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Exponential Stability of Implicit Euler, Discrete-Time Hopfield Neural Networks

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Abstract. The exponential stability of continuous-time Hopfield neural networks is not preserved when implemented on digital computers by means of explicit numerical methods, whereas the implicit (or backward) Euler method preserves this exponential stability under exactly the same sufficient conditions as those previously obtained for the continuous model. The proof is based on the nonlinear measure approach, here extended to discrete-time systems. This approach also allows the estimation of the exponential convergence rate of the discrete solutions.

1 Introduction

The application of Hopfield neural networks to both associative memories and the solution of combinatorial optimization problems requires the estimation of the stability of their equilibrium points, the size of their basins of attraction, and the corresponding convergence rates. There is considerable interest in the determination of sufficient conditions that assure the exponential, absolute or asymptotic stability of neural networks, both globally and locally (see Ref. [1] and references there in). The standard technique for stability analysis is based on the use of Lyapunov functions and requires some conditions on both monotonicity and differentiability of the transfer functions. Recently, a new approach based on nonlinear measures has been introduced with considerable success [2, 3, 4]. This new technique has been only applied to continuous-time neural networks and only requires the Lipschitz continuity of the transfer functions.

The computer simulation of continuous-time neural networks by means of numerical methods for differential equations yields (synchronous) discrete-time neural networks. The standard discretization of the continuous-time Hopfield neural network is based on the explicit (or forward) Euler method yielding the Takeda-Goodman model [5]. Usually, the time step used in the simulations is fixed to unity. This practice introduces instabilities in the synchronous mode, so some authors recommend an asynchronous implementation, which is more stable in this case. However, the stability properties of the original continuous-time model are not preserved by this kind of discretization.

Recently, Atencia et al. [6] have introduced a new discrete-time network based on the discretization of the Hopfield network by means of the implicit (or backward) Euler method. This new discrete-time network allows a larger time

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step than explicit Euler discretization and also, for a time step smaller than the inverse of the one-sided Lipschitz constant of the nonlinearity, possesses exactly the same Lyapunov function as that of the continuous-time model [6].

In this paper, the stability analysis based on nonlinear measures is extended to discrete-time systems and applied to the implicit Euler discretization of Hopfield neural networks, showing that the sufficient conditions which assure both their global and their local exponential stability are exactly preserved. In Section 2, the nonlinear measure stability analysis of the implicit Euler, discrete-time, dynamical system is presented. 3 is devoted to the application of these results to Hopfield neural networks. Finally, Section 4 provides some conclusions and directions of further research.

2 Nonlinear measures and the implicit Euler method

Nonlinear measures are a generalization of matrix norms for nonlinear functions. Qiao et al. [2] introduced the nonlinear measure associated with the l^1-norm, the one used in this paper, and Ruan et al. [4] the one associated with the l^2-norm.

Definition 1. Take an open set $\Omega \subset \mathbb{R}^n$, an operator $F : \Omega \to \mathbb{R}^n$, and a fixed vector $x^0 \in \Omega$. Qiao et al. [2] define the nonlinear l^1-measure of F in Ω as

$$m_{1,\Omega}(F) = \sup_{x \neq y \in \Omega} \frac{\langle F(x) - F(y), \operatorname{sign}(x-y) \rangle}{\|x-y\|_1},$$

and the relative nonlinear l^1-measure of F in Ω at x^0 as

$$m_{1,\Omega}(F, x^0) = \sup_{x^0 \neq x \in \Omega} \frac{\langle F(x) - F(x^0), \operatorname{sign}(x-x^0) \rangle}{\|x-x^0\|_1},$$

where the l^1-norm in $\mathbb{R}^n$ is defined by $\|x\|_1 = \sum_i |x_i| = \langle x, \operatorname{sign}(x) \rangle$, the sign function is defined componentwise as $\operatorname{sign}(x) = (\operatorname{sign}(x_1), \ldots, \operatorname{sign}(x_n))^\top$, each component being equal to 1, 0, and -1 for positive, null and negative arguments, respectively, and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product.

Nonlinear measures can be used to characterize the uniqueness, the stability, the basin of attraction and the rate of convergence for equilibrium points of nonlinear systems. In Refs. [2] and [4] only differential equations were considered. In this paper, we focus on discrete-time nonlinear systems.

Let $dx/dt = F(x)$ be an autonomous system of differential equations, the discretization of which by the implicit Euler rule yields

$$\frac{x^{k+1} - x^k}{\Delta t} = F(x^{k+1}), \tag{1}$$

where x^k is an approximation to $x(t^k) \equiv x(k\,\Delta t)$, and the nonlinear operator is $F : \mathbb{R}^n \supset \Omega \longrightarrow \mathbb{R}^n$. From here on, $\Delta t = 1$, as usual for neural networks.

The key properties of nonlinear measures is that if $m_{1,\Omega}(F, x^*) < 0$, then, any fixed point x^* of Eq. (1), i.e., $F(x^*) = 0$, is unique in Ω (proved in Ref. [2]), and that $m_{1,\Omega}(F) < 0$ is sufficient condition for global exponential stability, as shown in the following theorem.

Theorem 1. If $m_{1,\Omega}(F) < 0$, then there is at most one fixed point in Ω, and any two solutions x^k and y^k, from initial values x^0 and y^0, respectively, satisfy

$$\left\|x^k - y^k\right\|_1 \leq (1 + |m_{1,\Omega}(F)|)^{-k} \left\|x^0 - y^0\right\|_1 = e^{-\ln(1+|m_{1,\Omega}(F)|)\,k} \left\|x^0 - y^0\right\|_1 .$$

Proof. Since $m_{1,\Omega}(F, z) \leq m_{1,\Omega}(F)$, and $-\|x\|_1 \leq \langle -x, \operatorname{sign}(z)\rangle$, $\forall z \in \Omega$, then,

$$\begin{aligned}
&\left\|x^{k+1} - y^{k+1}\right\|_1 - \left\|x^k - y^k\right\|_1 \\
&\qquad \leq \langle x^{k+1} - y^{k+1} - (x^k - y^k), \operatorname{sign}(x^{k+1} - y^{k+1})\rangle \\
&\qquad = \langle F(x^{k+1}) - F(y^{k+1}), \operatorname{sign}(x^{k+1} - y^{k+1})\rangle \\
&\qquad \leq m_{1,\Omega}(F) \left\|x^{k+1} - y^{k+1}\right\|_1 ,
\end{aligned}$$

which finally results in

$$\left\|x^{k+1} - y^{k+1}\right\|_1 \leq \frac{\left\|x^k - y^k\right\|_1}{1 - m_{1,\Omega}(F)} \leq \frac{\left\|x^0 - y^0\right\|_1}{(1 + |m_{1,\Omega}(F)|)^{k+1}}.$$

□

Let $F \cdot G$ be the composition of two operators F and G, i.e., $F \cdot G(x) \equiv F(G(x))$. Since, if D is a strictly positive diagonal matrix, then, $m_{1,\Omega}(F) < 0$ if and only if $m_{1,D(\Omega)}(F \cdot D^{-1}) < 0$, the following theorem immediately follows.

Theorem 2. If $m_{1,D(\Omega)}(F \cdot D^{-1}) < 0$, for a diagonal matrix D with $(D)_{ii} = d_i > 0$, then, any two solutions x^k and y^k of Eq. (1) satisfy

$$\left\|x^k - y^k\right\|_1 \leq \frac{\left\|x^0 - y^0\right\|_1}{\left(1 + |m_{1,D(\Omega)}(F \cdot D^{-1})|\ \|D\|_1\right)^k}, \qquad \|D\|_1 = \max_{1\leq i\leq n} d_i.$$

Proof. From the initial part of the proof of Theorem 1,

$$\begin{aligned}
&\left\|x^{k+1} - y^{k+1}\right\|_1 - \left\|x^k - y^k\right\|_1 \\
&\qquad \leq \langle F(x^{k+1}) - F(y^{k+1}), \operatorname{sign}(x^{k+1} - y^{k+1})\rangle \\
&\qquad = \langle F \cdot D^{-1}(D\,x^{k+1}) - F \cdot D^{-1}(D\,y^{k+1}), \operatorname{sign}(D\,x^{k+1} - D\,y^{k+1})\rangle \\
&\qquad \leq m_{1,D(\Omega)}(F \cdot D^{-1}) \left\|D\,x^{k+1} - D\,y^{k+1}\right\|_1 \\
&\qquad \leq m_{1,D(\Omega)}(F \cdot D^{-1}) \ \|D\|_1 \left\|x^{k+1} - y^{k+1}\right\|_1 .
\end{aligned}$$

□

Local exponential stability follows using the relative nonlinear measure.

Theorem 3. Let x^* be a fixed point of Eq. (1), and Ω^* an open l^1-ball centered at x^*. If $m_{1,\Omega^*}(F, x^*) < 0$, then, (1) if $x^0 \in \Omega^*$, then $x^k \in \Omega^*$, $\forall k > 0$, (2) x^* is exponentially stable with Ω^* contained in its basin of attraction, and (3) the following exponential estimate holds for every trajectory

$$\left\|x^k - x^*\right\|_1 \leq (1 + |m_{1,\Omega^*}(F, x^*)|)^{-k} \left\|x^0 - x^*\right\|_1 .$$

Corollary 4. Theorem 3 also applies if $m_{1,D(\Omega^*)}(F \cdot D^{-1}, D\, x^*) < 0$, with D a diagonal matrix with $(D)_{ii} > 0$, with the following decay estimate

$$\left\|x^k - x^*\right\|_1 \leq (1 + |m_{1,D(\Omega^*)}(F \cdot D^{-1}, D\, x^*)|\ \|D\|_1)^{-k} \left\|x^0 - x^*\right\|_1 .$$

3 Exponential stability of discrete-time Hopfield networks

The continuous-time Hopfield neural network can be discretized by the implicit (or backward) Euler method, as introduced by Atencia et al. [6], resulting in

$$\frac{u_i^{k+1} - u_i^k}{\Delta t} = F_i(u^{k+1}) = -\frac{u_i^{k+1}}{R_i} + \sum_{j=1}^{n} w_{ij}\, f_j(u_j^{k+1}) + I_i, \quad 1 \leq i \leq n, \qquad (2)$$

where u_i^k is an approximation to $u(t^k)$, $t^k = k\,\Delta t$, $u_i(t)$ are the neural voltages, R_i are the resistances, $W = (w_{ij})$ is the connection weight matrix, f_i are the transfer functions, and I_i are the constant external inputs. Here on, $\Delta t = 1$, and any f_i is one-sided Lipschitz continuous, i.e., there exists m_i such that

$$m_i = \sup_{x \neq y \in \mathbb{R}} \frac{\langle f_i(x) - f_i(y), x - y\rangle}{\|x - y\|^2} < \infty.$$

The next theorem shows that the sufficient condition for global exponential stability of the solutions of Eq. (2) is exactly the same as the obtained by Qiao et al. [2] for continuous-time Hopfield neural networks.

Theorem 5. If there exists a set of real numbers $d_i > 0$ such that

$$m_j\, R_j \sum_{i=1}^{n} \frac{d_j}{d_i} |w_{ij}| < 1, \qquad j = 1, 2, \ldots, n, \qquad (3)$$

then, for every set of external inputs I_i, the discrete-time neural network (2) is globally exponential stable and its solution u^k with initial value u^0 satisfies

$$\left\|u^k - u^*\right\|_1 \leq \alpha \exp(-\ln(1 + b/(\min_i R_i))\, k) \left\|u^0 - u^*\right\|_1 , \qquad \forall k > 0, \qquad (4)$$

$$\alpha = \frac{\max_i d_i}{\min_i d_i}, \qquad b = 1 - \max_{1 \leq j \leq n} \left(m_j\, R_j \sum_{i=1}^{n} \frac{d_j}{d_i} |w_{ij}| \right), \qquad 0 < b < 1. \qquad (5)$$

Proof. Let $F(x) = (F_1(x), \ldots, F_n(x))^\top$, $x = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$ and, D and R^{-1} be diagonal matrices with $(D)_{ii} = d_i$, and $(R^{-1})_{ii} = R_i$, respectively. Since

$$\begin{aligned}
&\langle D^{-1} F(D\, R^{-1}\, x) - D^{-1} F(D\, R^{-1}\, y), \mathrm{sign}(x - y)\rangle \\
&\quad = -\sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} \frac{\mathrm{sign}(x_i - y_i)}{d_i} \sum_{j=1}^{n} w_{ij}\ (f_j(d_j\, R_j\, x_j) - f_j(d_j\, R_j\, y_j)) \\
&\quad \leq -\sum_{i=1}^{n} |x_i - y_i| + \sum_{j=1}^{n} m_j\, d_j\, R_j\, |x_j - y_j| \sum_{i=1}^{n} \frac{|w_{ij}|}{d_i} \leq -b\, \|x - y\|_1 \,,
\end{aligned}$$

showing that $m_{1,\mathbb{R}^n}(D^{-1}\, F\, D\, R^{-1}) \leq -b < 0$. Then, using Theorem 2, the equation $x^{k+1} - x^k = D^{-1}\, F(D\, x^{k+1})$, has a unique fixed point x^* globally exponential stable with decay estimate $\left\|x^k - x^*\right\|_1 \leq (1+b\, \|R\|_1)^{-k}\, \left\|x^0 - x^*\right\|_1$, where $\|R\|_1 = \max_i(1/R_i) = 1/(\min_i R_i)$. Clearly, the unique solution of this equation is $u^k = D\, x^k$, whenever u^k is that of Eq. (2). Finally, taking into account that

$$\frac{\left\|u^k - u^*\right\|_1}{\|D\|_1} \leq \left\|D^{-1}(u^k - u^*)\right\|_1, \quad \left\|D^{-1}(u^0 - u^*)\right\|_1 \leq \left\|D^{-1}\right\|_1\, \left\|(u^0 - u^*)\right\|_1,$$

$\|D\|_1 = \max_i d_i$, and $\left\|D^{-1}\right\|_1 = 1/(\min_i d_i)$, the theorem is proved. □

Many known global exponential stability criteria published in the literature can be obtained from Theorems 5 by a proper choice of the adjustable parameters d_i appearing in Eq. (3). The following corollary provides some examples.

Corollary 6. Let each transfer function f_i of Eq. (2) being differentiable with $0 \leq f_i'(x) \leq \beta_i$, $\forall x \in \mathbb{R}$, and some $\beta_i \in \mathbb{R}$, then, for every set of external inputs I_i, the discrete-time neural network (2) is globally exponential stable for either (1) $w_{jj}\, \beta_j\, R_j + \beta_j\, R_j \sum_{i\neq j}^{n} |w_{ij}| < 1$, or (2) $w_{jj}\, \beta_j\, R_j + \sum_{i\neq j}^{n} \beta_i\, R_i\, |w_{ij}| < 1$, or (3) $w_{jj}\, \beta_j\, R_j + \beta_j \sum_{i\neq j}^{n} R_i\, |w_{ij}| < 1$, or (4) $w_{jj}\, \beta_j\, R_j + R_j \sum_{i\neq j}^{n} \beta_i\, |w_{ij}| < 1$, with $j = 1, 2, \ldots, n$, satisfying the exponential decay estimate given by Eq. (4).

Proof. Apply Theorem 5 with f_i monotonically increasing, $m_i \leq \beta_i$, and $d_i = 1$, $d_i = \beta_i\, R_i$, $d_i = R_i$, and $d_i = \beta_i$, for cases 1), 2), 3), and 4), respectively. □

Local exponential stability analysis of discrete-time Hopfield neural networks can be carried out by using a relative nonlinear measure approach.

Theorem 7. Let $u^* \in \Omega^*$ be a fixed point of Eq. (2), and Ω^* a l^1-ball, if

$$R_j\, m_j(\Omega^*) \sum_{i=1}^{n} |w_{ij}| < 1, \qquad m_j(\Omega^*) = \sup_{u_j^* \neq x \in \Omega_j^*} \frac{|f_j(x) - f_j(u_j^*)|}{|x - u_j^*|},$$

for $j = 1, 2, \ldots, n$, where Ω_i^* is the projection of Ω^* on the i-th axis of $\mathbb{R}^n$. Then, u^* is exponentially stable and its basin of attraction contains Ω^*. Moreover,

$$\left\|u^k - u^*\right\|_1 \leq \exp(-\ln(1 + b/(\min_i R_i))\, k)\, \left\|u^0 - u^*\right\|_1, \qquad \forall k > 0, \qquad (6)$$

is satisfied by u^k, the solution of Eq. (2) with initial value u^0, where

$$b = 1 - \max_{1 \leq j \leq n} R_j \, m_j(\Omega^*) \sum_{i \neq j}^{n} |w_{ij}|.$$

Proof. Taking $F(x)$ and R^{-1} as in the proof of Theorem 5, for any $x \in R(\Omega^*)$,

$$\langle F(R^{-1}\, x) - F(R^{-1}\, R\, u^*), \mathrm{sign}(x - R\, u^*) \rangle \leq -b \, \|x - R\, u^*\|_1 \, ,$$

implying $m_{1,R(\Omega^*)}(F \cdot R^{-1}, R\, u^*) \leq b < 0$. The rest follows from Corollary 4. □

This result applies even for nonmonotonic and/or nondifferentiable transfer functions and could be easily sharpened for both monotonically increasing and differentiable ones. However, the details are omitted here for brevity.

4 Conclusions

The stability analysis based on nonlinear measures has been extended to discrete-time nonlinear systems based on the implicit Euler discretization. Sufficient conditions assuring both the global and the local exponential stability of implicit Euler, discrete-time, Hopfield neural networks have been obtained. These conditions are exactly the same as those which can be obtained for continuous-time networks, showing that the implicit Euler method preserves the exponential stability of the continuous model.

The application of the nonlinear measures approach for the stability analysis of other discrete-time nonlinear neural networks, such as recurrent back-propagation, discrete-time Cohen-Grossberg or delayed Hopfield neural networks, are interesting topics for further research.

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