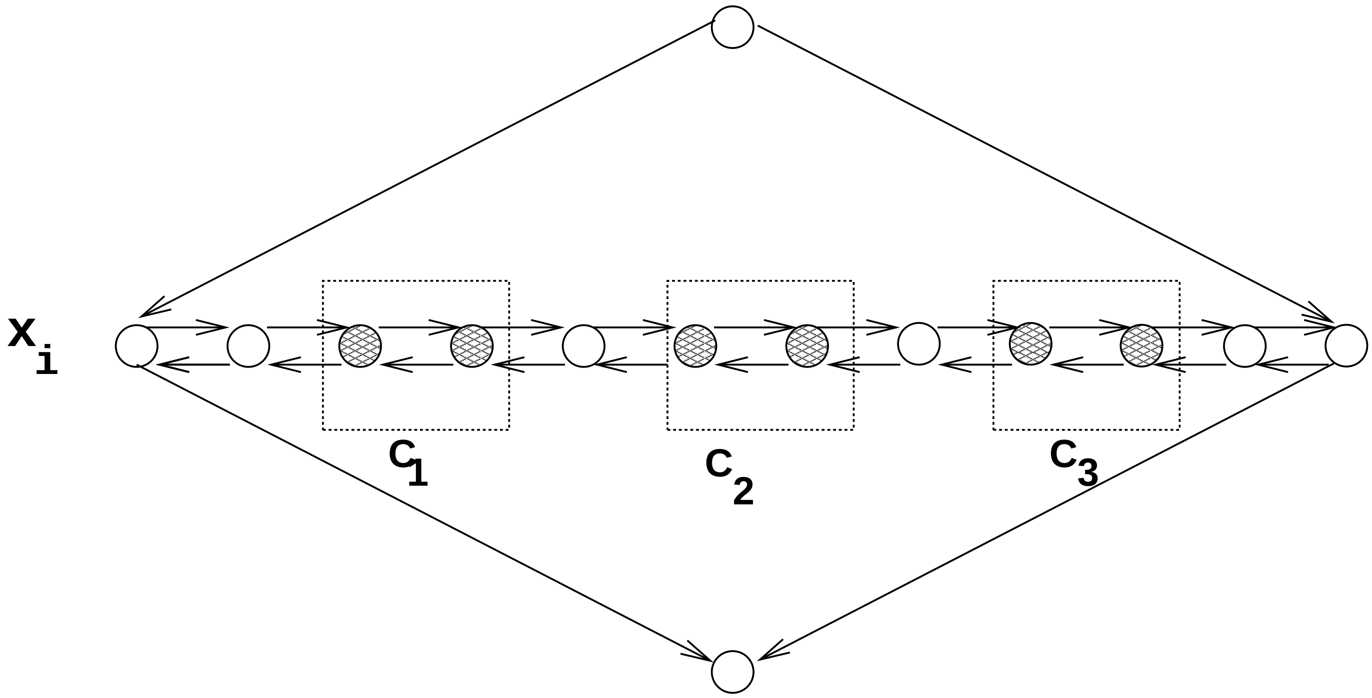
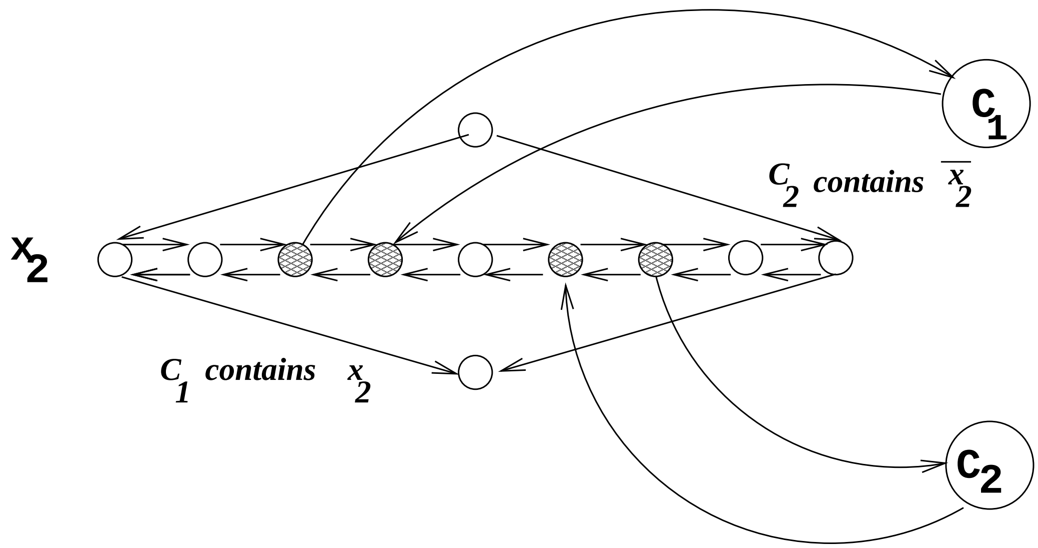


x_i
C_1
C_2
C_3

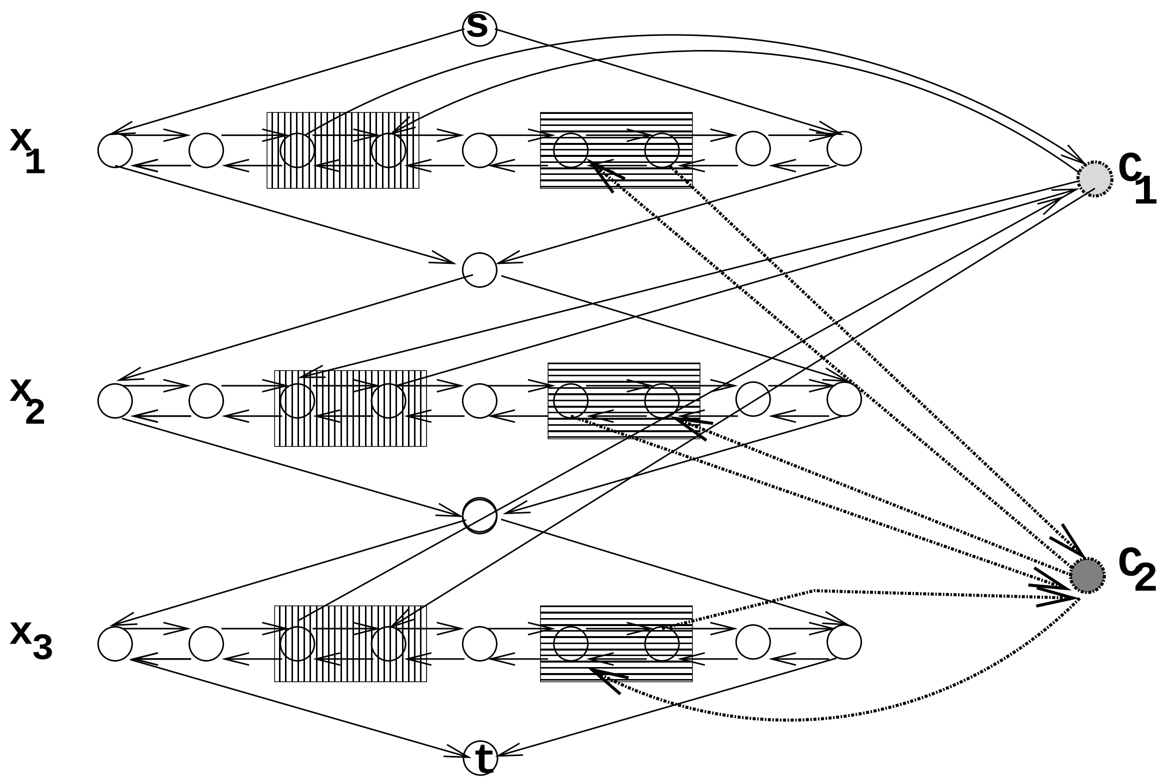
The **second (and last) stage** of the construction is to introduce edges representing the occurrences of literals in the clauses:



Example. The graph corresponding to formula

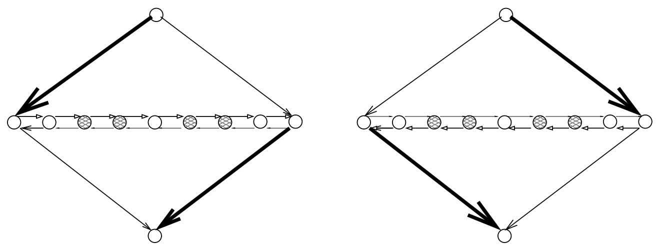
$$\mathcal{F} = (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3})$$

Find an assignment satisfying $\mathcal{F}$ and use it to construct a Hamiltonian cycle connecting s with t.



$\mathcal{F}$ is satisfiable $\implies$ $\langle G, s, t\rangle$ is Hamiltonian.

The rule for the Hamiltonian path:

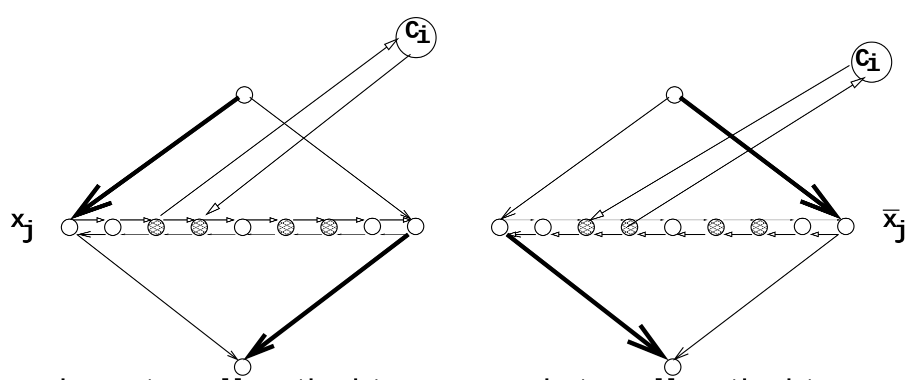


zig−zag thru the diamond if x = TRUE *zag−zig thru the diamond if x = FALSE*

To include the vertices corresponding to the clauses: for every clause C_i ($i = 1, \ldots, k$), select a literal l_i which gives value 1 to the clause.

If $l_i = x_j$ for some j, then $x_j = 1$ and the diamond corresponding to x_j is zig-zag-ed; detour from the diamond to include C_i into the path.

If $l_i = \overline{x_j}$ for some j, then $x_j = 0$ and the diamond corresponding to x_j is zag-zig-ed; detour from the diamond to include C_i into the path.



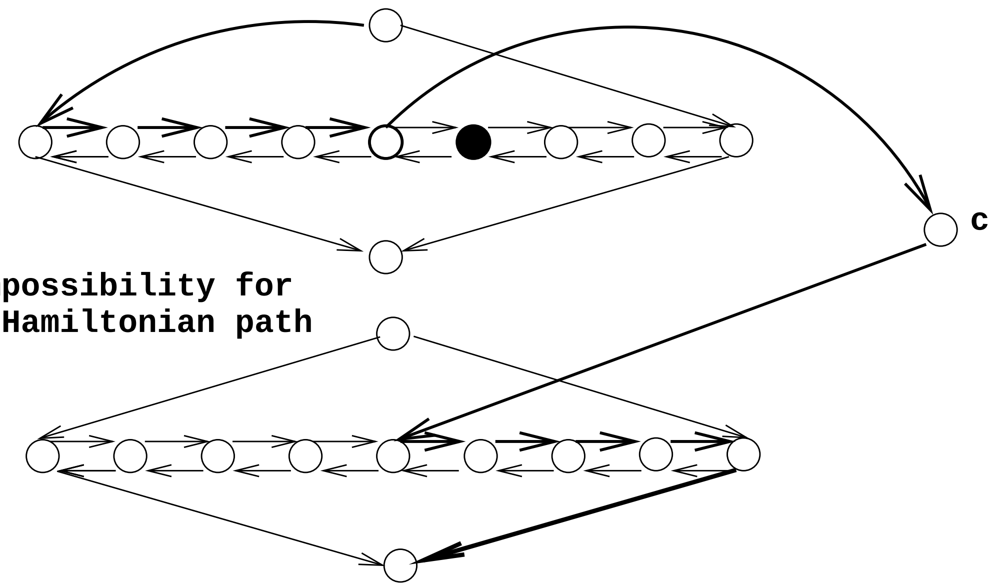
zig−zag tour allows the detour zag−zig tour allows the detour

$\langle G, s, t\rangle$ is Hamiltonian $\implies$ $\mathcal{F}$ is satisfiable
If the Hamiltonian path from s to t traverses all diamonds in without interruption, except for the detours to the clauses-vertices, then the satisfying assignment for $\mathcal{F}$ is obtained by the following rule:

> if the path zig-zags (resp. zag-zigs) the diamond corresponding to the variable x_i, then it is assigned the value 1 (resp. 0)

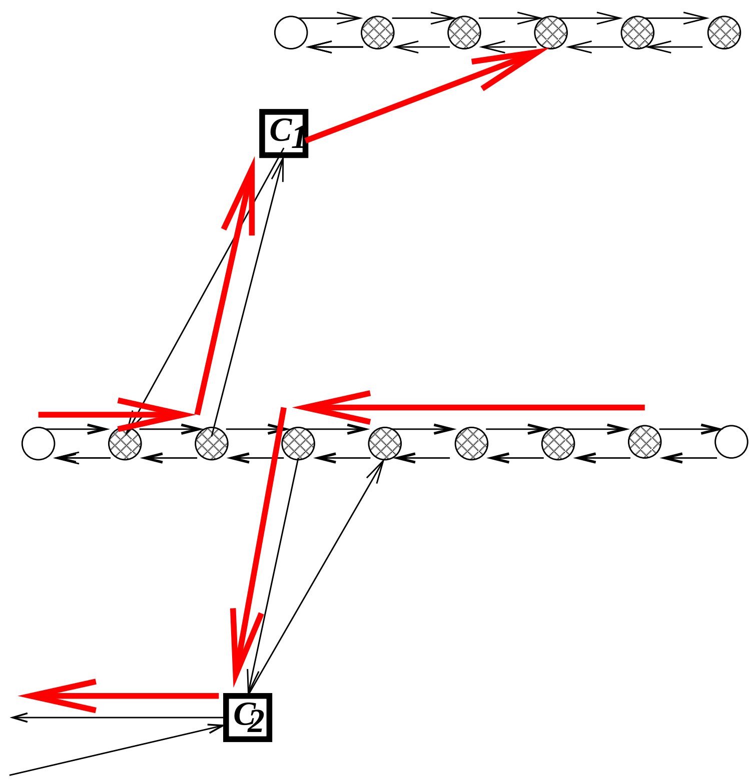
Claim. Every Hamiltonian path from s to t has the following property: for every diamond, the path starts from one end and reaches the other end before going through any not-top vertex of the next diamond.

Proof. The only way for the path to violate the Claim is to enter a clause-vertex from one diamond, but return to another diamond (see Figure below). It is then easy to show that a vertex adjacent to the departure-vertex on the diamond cannot be traced by the path (the shaded vertex on the Figure).

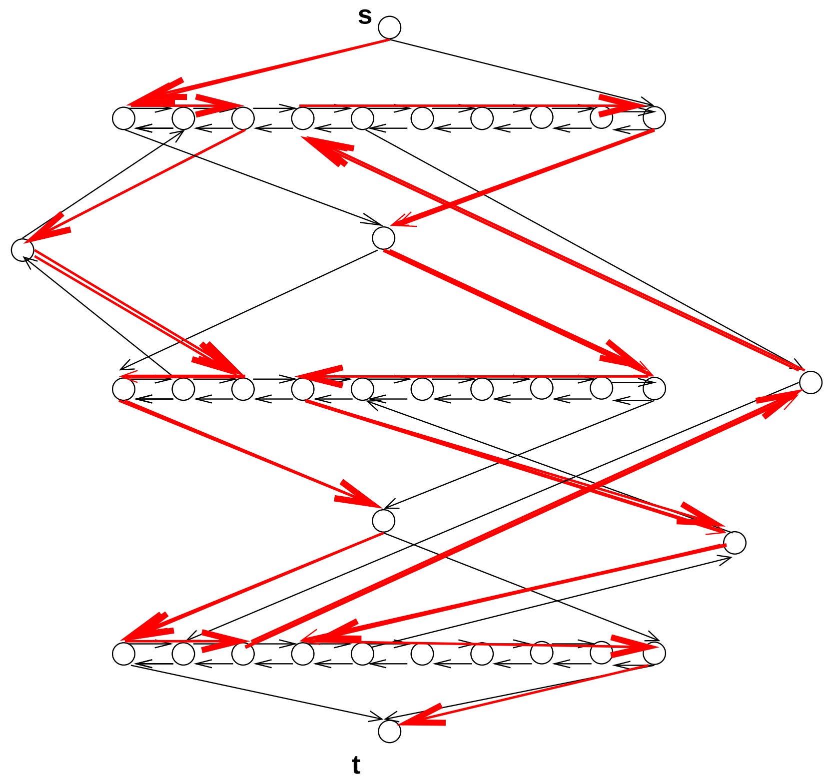


Impossibility for a Hamiltonian path

Property of the construction. In every diamond, the pairs of adjacent vertices that are associated with clause-vertices are separated by intermediate vertices. These separating vertices are essential when we are proving the second part of the "iff" statement: $\langle G, s, t\rangle$ is Hamiltonian $\implies$ $\mathcal{F}$ is satisfiable. Without these vertices, we cannot guarantee that graph G has Hamiltonian s, t-path which traverses the diamonds sequentially (see the next Figure.) The red edges form fragments of such a Hamiltonian s, t-path.



Also see the Figure below


s
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