Efficient Approach for Sinusoidal Frequency Estimation of Gapped Data

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***Abstract*—The problem of frequency estimation for noisy sinusoidal signals from multiple segments or channels, which are referred to as gapped data, is addressed. Based on linear prediction and weighted least squares techniques, an iterative relaxation-based frequency estimator is devised and analyzed. The proposed algorithm is also extended to harmonically related frequencies. Computer simulations are conducted to compare the estimation performance of the developed approach with an existing multichannel frequency estimator and Cramér-Rao lower bound.**

***Index Terms*—Frequency estimation, gapped data, multichannel parameter estimation, weighted least squares.**

I. Introduction

THE problem of sinusoidal parameter estimation from noisy data has attracted considerable attention [1][2] because of its important applications in many fields such as astronomy, radar, sonar, digital communications, biomedical engineering, speech and music analysis as well as instrumentation and measurement. The crucial step is to estimate the frequencies because they are nonlinear functions in the received data. Once the frequency estimates are obtained, the remaining parameters can then be computed straightforwardly. Although numerous frequency estimators have been proposed in the literature, limited attention has been paid to the scenarios of multiple segments or channels. The former occurs when the data sequence acquired by a single receiver has gaps which are due to the failure of the measuring device or the impossibility of performing measurements for certain periods such as in astronomical and radar applications [1][2]. On the other hand, a representative application example for the latter is quantification of magnetic resonance spectroscopy (MRS) signals from multiple detector coils [3]. Basically, the signal models in both cases are identical although the first and second correspond to the temporal and spatial domains, respectively, and we refer them to as gapped data [1][2]. The main contribution of this work is to develop an efficient estimation approach for gapped sinusoidal signals.

Manuscript received May 08, 2009; revised July 05, 2009. Date of publication March 22, 2010; date of current version May 12, 2010. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Andreas Jakobsson

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Digital Object Identifier 10.1109/LSP.2010.2046094

Consider the temporal gapped data scenario, the signal of the kth segment at time n can be modeled as

$$x_{k,n} = s_{k,n} + q_{k,n},$$
$$s_{k,n} = \sum_{m=1}^{M} \alpha_{k,m} \cos(\omega_m n + \phi_{k,m}) \tag{1}$$

with $k = 1, \ldots, K$ and $n = 1, \ldots, N_k$. The K, N_k and M are the number of segments, size of the kth segment and number of sinusoids, respectively. It is assumed that M is known *a priori* or accurately estimated by Akaike information criterion or minimum description length while $\{N_k\}$ are allowed to be distinct among segments. The mth frequency is represented as $\omega_m \in (0, \pi)$, and the amplitude and phase of the mth sinusoid in the kth segment are denoted by $\alpha_{k,m} > 0$ and $\phi_{k,m} \in (0, 2\pi]$, respectively. Note that for notation simplicity, we have aligned the time index n in all segments by absorbing the effect in the phase parameters. The noises $\{q_{k,n}\}$ are assumed uncorrelated zero-mean white Gaussian random processes with unknown variances σ^2, and extension to general Gaussian distribution is straightforward as long as the noise covariance matrix is known up to a multiplying constant. Our objective is to estimate ω_m, $m = 1, 2, \ldots, M$ given $x_{k,n}$, $k = 1, \ldots, K$, and $n = 1, \ldots, N_k$.

The rest of this letter is organized as follows. Based on linear prediction (LP) property of sinusoids and weighted least squares (WLS) technique, an accurate frequency estimator for gapped data is developed in Section II. The devised estimation algorithm can be considered as an extension of our previous work for the complete-data case [4]. The special case of harmonic sinusoidal signals, which corresponds to speech and music applications [5]–[7], is studied in Section III. Simulation results are presented in Section IV to evaluate the performance of the proposed approach by comparing with the multichannel parametric Hankel-(matrix) singular value decomposition (MC-P-HSVD) method [3] and Cramér-Rao lower bound (CRLB). Finally, conclusions are drawn in Section V.

II. Algorithm Development

Based on LP property of $\{s_{k,n}\}$, we have

$$\sum_{i=0}^{M-1} a_i (s_{k,n-i} + s_{k,n-2M+i}) + a_M s_{k,n-M} = 0 \tag{2}$$

with $a_0 = 1$ and $n = 2M+1, \ldots, N_k$ where $a_i, i = 1, \ldots, M$ are called LP coefficients. The frequencies $\{\omega_m\}$ are related to the following polynomial:

$$\sum_{i=0}^{M-1} a_i \left(z^i + z^{2M-i}\right) + a_M z^M = 0 \tag{3}$$

whose roots are $z = \exp(\pm j\omega_m), m = 1, \ldots, M$. With the use of $z = \exp(j\omega)$ and multiplying both sides of (3) by z^{-M}, $\{\omega_m\}$ can be determined in a more straightforward manner, namely, their values are given by the roots of $f(\omega)$:

$$f(\omega) = 2\sum_{i=0}^{M-1} a_i \cos\left((M-i)\omega\right) + a_M. \tag{4}$$

Using (2), a LP error vector for the kth segment, denoted by $\mathbf{e}_k$, can be set up:

$$\mathbf{e}_k = \mathbf{X}_k\mathbf{a} - \mathbf{b}_k, \quad k = 1, \ldots, K \tag{5}$$

where (see the equation at the bottom of the page). Here, ^{T} is the transpose operator. The WLS cost function is then

$$\mathbf{e}_k^T \mathbf{W}_k \mathbf{e}_k \tag{6}$$

where $\mathbf{W}_k$ is a weighting matrix and its optimal form is [4]

$$\mathbf{W}_k = \sigma^2 \left\{E\left(\mathbf{e}_k\mathbf{e}_k^T\right)\right\}^{-1} = \left(\mathbf{A}_k\mathbf{A}_k^T\right)^{-1} \tag{7}$$

with E represents the expectation operator and

$$\mathbf{A}_k = \text{Toeplitz}\Big(\left[1 \quad \mathbf{0}_{1\times(N-2M-1)}\right]^T, \\ \times \left[1 \quad a_1 \quad \ldots \quad a_M \quad a_{M-1} \quad \ldots \quad 1 \quad \mathbf{0}_{1\times(N-2M-1)}\right]\Big).$$

Here $^{-1}$ stands for matrix inverse, $\mathbf{0}_{u\times v}$ is the $u \times v$ zero matrix, $\text{Toeplitz}(\mathbf{u}, \mathbf{v}^T)$ is the Toeplitz matrix with $\mathbf{u}$ and $\mathbf{v}^T$ being the first column and first row, respectively. Stacking $\mathbf{X}_k$ and $\mathbf{b}_k$, $k = 1, 2, \ldots, K$ to form $\mathbf{X}$ and $\mathbf{b}$, respectively, the WLS estimate of $\mathbf{a}$, denoted by $\hat{\mathbf{a}}$, is obtained by extending (6) [8]:

$$\hat{\mathbf{a}} = \arg\min_{\mathbf{a}} \mathbf{e}^T\mathbf{W}\mathbf{e} = \left(\mathbf{X}^T\mathbf{W}\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{W}\mathbf{b} \\ = \left(\sum_{k=1}^{K}\mathbf{X}_k^T\mathbf{W}_k\mathbf{X}_k\right)^{-1}\left(\sum_{k=1}^{K}\mathbf{X}_k^T\mathbf{W}_k\mathbf{b}_k\right) \tag{8}$$

where $\mathbf{e} = \mathbf{X}\mathbf{a} - \mathbf{b}$, $\mathbf{X} = [\mathbf{X}_1^T \ \ldots \ \mathbf{X}_K^T]^T$ $\mathbf{b} = [\mathbf{b}_1^T \ \ldots \ \mathbf{b}_K^T]^T$ and $\mathbf{W} = \sigma^2\left\{E\left(\mathbf{e}\mathbf{e}^T\right)\right\}^{-1} =$ $\text{blkdiag}(\mathbf{W}_1, \ldots, \mathbf{W}_K)$ with $\text{blkdiag}(\cdot)$ denotes block diagonal matrix. As the weighting matrices $\mathbf{W}_k$, $k = 1, 2, \ldots, K$, depend on the unknown $\mathbf{a}$, we follow [4] to estimate $\mathbf{a}$ in an iterative relaxation manner and the procedure is summarized in Table I.

TABLE I
ALGORITHM FOR FREQUENCY ESTIMATION OF GAPPED DATA

(i) Set $\mathbf{W}_k = \mathbf{I}_{N_k}$, $k = 1, 2, \cdots, K$ where $\mathbf{I}_u$ denotes the $u \times u$ identity matrix
(ii) Estimate $\hat{\mathbf{a}}$ using (8)
(iii) Construct $\mathbf{W}_k$, $k = 1, 2, \cdots, K$, using (7)
(iv) Repeat Steps (ii) and (iii) until parameter convergence
(v) Substitute $\hat{\mathbf{a}}$ into (3) or (4) and solve the roots to obtain $\hat{\omega}_m$, $m = 1, 2, \cdots, M$

To analyze the variances of the frequency estimates, we first notice that when the data length and signal-to-noise ratio (SNR) are sufficiently large, the mean and covariance matrix of $\hat{\mathbf{a}}$, $E\{\hat{\mathbf{a}}\}$ and $\text{cov}(\hat{\mathbf{a}})$, can be approximated as [4]:

$$E\{\hat{\mathbf{a}}\} \approx \mathbf{a} \tag{9}$$

and

$$\text{cov}(\hat{\mathbf{a}}) \approx \sigma^2\left(\mathbf{S}^T\mathbf{W}\mathbf{S}\right)^{-1} \tag{10}$$

where $\mathbf{S}$ is the noise-free version of $\mathbf{X}$ and $\mathbf{W}$ is assumed the ideal form which is characterized by the true $\{a_i\}$. Assuming that $f(\omega)$ is sufficiently smooth around $\omega = \omega_m$ and the corresponding estimate $\hat{\omega}_m$ is located at a reasonable proximity of ω_m, the variance of $\hat{\omega}_m$, $\text{var}(\hat{\omega}_m)$, is computed as [9]:

$$\text{var}(\omega_m) \approx \left.\frac{E\{f^2(\omega)\}}{E\{(f'(\omega))^2\}}\right|_{\omega=\omega_m} \tag{11}$$

where $\{a_i\}$ are replaced by $\{\hat{a}_i\}$ and their first-order and second-order statistics are obtained from (9) and (10). Although there is no closed-form expression for (11), empirical studies show that (11) is identical to the CRLB for $\{\omega_m\}$ (See Appendix).

III. MODIFICATION FOR HARMONIC FREQUENCIES

In this section, the proposed algorithm is applied to fundamental frequency estimation [5]–[7] which is important for music and speech signal processing. Though both [5] and the proposed algorithm utilize WLS technique, we consider real sinusoids instead of complex cisoids. Moreover, results of [5] indicates its suboptimality while it is shown in Section IV that the performance of the proposed algorithm can attain the CRLB when the SNR is sufficiently high. Unlike Section III, the frequencies considered here are in harmonic relation, that is, $\omega_m = m\omega_1$, $m = 1, 2, \ldots, M$ with ω_1 is the fundamental

$$\mathbf{X}_k = \begin{bmatrix} x_{k,N_k-1} + x_{k,N_k-2M+1} & \cdots & x_{k,N_k-M+1} + x_{k,N_k-M-1} & x_{k,N_k-M} \\ x_{k,N_k-2} + x_{k,N_k-2M} & \cdots & x_{k,N_k-M} + x_{k,N_k-M-2} & x_{k,N_k-M-1} \\ \vdots & \ddots & \vdots & \vdots \\ x_{k,2M-1} + x_{k,2} & \cdots & x_{k,M+1} + x_{k,M-1} & x_M \end{bmatrix}$$

$$\mathbf{b}_k = -\left[x_{k,N_k} + x_{k,N_k-2M} \quad x_{k,N_k-1} + x_{k,N_k-2M-1} \quad \ldots \quad x_{k,2M} + x_{k,1}\right]^T, \mathbf{a} = \left[a_1 \quad a_2 \quad \ldots \quad a_M\right]^T.$$

TABLE II
SIMULATION SETTINGS OF $\lambda_{k,m}$

m \ k	1	2	3	4	5
1	1	2.5	1.5	4	3
2	2	3	1.5	3	2
3	2	1.5	1	1	2
4	3	2	2	2	1

TABLE III
SIMULATION SETTINGS OF $\phi_{k,m}$

m \ k	1	2	3	4	5
1	1	0.7	0.5	2.7	2.1
2	1.5	1.2	1	1.5	3
3	2	0.5	2.9	0.5	2.4
4	2.5	1	2.5	1.2	1.5

TABLE IV
ω_m/π IN THE FIRST TEST

m	1	2	3	4
$\frac{\omega_m}{\pi}$	0.1	0.26	0.52	0.8

TABLE V
SIMULATION SETTINGS OF N_k

k	1	2	3	4	5
N_k	20	30	50	100	200

frequency. In the first stage, we ignore the harmonic structure of the frequencies and get a set of coarse frequency estimates using the algorithm in Table I and denote the corresponding vector by $\hat{\boldsymbol{\omega}} = [\hat{\omega}_1 \quad \cdots \quad \hat{\omega}_M]^T$ such that $\hat{\omega}_1 < \cdots < \hat{\omega}_M$. In the second stage, the harmonic relation is exploited to obtain a fine-tune estimate of ω_1 by solving another WLS cost function:

$$\hat{\omega} = \arg\min_{\omega} \boldsymbol{\tau}^T \mathbf{R} \boldsymbol{\tau} = \left(\boldsymbol{\xi}^T \mathbf{R} \boldsymbol{\xi}\right)^{-1} \boldsymbol{\xi}^T \mathbf{R} \hat{\boldsymbol{\omega}} \qquad (12)$$

where $\boldsymbol{\tau} = \boldsymbol{\xi}\omega - \hat{\boldsymbol{\omega}}$, $\boldsymbol{\xi} = [1 \ \ldots \ M]^T$, $\mathbf{R} = \sigma^2 \left\{E\left(\boldsymbol{\tau}\boldsymbol{\tau}^T\right)\right\}^{-1} = \sigma^2 \left(\mathrm{cov}\left(\hat{\boldsymbol{\omega}}\right)\right)^{-1}$ with $\mathrm{cov}(\hat{\boldsymbol{\omega}})$ denotes the unknown covariance matrix of $\hat{\boldsymbol{\omega}}$. It is seen in (12) that the value of σ^2 is not required as it will be canceled out. If SNR is sufficiently high, we expect that the WLS estimate gives optimum performance and thus $\mathrm{cov}(\hat{\boldsymbol{\omega}})$ is approximately equal to the CRLB matrix of $\boldsymbol{\omega}$ where $\boldsymbol{\omega} = [\omega_1 \ldots \quad \omega_M]^T$ with $\omega_1 < \cdots < \omega_M$. In the Appendix, the CRLB matrix of all unknown parameters, denoted by $\boldsymbol{\eta} = [\boldsymbol{\omega}^T \quad \boldsymbol{\zeta}^T]^T$ where $\boldsymbol{\zeta} = [\boldsymbol{\alpha}_1^T \quad \boldsymbol{\phi}_1^T \quad \ldots \quad \boldsymbol{\alpha}_K^T \quad \boldsymbol{\phi}_K^T]^T$, $\boldsymbol{\alpha}_k = [\alpha_{k,1} \quad \ldots \quad \alpha_{k,M}]^T$ and $\boldsymbol{\phi}_k = [\phi_{k,1} \quad \ldots \quad \phi_{k,M}]^T$, is derived and it is employed for the estimation of $\mathrm{cov}(\hat{\boldsymbol{\omega}})$. Noting that the complete CRLB matrix is also parameterized by $\boldsymbol{\alpha}_k$ and $\boldsymbol{\phi}_k$, we suggest to use the least squares (LS) technique to solve for them as follows. With the estimated fundamental frequency, we construct a system of linear equations [8]:

$$\mathbf{G}_k \boldsymbol{\rho}_k \approx \mathbf{x}_k, \quad k = 1, \ldots, K \qquad (13)$$

where (see the equation at the bottom of the page). The LS estimate of $\boldsymbol{\rho}_k$, denoted by $\hat{\boldsymbol{\rho}}_k$, is

$$\hat{\boldsymbol{\rho}}_k = \left(\mathbf{G}_k^T \mathbf{G}_k\right)^{-1} \mathbf{G}_k^T \mathbf{x}_k. \qquad (14)$$

Based on (14), the estimates of $\alpha_{k,m}$ and $\phi_{k,m}$ are then $\hat{\lambda}_{k,m} = \sqrt{\hat{\boldsymbol{\rho}}_{k,2m-1}^2 + \hat{\boldsymbol{\rho}}_{k,2m}^2}$ and $\hat{\phi}_{k,m} = \tan^{-1}(\hat{\boldsymbol{\rho}}_{k,2m}/\hat{\boldsymbol{\rho}}_{k,2m-1})$ where $\hat{\boldsymbol{\rho}}_{k,u}$ is the uth element of $\hat{\boldsymbol{\rho}}_k$. Substituting $\hat{\omega}_k$, $\hat{\alpha}_{k,m}$ and $\hat{\phi}_{k,m}$ into the Fisher information matrix (FIM) of $\boldsymbol{\eta}$, $\mathrm{cov}(\hat{\boldsymbol{\omega}})$ is estimated as the upper left $M \times M$ submatrix of the FIM inverse.

IV. SIMULATION RESULTS

Computer simulations have been conducted to evaluate the frequency estimation performance of the proposed approach for gapped sinusoidal signals in additive zero-mean white Gaussian noise. The algorithm is terminated if the norm of parameter difference in two successive iterations is less than 0.01. In our study, $K = 5$, $M = 4$ and the values of $\{\alpha_{k,m}\}$, $\{\phi_{k,m}\}$, $\{\omega_m\}$ and N_k are listed in Tables II–V. The SNR is defined as the average signal power (ASP) divided by σ^2 where $\mathrm{ASP} = 1/2K \sum_{m=1}^{M} \sum_{k=1}^{K} \alpha_{k,m}^2$. We scale the noises $\{q_{k,n}\}$ to produce different SNR conditions. The mean square frequency error (MSFE) is employed as the performance measure to contrast with the MC-P-HSVD algorithm and CRLB. All results provided are averages of 1000 independent runs.

In the first test, there is no harmonic relationship in the frequency parameters. Fig. 1 shows the MSFEs for $\{\omega_m\}$. It is seen that the MSFE performance of the proposed approach is close to the CRLB when $\mathrm{SNR} \geq 6$ dB while that of the MC-P-HSVD algorithm deviates from the CRLB by more than 5 dB. The variance expression of (11) is also validated for sufficiently high SNR conditions. In the second test, all the simulation settings remain unchanged except that the frequencies are harmonically related and the structure is known *a priori* so that the two-step approach is employed in this scenario. The fundamental frequency is set to $\omega_1 = 0.1\pi$. The MSFE results are shown in

$$\mathbf{G}_k = \begin{bmatrix} \cos(\hat{\omega}_1) & -\sin(\hat{\omega}_1) & \ldots & \cos(\hat{\omega}_M) & -\sin(\hat{\omega}_M) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \cos(N_k\hat{\omega}_1) & -\sin(N_k\hat{\omega}_1) & \ldots & \cos(N_k\hat{\omega}_M) & -\sin(N_k\hat{\omega}_M) \end{bmatrix}$$

$$\boldsymbol{\rho}_k = [\alpha_{k,1}\cos(\phi_{k,1}) \quad \alpha_{k,1}\sin(\phi_{k,1}) \quad \ldots \quad \alpha_{k,M}\cos(\phi_{k,M}) \quad \alpha_{k,M}\sin(\phi_{k,M})]^T$$

$$\mathbf{x}_k = [x_{k,1} \quad x_{k,2} \quad \ldots \quad x_{k,N_k}]^T.$$

$$\mathbf{D} = [\,\mathbf{D}_{\boldsymbol{\omega}}^T \quad \mathbf{D}_{\boldsymbol{\alpha\phi}}^T\,]^T$$

$$\text{where} \quad \mathbf{D}_{\boldsymbol{\omega}} = [\,\boldsymbol{\Omega}_1 \quad \boldsymbol{\Omega}_2 \quad \dots \quad \boldsymbol{\Omega}_K\,]$$

$$\mathbf{D}_{\boldsymbol{\alpha\phi}} = \text{blkdiag}\left([\,\boldsymbol{\Lambda}_1^T \quad \boldsymbol{\Phi}_1^T\,]^T, [\,\boldsymbol{\Lambda}_2^T \quad \boldsymbol{\Phi}_2^T\,]^T, \dots, [\,\boldsymbol{\Lambda}_K^T \quad \boldsymbol{\Phi}_K^T\,]^T\right)$$

$$\boldsymbol{\Omega}_k = \begin{bmatrix} -\alpha_{k,1}\sin(\omega_1+\phi_{k,1}) & -2\alpha_{k,1}\sin(2\omega_1+\phi_{k,1}) & \dots & -N_k\alpha_{k,1}\sin(N_k\omega_1+\phi_{k,1}) \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{k,M}\sin(\omega_M+\phi_{k,M}) & -2\alpha_{k,M}\sin(2\omega_M+\phi_{k,M}) & \dots & -N_k\alpha_{k,M}\sin(N_k\omega_M+\phi_{k,M}) \end{bmatrix}$$

$$\boldsymbol{\Lambda}_k = \begin{bmatrix} \cos(\omega_1+\phi_{k,1}) & \cos(2\omega_1+\phi_{k,1}) & \dots & \cos(N_k\omega_1+\phi_{k,1}) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(\omega_M+\phi_{k,M}) & \cos(2\omega_M+\phi_{k,M}) & \dots & \cos(N_k\omega_M+\phi_{k,M}) \end{bmatrix}$$

$$\boldsymbol{\Phi}_k = \begin{bmatrix} -\alpha_{k,1}\sin(\omega_1+\phi_{k,1}) & -\alpha_{k,1}\sin(2\omega_1+\phi_{k,1}) & \dots & -\alpha_{k,1}\sin(N_k\omega_1+\phi_{k,1}) \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{k,M}\sin(\omega_M+\phi_{k,M}) & -\alpha_{k,M}\sin(2\omega_M+\phi_{k,M}) & \dots & -\alpha_{k,M}\sin(N_k\omega_M+\phi_{k,M}) \end{bmatrix}.$$

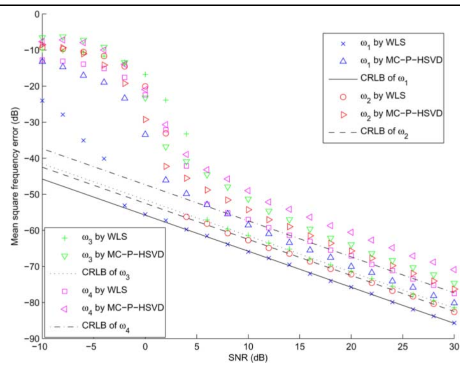

Fig. 1. Mean square error of $\{\omega_i\}$ versus SNR.

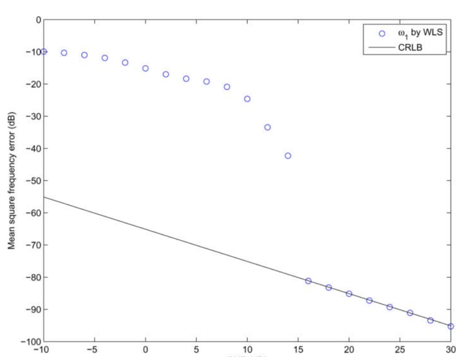

Fig. 2. Mean square error of fundamental frequency versus SNR.

Fig. 2 and we observe again that the proposed method gives optimum estimation performance for sufficiently high SNR condition, say, $\text{SNR} \geq 16$ dB.

V. Conclusion

An iterative relaxation-based algorithm has been devised to estimate the frequencies of gapped sinusoidal signals in additive noise. The main ideas are to utilize the linear prediction property of sinusoids and WLS technique. The variances of the frequency estimates are derived and validated. Algorithm modification to the periodic signals is achieved by using a second WLS step to exploit the harmonic relationship. Simulation results show that the mean square frequency errors of the proposed algorithm are smaller than those of [3] and able to attain the CRLB when the signal-to-noise ratio is sufficiently high.

Appendix

The probability density function of $\mathbf{x} = [\,\mathbf{x}_1^T \quad \dots \quad \mathbf{x}_K^T\,]^T$ where $\mathbf{x}_k = [\,x_{k,1} \quad \dots \quad x_{k,N_k}\,]^T$, is

$$\frac{1}{(2\pi)^{N/2}\sigma^N} \exp\left(\frac{-(\mathbf{x}-\mathbf{s})^T(\mathbf{x}-\mathbf{s})}{2\sigma^2}\right) \qquad \text{(A.1)}$$

where $N = \sum_{k=1}^{K} N_k$ and $\mathbf{s}$ is the noise-free version of $\mathbf{x}$. The FIM of $\boldsymbol{\eta}$ is $\text{FIM}(\boldsymbol{\eta}) = 1/\sigma^2 \mathbf{D}\mathbf{D}^T$ where $\mathbf{D} = \partial \mathbf{s}^T / \partial \boldsymbol{\eta}$. The mathematical form of $\mathbf{D}$ is given by the equation shown at the top of the page.

The CRLB of $\boldsymbol{\eta}$ is given by the diagonal elements of the inverse of $\text{FIM}(\boldsymbol{\eta})$.

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