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Nonlinear prediction of spatio-temporal time series

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Abstract. A prediction scheme for spatio-temporal time series is presented that is based on reconstructed local states. As a numerical example the evolution of a Kuramoto-Sivashinsky equation is forecasted using previously sampled data.

1. Introduction

The starting point for most analysis methods used in *Nonlinear Time Series Analysis* are measurements of a single observable of the system of interest [1]. Many interesting dynamical systems, however, are spatially extended and thus any description using only a few local or global observables may be incomplete. To overcome these limitations we present an approach for local state space reconstructions that turned out to be very useful for analysis of (large) data sets from spatially extended systems.

Let $\{\mathbf{s}^n\}$ be the temporal sequence of spatial patterns (snapshots) of the spatio-temporal evolution with $n = 1, ..., N$. Each pattern $\mathbf{s}^n$ consists of M elements and may be represented by a M-dimensional vector with elements s_m^n $(m = 1, ..., M)$. Most of the techniques applied so far to spatio-temporal time series (STTS) are based on linear decompositions into spatial modes that constitute (orthogonal) bases in a high dimensional vector space [2, 3]. However, not always can such a decomposition yield a low-dimensional description of the data, even in cases, where the STTS is governed by a low dimensional attractor [4]. Another approach for analysing and modelling STTS consists in the application of system identification tools and was successfully applied in cases where the underlying spatio-temporal system can be described by a partial differential equation (PDE) [5].

An alternative to decomposition into global linear modes or identifying global nonlinear models is the construction of local states and models [6]. This approach may be applied in all cases where the dynamics of the spatio-temporal system of interest is governed by spatially local (inter-) actions and where we may assume that the (local) state of the system in a small region of space

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can be represented by a vector $\mathbf{x}$. Such a reconstruction of a local state space can be done in different ways and the basic idea was suggested for the first time by K. Kaneko in Ref. [7]. Later Rubin [8] used a similar approach for characterizing dynamic and static pattern und recently Orstavik and Stark [8] used spatio-temporal embedding techniques for cross-prediction of coupled map lattices. In the following we shall assume that local states exist in a unique and deterministic sense that allows in principle exact predictions (without stochastic components). This conjecture is motivated by the case of coupled map lattices where it can be easily verified [6]. Numerical simulations with coupled oscillators and PDEs show, however, that it seems to be correct also for other classes of systems. We want to stress that the latter is a necessary condition for the reconstruction techniques to be useful in time series analysis of real world data, because as soon as the local states are succesfully (re)constructed they can be used for subsequent analysis or, as in the example given below, for predicting the underlying dynamics.

In the following we will discuss only the case of one dimensional spatial pattern. Generalizations for higher dimensional cases are straight forward.

2. Reconstruction of local states

The entire STTS may be represented by a $N \times M$-matrix S as shown in Fig. 1. The state of the system at position m and time n is reconstructed in analogy to the delay embedding of scalar time series [1]. Here we use the center element s_m^n, some of its neighbors and the corresponding values in the past to construct the *state vector*

$$\mathbf{x}_m^n = (s_{m-IK}^n, ..., s_m^n, ..., s_{m+IK}^n, ..., s_{m-IK}^{n-JL}, ..., s_m^{n-JL}, ..., s_{m+IK}^{n-JL}) \qquad (1)$$

where I is the number of spatial neighbors, J is the number of temporal neighbors (in the past), K is the spatial shift which has a similar meaning as the time delay L (time lag) known from the delay embedding of scalar time series. This construction is visualized in Fig. 1 for $I = 1$, $J = 3$, $K = 2$ and $L = 2$. The dimension d of the state vector $\mathbf{x}_m^n \in R^d$ equals $d = (J + 1)(1 + 2I)$.

3. Boundary conditions

Spatial boundaries of the dynamical process generating the STTS can be taken into account by constructing the corresponding states separately in a way analogous to that described above. In principle all states $\mathbf{x}_m^n$ containing boundary values of the STTS have to be devided into different classes where each class is characterized by the distance of the center of the region from the boundary. For predicting or modelling only states of the relevant class have then to be used. This decomposition into classes can be implemented in different ways [6]. In order to take into account the influence of the boundary we shall use in the

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follo wing example a *penalty function*

$$w(x) = a(x_c - x)^b \tag{2}$$

which serves as an additional coordinate for the selection of appropriate neighbors in state space. Depending on the center point x_c of the spatial domain and the free parameters a and b preferentially those neighboring states are selected which hav e a similar location with respect to the boundaries.

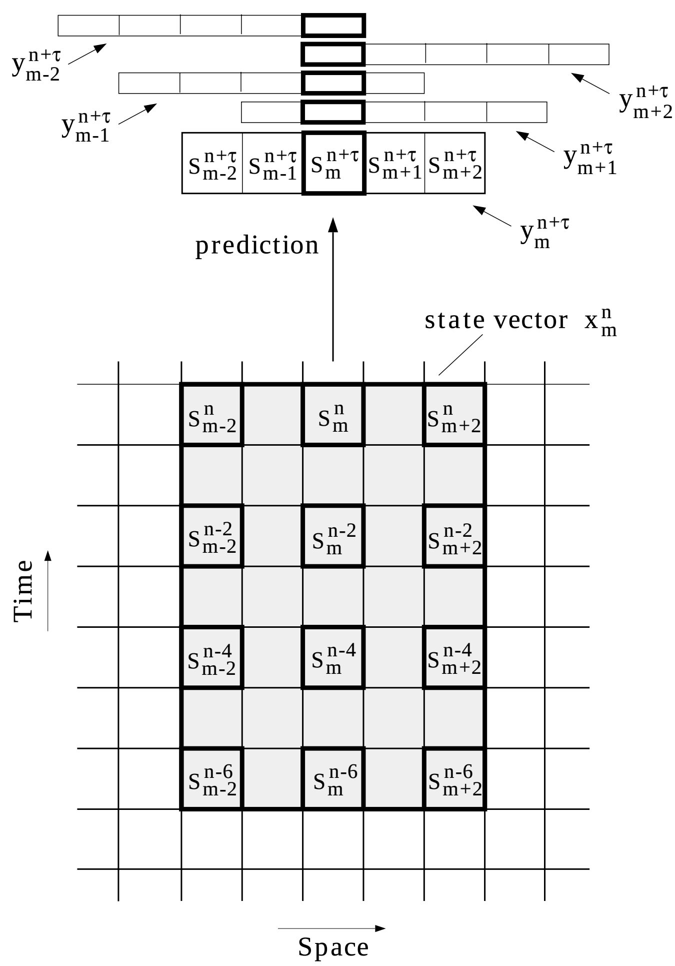

Figure 1: Local reconstruction of states from regions of the spatio-temporal time series $S = \{\mathbf{s}^n\}^{n=1,\ldots,N} = \{s_m^n\}_{m=1,\ldots,M}^{n=1,\ldots,N}$ and prediction of the future v alue of the center element $s_m^{n+\tau}$ using forecasts of ov erlapping regions ..., $\mathbf{y}_{m-2}^{n+\tau}$, $\mathbf{y}_{m-1}^{n+\tau}$, $\mathbf{y}_{m}^{n+\tau}$, $\mathbf{y}_{m+1}^{n+\tau}$, $\mathbf{y}_{m+2}^{n+\tau}$, ... that give the future evolution of the center region and of neighboring regions.

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4. Nonlinear prediction

As an application of the local state reconstruction we consider here the prediction of STTS that can be implemented in different ways. Either only the future value of the central element $s_m^{n+\tau}$ is predicted [6] or a complete region $\mathbf{y}_m^{n+\tau} = (s_{m-I}^{n+\tau}, ..., s_{m-1}^{n+\tau}, s_m^{n+\tau}, s_{m+1}^{n+\tau}, ..., s_{m+I}^{n+\tau})$ "in front" of the reconstruction region is forecasted (see Fig. 1) where τ gives the prediction time interval. For this purpose a training set of states $\mathcal{A} = \{\mathbf{x}_m^n\}$ is derived from N_{train} samples $\mathbf{s}^n$ of the STTS. For these states the preimage-image relation $\mathbf{x}_m^n \mapsto \mathbf{y}_m^{n+\tau}$, is known and is assumed to represent a nonlinear map F. To approximate this map the reference state $\mathbf{x}_m^n$ is reconstructed and its nearest neighbor $\mathbf{x}_j^i$ is selected from the training set $\mathcal{A}$. Using the indices (i, j) of the nearest neighbor the underlying map F is locally approximated by the future values $\mathbf{y}_j^{i+\tau}$ of the region "in front " of the state $\mathbf{x}_j^i$. In this way we obtain $1 + 2I$ estimates for a single future element $s_m^{n+\tau}$ of the STTS which are then averaged using an overlap-add approach as illustrated in Fig. 1. (For locations close to the boundary the number of estimates decreases to 1.) Alternatively, locally linear or nonlinear maps may also be used to approximate the dynamics F. Furthermore, predictions over longer periods of time ($\tau > 1$) can be computed as a single large step or iteratively by concatenating steps with $\tau = 1$.

5. Numerical example

As an example we shall use in the following a STTS that is generated using the *Kuramoto-Sivashinsky* (KS) equation [9]:

$$u_t = -2uu_x - u_{xx} - u_{xxxx} \qquad (3)$$

in the interval $[0, L]$ with $u = u_x = 0$ at the boundaries $x = 0$ and $x = L = 200$. The spatio-temporal dynamics of this system is governed by a hyperchaotic attractor with Lyapunov dimension $D_L \approx 43$. For predicting the dynamics of this PDE states close to boundaries have been selected using the penalty function (2) with $a = 3$ and $b = 7$ and the prediction is performed iteratively using the overlap-add approach based on predicted regions $\mathbf{y}_m^{n+\tau}$ (see Fig. 1).

Figure 2a shows the spatio-temporal evolution of the KS-equation in the time interval that is used as a training set for the prediction of the test data given in Fig. 2b. Figure 2c shows the results of a iterative prediction ($\tau = 1$) based on a reconstruction of local states with $I = 2$, $K = 4$, $J = 2$ and $L = 1$. As can be seen in Figs. 2b and 2c the essential features of the time evolution are correctly predicted including the splitting and merging of structures.

Similar to the case of delay embedding of scalar time series [1] the choice of proper embedding parameters is crucial for succesful applications. For our simillations we proceeded in two steps. First the values for the spatial shift K and the temporal delay L are estimated using the (averaged) mutual information H [1] of spatial or temporal neighbors in the STTS as a function of K or L, respectively, in order to minimize the redundancy of the components of the local state

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vectors. For the Kuramoto-Sivashinsky data the H-K-curve possesses a local minimum at $K = 4$ whereas the H-L-curve has no pronounced minimum but decays sufficiently for $L = 1$ indicating a low redundancy of temporal neighbors for this value of the lag. These values for K and L are used then to determine the necessary number of spatial I and temporal J neigbors (i.e. the dimension) of the reconstruction. This was done in the present study by increasing these values until the average prediction error decreased significantly.

6. Conclusion

The numerical examples show that the local reconstruction of states is a powerful method for predicting spatio-temporal time series. It may also serve as a starting point for deriving local mathematical models in terms of polynomials, radial basis functions or neural networks. Based on these reconstructions subsequent bifurcation analysis or noise reduction [10] is possible. The scheme discussed in this paper may be generalized in different directions. The dimension of the reconstructed states can be reduced if the STTS stems from a dynamical system that possesses additional spatial symmetries that can be exploited when constructing the state vectors. If the process generating the

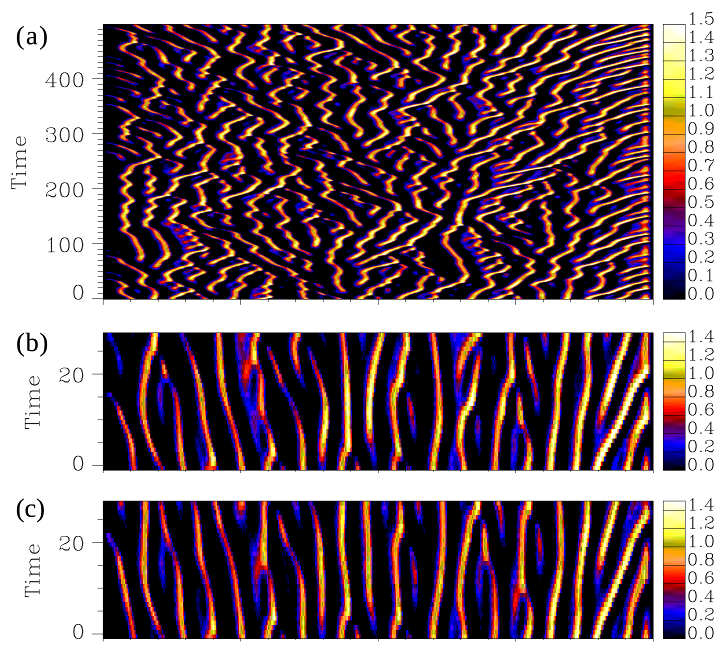

Figure 2: Spatio-temporal time series generated by the Kuramoto-Sivashinsky equation (3). Values of the variable $u(x, t)$ are plotted gray scaled vs. space x and time t. (a) Training set. (b) Original time series to be predicted. (c) Predicted time series.

STTS is not spatially homogeneous one may add to the dimension d of the reconstruction space the number d_S of spatial dimensions of the problem (i.e. $d_S = 1, 2$ or 3) and work then in the extended $d + d_S$ dimensional space. The reconstruction and prediction methods worked also well for data that were not sampled simultaneously but (slowly) scanned spatially as it is the case in many experimental measurements of extended systems. Furthermore, one may take into account that any physical information spreads with some maximum speed and a triangle ("light cone") may be more efficient for reconstructing local states instead of using a rectangular region of the matrix S (comp. Fig.1).

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