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Separability of analytic postnonlinear blind source separation with bounded sources

F.J. Theis and P. Gruber

Institute of Biophysics, University of Regensburg,
D-93040 Regensburg, Germany

Abstract. The aim of blind source separation (BSS) is to transform a mixed random vector such that the original sources are recovered. If the sources are assumed to be statistically independent, independent component analysis (ICA) can be applied to perform BSS. An important aspect of successfully analysing data with BSS is to know the indeterminacies of the problem, that is how the separating model is related to the original mixing model. In the case of linear ICA-based BSS it is well known that the mixing matrix can be found except for permutation and scaling [3], but for more general settings not many results exist. In this work we only consider random variables with bounded densities. We will shortly describe the bounded BSS problem for linear mixtures. Then, based on [1], we generalize these ideas to the postnonlinear mixing model with analytic nonlinearities and calculate its indeterminacies.

1 Introduction

Independent component analysis (ICA) finds statistically independent data within a given random vector. It is often applied to blind source separation (BSS), where it is furthermore assumed that the given vector has been mixed using a fixed set of independent sources. Good textbook-level introductions to ICA are given in [4] and [2]. In this work, we will analyze separability of linear and postnonlinear [5] mixtures. The postnonlinear model corresponds to an often occurring real situation, when the mixture is in principle linear, but the sensors introduce an additional nonlinearity during the recording. We refer to [5] for simulations and applications thereof. Here, we additionally assume that the source signals have bounded range. Section 2 gives a result about homogeneous functions, and section 3 covers the postnonlinear separability problem.

2 Basics

For $m, n \in \mathbb{N}$ let $\mathrm{Mat}(m \times n)$ be the $\mathbb{R}-$vectorspace of real $m \times n$ matrices, and $\mathrm{Gl}(n) := \{\mathbf{W} \in \mathrm{Mat}(n \times n) \mid \det(\mathbf{W}) \neq 0\}$ be the general linear group of $\mathbb{R}^n$. An invertible matrix $\mathbf{L} \in \mathrm{Gl}(n)$ is said to be a *scaling matrix*, if it is diagonal. We say two matrices $\mathbf{B}, \mathbf{C} \in \mathrm{Mat}(m \times n)$ are *equivalent*, $\mathbf{B} \sim \mathbf{C}$, if $\mathbf{C}$ can

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be written as $\mathbf{C} = \mathbf{BPL}$ with an scaling matrix $\mathbf{L} \in \mathrm{Gl}(n)$ and an invertible matrix with unit vectors in each row (permutation matrix) $\mathbf{P} \in \mathrm{Gl}(n)$.

Denote $\mathcal{C}^1(U)$ and $\mathcal{C}^\omega(U)$, $U \subset \mathbb{R}^n$ the set of all continuously differentiable respectively analytic functions $U \to \mathbb{R}$. If we write $\mathbf{f} \equiv \mathbf{g}$, we mean that $\mathbf{f}(\mathbf{x}) = \mathbf{g}(\mathbf{x})$ for all $\mathbf{x} \in U$.

Definition 1. *Given a function $\mathbf{f} : U \to \mathbb{R}$ assume there exist $a, b \in \mathbb{R}$ such that not both are in $\{-1, 0, 1\}$ and $\mathbf{f}(a\mathbf{x}) = b\mathbf{f}(\mathbf{x})$ for all $\mathbf{x} \in U$ with $a\mathbf{x} \in U$. Then $\mathbf{f}$ is said to be* (a, b)-homogeneous *or simply* homogeneous.

This following lemma is from [1] with the correction of taking into account the cases $a, b \in \{-1, 0, 1\}$ in which homogeneity does not induce such strong results. This lemma can be generalized to $\mathcal{C}^1$-functions, so the strong assumption of analyticity is not needed, but shortens the proof.

Lemma 2. *Let $f : U \to \mathbb{R}$, be an analytic function that is (a, b)-homogeneous on $[0, \varepsilon)$ with $\varepsilon > 0$. Then there exist $c \in \mathbb{R}$, $n \in \mathbb{N}_0$ (possibly 0) such that $f(x) = cx^n$ for all $x \in U$.*

Proof. If a is in $\{-1, 0, 1\}$ or $b = 0$ then obviously $f \equiv 0$. If $b = -1$ then $f \equiv 0$, since $a \notin \{-1, 0, 1\}$, $f(a^2x) = f(x)$ and f continuous, f is constant, but $f(ax) = -f(x)$ implies that $f \equiv 0$. In the case that $b = 1$ again f is constant since $f(ax) = f(x)$ and $a^0 = 1 = b$.

By m-times differentiating the homogeneity equation we get $bf^{(m)}(x) = a^m f^{(m)}(ax)$, where $f^{(m)}$ denotes the m-th derivative of f. Evaluating this at 0 yields $bf^{(m)}(0) = a^m f^{(m)}(0)$. Since f is assumed to be analytic, f is determined uniquely by its derivatives at 0. Now either there exists an $n \geq 0$ with $b = a^n$, then $f(x) = cx^n$ or else $f \equiv 0$. □

Definition 3. *We call a random vector $\mathbf{X}$ with density $p_\mathbf{X}$* bounded*, if its density $p_\mathbf{X}$ is* bounded. *Denote* $\operatorname{supp} p_\mathbf{X} := \overline{\{\mathbf{x} | p_\mathbf{X}(\mathbf{x}) \neq 0\}}$ *the* support *of $p_\mathbf{X}$ i.e. the closure of the non-zero points of $p_\mathbf{X}$.*

Therefore it makes sense to introduce the following notion: An independent random vector $\mathbf{X}$ is said to be *fully bounded*, if $p_{X_i}(x) \neq 0$ for all $x \in (a_i, b_i)$. And in this case we get $\operatorname{supp} p_\mathbf{X} = [a_1, b_1] \times \ldots \times [a_n, b_n]$.

In the following we will always assume to have fully bounded densities, so $\mathbf{S}$ is assumed to have a fully bounded density $p_\mathbf{S} : \mathbb{R}^n \to \mathbb{R}$. In the case of the linear instantaneous Blind Source Separation (BSS) problem the following separability result is well known and can be derived from a more general version of this theorem for non-bounded Comon [3]. But in the context of fully bounded random vectors, this follows already from the fact that in this case independence is equivalent to having support within a cube with sides parallel to the coordinate planes, and only matrices equivalent to the identity leave this property invariant:

Theorem 4 (Separability of bounded linear BSS). *Let $\mathbf{A} \in \mathrm{Gl}(n)$ and $\mathbf{S}$ a fully bounded independent random vector. If $\mathbf{AS}$ is again independent, then $\mathbf{A}$ is equivalent to the identity.*

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This theorem indeed proves separability of the linear ICA model as above, because if $\mathbf{X} = \mathbf{AS}$ and $\mathbf{W}$ is a demixing matrix such that $\mathbf{WX}$ is independent, then $\mathbf{WA} \sim \mathbf{I}$, so $\mathbf{W}^{-1} \sim \mathbf{A}$ as desired. As the model is invertible and the indeterminacies are trivial, identifiability and uniqueness follow directly.

3 Separability of postnonlinear BSS

Definition 5. *A function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is called* diagonal *or* component-wise *if each component $f_i(\mathbf{x})$ of $\mathbf{f}(\mathbf{x})$ depends only on the variable x_i.*

In this case we often omit the other variables and write $\mathbf{f}(x_1, \ldots, x_n) = \big(f_1(x_1), \ldots, f_n(x_n)\big)$.

Consider now the postnonlinear Blind Source Separation model:

$$\mathbf{X} = \mathbf{f}(\mathbf{AS})$$

where again $\mathbf{S}$ is an independent random vector, $\mathbf{A} \in \mathrm{Gl}(n)$ and $\mathbf{f}$ is a diagonal nonlinearity. We assume the components f_i of $\mathbf{f}$ to be injective analytic functions with non-vanishing derivative. Then also f_i^{-1} is analytic.

Definition 6. *Let $\mathbf{A} \in \mathrm{Gl}(n)$ be an invertible matrix. Then $\mathbf{A}$ is said to be* mixing *if $\mathbf{A}$ has at least two nonzero entries in each row. And $\mathbf{A} = (a_{ij})_{i,j=1\ldots n}$ is said to be* absolutely degenerate *if there are two columns $l \neq m$ such that $a_{il}^2 = \lambda a_{im}^2$ for a $\lambda \neq 0$ i.e. the normalized columns differ only by the sign of the entries.*

Postnonlinear BSS is a generalization of linear BSS, so the indeterminacies of postnonlinear ICA contain at least the indeterminacies of linear BSS: $\mathbf{A}$ can only be reconstructed up to scaling and permutation. Here of course additional indeterminacies come into play because of translation: f_i can only be recovered up to a constant. Also, if $\mathbf{L} \in \mathrm{Gl}(n)$ is a scaling matrix, then $\mathbf{f}(\mathbf{AS}) = (\mathbf{f} \circ \mathbf{L})\big((\mathbf{L}^{-1}\mathbf{A})\mathbf{S}\big)$, so $\mathbf{f}$ and $\mathbf{A}$ can interchange scaling factors in each component. Another obvious indeterminacy could occur if $\mathbf{A}$ is not general enough: If for example $\mathbf{A} = \mathbf{I}$, then $\mathbf{f}(\mathbf{S})$ is already again independent, because independence is invariant under component-wise nonlinear transformation; so $\mathbf{f}$ cannot be found using this method.

A not so obvious indeterminacy occurs if $\mathbf{A}$ is absolutely degenerate. Then only the matrix $\mathbf{A}$ but not the nonlinearities can be recovered from looking at the edges of the support of the fully-bounded random vector. For example consider the case $n = 2$, $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$ and the analytic function $\mathbf{f}(x_1, x_2) = \big(x_1 + \frac{1}{2\pi}\sin(\pi x_1), x_2 + \frac{1}{\pi}\sin(\pi \frac{x_2}{2})\big)$. Then $\mathbf{f} \circ \mathbf{A}$ maps $[0,1]^2$ onto $[0,1]^2$ as can be shown by calculation, see for example figure 1. But $\mathbf{f}$ is not affine linear. Nonetheless this is no indeterminacy of the model itself, since $\mathbf{A}^{-1}\mathbf{f}(\mathbf{AS})$ is obviously not independent.

If we however assume that $\mathbf{A}$ is mixing and not absolutely degenerate, then we will show for all fully-bounded sources $\mathbf{S}$ that except for scaling interchange between $\mathbf{f}$ and $\mathbf{A}$ no more indeterminacies as in the affine linear case exist. If $\mathbf{f}$ is only assumed to be $\mathcal{C}^1$, then additional indeterminacies come into play.

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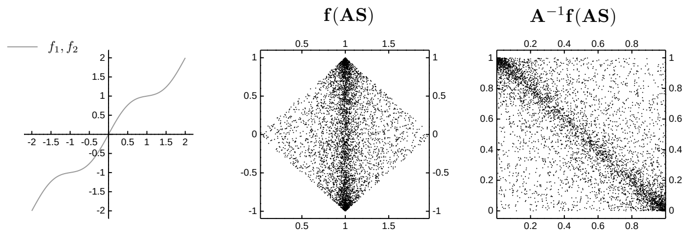

Figure 1: Example for a postnonlinear transformation using a absolutely degenerate Matrix $\mathbf{A}$ and in $[0,1]^2$ uniform sources $\mathbf{S}$.

Theorem 7 (Separability of bounded postnonlinear BSS). *Let* $\mathbf{A}, \mathbf{W} \in \mathrm{Gl}(n)$ *and one of them mixing and not absolutely degenerate,* $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^n$ *be a diagonal injective* $\mathcal{C}^\infty$*-function such that* $h_i' \neq 0$ *and let* $\mathbf{S}$ *be a fully-bounded independent random vector. If* $\mathbf{W}\big(\mathbf{h}(\mathbf{AS})\big)$ *is independent, then there exists a scaling* $\mathbf{L} \in \mathrm{Gl}(n)$ *and* $\mathbf{v} \in \mathbb{R}^n$ *with* $\mathbf{LA} \sim \mathbf{W}^{-1}$ *and* $\mathbf{h}(\mathbf{x}) = \mathbf{Lx} + \mathbf{v}$ *for* $\mathbf{x} \in \operatorname{supp} \mathbf{S}$*.*

If $\mathbf{f} \circ \mathbf{A}$ is the mixing model, $\mathbf{W} \circ \mathbf{g}$ is the separating model. Putting the two together we get the above mixing-separating model. Note that $\mathbf{h}$ can be determined only on the cross containing $\mathbf{A} \operatorname{supp} \mathbf{S}$ because $\mathbf{S}$ is bounded. For ease of notation we only fix $\mathbf{h}$ on $\mathbf{A} \operatorname{supp} \mathbf{S}$. As usual, because the model was assumed to be invertible, identifiability and uniqueness of the model follow from the separability.

Definition 8. *A subset* $P \subset \mathbb{R}^n$ *is called* parallelepipeds*, if it is the linear image of a square, that is*

$$P = \mathbf{A}([a_1, b_1] \times \ldots \times [a_n, b_n])$$

for $a_i < b_i$*,*$i = 1, \ldots, n$ *and* $\mathbf{A} \in \mathrm{Gl}(n)$*. A parallelepipeds* P *is said to be* tilted*, if* $\mathbf{A}$ *is mixing and no* 2×2*-minor is absolutely degenerate.*

Let $i \neq j \in \{1, \ldots, n\}$ *and* $c \in \{a_1, b_1\} \times \ldots \times \{a_n, b_n\}$*, then*

$$\mathbf{A}\big(\{c_1\} \times \ldots \times [a_i, b_i] \times \ldots \times [a_j, b_j] \times \ldots \times \{c_n\}\big)$$

is called a 2-face *of* P *and* $\mathbf{A}(c)$ *is called a* corner *of* P*.*

If $n = 2$ *the parallelepipeds are called* parallelograms*.*

Lemma 9. *Let* $f_1, \ldots, f_n \in \mathcal{C}^\omega(\mathbb{R})$ *be* n *analytic injective functions with* $f_i' \neq 0$*, and let* $\mathbf{f} := f_1 \times \ldots \times f_n$ *be the induced injective mapping on* $\mathbb{R}^n$*. Let* $P, Q \subset \mathbb{R}^n$ *be two parallelepipeds, one of them tilted. If* $\mathbf{f}(P) = Q$ *(or equivalently for the boundaries* $\mathbf{f}(\partial P) = \partial Q$*), then* $\mathbf{f}|P$ *is affine linear diagonal.*

In the proof we see that the requirement for P or Q being tilted is too strong. It would suffice that enough 2-minors are not absolutely degenerate.

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Nevertheless the set of mixing matrices which having no absolutely degenerate 2×2-minors is dense in $\mathrm{Gl}(n)$.

We can easily reduce the prove of this lemma to the case where $n = 2$ and the next lemma yields the desired result.

Lemma 10. *Let $f_1, f_2 \in \mathcal{C}^\omega(\mathbb{R})$ be two analytic injective functions on $\mathbb{R}$, and let $\mathbf{f} := f_1 \times f_2$ be the induced injective mapping on $\mathbb{R}^2$. Let $P, Q \subset \mathbb{R}^2$ be two parallelograms such that one of them is mixed. If $\mathbf{f}(P) = Q$ (or equivalently for the boundaries $\mathbf{f}(\partial P) = \partial Q$), then $\mathbf{f}$ is affine linear diagonal.*

Note that the tiltedness is essential, for example let $P = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 0 \end{smallmatrix}\right)[0,1]^2$ and f_1 such that

$$f_1(x) = \begin{cases} \frac{3}{2}x & \text{for } x < 1 \\ \frac{3}{2}x - 1 & \text{for } x > 2 \end{cases}$$

and $f_2(x) := x$. Then Q is a parallelogram and its corners are $(0,0)$, $(\frac{3}{2}, 1)$, $(0,2)$, $(\frac{7}{2}, 1)$ which is not a scaled version of P. We will prove this lemma together with the next two lemma. Also note that contrary to [1] an additional assumption (absolutely-degeneracy) occurs — this is in fact a necessary condition as shown in figure 1 above.

Proof of lemma 10. Obviously images of non-tilted parallelograms under diagonal mappings are again non-tilted. $\mathbf{f}$ is invertible, so we can assume that both P and Q are tilted. Without loss of generality using the scaling and translation invariance of our problem, we may assume that

$$\partial P = \begin{pmatrix} 1 & 1 \\ a_1 & a_2 \end{pmatrix} \partial([0,1] \times [0,c]), \quad \partial Q = \begin{pmatrix} 1 & 1 \\ b_1 & b_2 \end{pmatrix} \partial([0,1] \times [0,d]),$$

with $a_i, b_i \in \mathbb{R} \setminus \{0\}$ and $a_1^2 \neq a_2^2$ and $b_1^2 \neq b_2^2$ and $ca_2, db_2 > 0$ and $c \leq 1$, and

$$\mathbf{f}(\mathbf{0}) = \mathbf{0}, \mathbf{f}(1, a_1) = (1, b_1), \mathbf{f}(c, ca_2) = (d, db_2)$$

(i.e. the vertices of P are mapped onto the vertices of Q in the specified order). Note that the vertices of P have to be mapped onto vertices of Q because $\mathbf{f}$ is at continuously differentiable. Since the f_i are monotonously we also have $d \leq 1$ and that $a_1 < 0$ implies $b_1 < 0$.

It follows that $\mathbf{f}$ maps the four separate edges of ∂P onto the corresponding edges of ∂Q: $\mathbf{f}(t, a_1 t) = \big(g_1(t), b_1 g_1(t)\big)$, $\mathbf{f}(ct, ca_2 t) = \big(dg_2(t), db_2 g_2(t)\big)$ for $t \in [0,1]$. Here $g_i : [0,1] \to [0,1]$ is a strongly monotonously increasing parametrization of the respective edge. It follows that $g_1(t) = f_1(t)$ and $dg_2(t) = f_1(ct)$ and therefore $f_2(a_1 t) = b_1 f_1(t)$ and $f_2(ca_2 t) = b_2 f_1(ct)$ for $t \in [0,1]$. Therefore we get an equation for both components of f, e.g. for the second: $f_2\big(\frac{a_1}{a_2} t\big) = \frac{b_1}{b_2} f_2(t)$ for $t \in [0, ca_2]$.

So f_2 is $\big(\frac{a_1}{a_2}, \frac{b_1}{b_2}\big)$-homogeneous with coefficients not in $\{-1, 0, 1\}$ by assumption; according to lemma 2 f_2 and then also f_1 are homogeneous polynomials (everywhere due to analyticity). By assumption $f_i'(0) \neq 0$, hence the f_i are even linear.

We have used the translation invariance above, so in general $\mathbf{f}$ is an affine linear scaling. $\square$

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Proof of lemma 9. Again note that since diagonal maps preserve non-tiltedness we can assume that P and Q are tilted. Let $\pi_{ij} : \mathbb{R}^n \to \mathbb{R}^2$ be the projection onto the i, j-coordinates. Note that for any corner c and $i \neq j$ there is a 2-face P_{ijc} of P containing c such that $\pi_{ij}(P_{ijc})$ is a parallelogram. In fact since P is tilted $\pi_{ij}(P_{ijc})$ is also tilted. Since $\mathbf{f}$ is smooth $\pi_{ij}\big(\mathbf{f}(P_{ijc})\big)$ is also a 2-face of Q and again tilted.

For each corner c of P and $i \neq j \in \{0, \ldots, n\}$ we can apply lemma 10 to $\pi_{ij}(P_{ijc})$ and $\pi_{ij}\big(\mathbf{f}(P_{ijc})\big)$. Therefore f_i and f_j are affine linear on $\pi_i(P_{ijc})$ and $\pi_j(P_{ijc})$. Now $\pi_i(P) \subset \bigcup_{cj} \pi_i(P_{ijc})$ and hence f_i affine linear on $\pi_i(P)$ which proves that $\mathbf{f}$ is affine linear diagonal. □

Proof of theorem 7. $\mathbf{S}$ is bounded, and $\mathbf{W} \circ \mathbf{h} \circ \mathbf{A}$ is continuous, so $\mathbf{T} := \mathbf{W}\big(\mathbf{h}(\mathbf{AS})\big)$ is bounded as well. Furthermore, since $\mathbf{S}$ is fully bounded, $\mathbf{T}$ is also fully bounded. Then, as seen in section 2, $\operatorname{supp}\mathbf{S}$ and $\operatorname{supp}\mathbf{T}$ are rectangles with boundaries parallel to the coordinate axes. Hence $P := \mathbf{A}(\operatorname{supp}\mathbf{S})$ and $Q := \mathbf{W}^{-1}(\operatorname{supp}\mathbf{T})$ are parallelograms. One of them is tilted because otherwise $\mathbf{A}$ and $\mathbf{W}^{-1}$ would not be mixing.

As $\mathbf{W} \circ \mathbf{h} \circ \mathbf{A}$ maps $\operatorname{supp}\mathbf{S}$ onto $\operatorname{supp}\mathbf{T}$, $\mathbf{h}$ maps the set $\mathbf{A}\operatorname{supp}\mathbf{S}$ onto $\mathbf{W}^{-1}\operatorname{supp}\mathbf{T}$ i.e. $\mathbf{h}(P) = Q$. Then by lemma 9 $\mathbf{h}$ is affine linear diagonal, say $\mathbf{h}(\mathbf{x}) = \mathbf{Lx} + \mathbf{v}$ for $\mathbf{x} \in P$ with $\mathbf{L} \in \mathrm{Gl}(2)$ scaling and $\mathbf{v} \in \mathbb{R}^2$.

So $\mathbf{W}\big(\mathbf{h}(\mathbf{AS})\big) = \mathbf{WLAS} + \mathbf{Wv}$ is independent, and therefore also $\mathbf{WLAS}$. By theorem 4 $\mathbf{WLA} \sim \mathbf{I}$, so there exists a scaling $\mathbf{L}'$ and a permutation $\mathbf{P}'$ with $\mathbf{WLA} = \mathbf{L}'\mathbf{P}'$ as had to be shown. □

4 Conclusion

We have presented a new separability result for postnonlinear bounded mixtures that is based on the analysis of the borders of the mixture density. We hereby formalize and extend ideas already presented in [1]. We introduce the notion of absolutely degenerate mixing matrices. Using this we identify the restrictions of separability and also of algorithms that only use border analysis for postnonlinearity detection. In future works we will show how to relax the condition of analytic postnonlinearities to only $\mathcal{C}^1$-functions.

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