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A Stability Condition for Neural Network Control of Uncertain Systems

Pornchai Khlaeo-om[1] and Suwat Kuntanapreeda[2]

1- Department of Electrical Engineering
Rajamangala Institute of Technology Southern Campus, Songkhla 90000 Thailand
2- Research and Development Center for Intelligent Systems
King Mongkut's Institute of Technology North Bangkok, Bangkok 10800 Thailand

Abstract. This paper derives a stability condition for neural network control systems which the parameters of the controlled systems are uncertain. The stability condition can be imposed in training processes to guarantee the stability of the control systems. The controller is a single hidden layer, feedforward neural network. The controlled system is assumed to be full-state accessible and can be modeled as a linear uncertain system. The stability is confirmed by the existence of a Lyapunov function of the closed loop systems. A simulation result on Van der Pol's equation with parametric uncertainty presented to demonstrate an application of the condition. A modified backpropagation algorithm with a model reference technique is used to train the controller.

1 Introduction

Neural networks (NNs) have been proposed for use in a broad range of control applications. Nowadays, there are many approaches used to design a neural network controller (NNC) [1]-[4]. Regardless of the design approach, the stability of the control system needs to be systematically verified. Moreover, the problem becomes more complex when any parameter of the controlled system is uncertain.

Suykens, Vandewalle, and Moor [5] studied the stability of NN control systems by showing that the control systems could be represented as a two-hidden layer recurrent NN. As the results, they derived a sufficient condition for absolute stability and dissipativity of the recurrent NN from a Lur'e-Postnikov Lyapunov function. The condition was also expressed as a matrix inequality, which could be employed for controller synthesis. A similar approach presented in the framework of NLq theory was introduced in [6]. Kuntanapreeda and Fullmer [7] presented a stability sufficient condition for a class of NN control systems. The controller was a single hidden layer feedforward NN, with linear output functions at the output neurons. The controlled system was restricted to be locally hermitian, which was later removed in [8]. A modified backpropagation training algorithm for adjusting the weights of NNCs was also proposed in [7]. This modified algorithm imposed the stability condition as the training constraint so that the stability of the NN control system is guaranteed.

In this paper we extend the works in [7], [8] by deriving a new stability condition for neural network control systems which the parameters of the controlled systems are uncertain.

2 Neural Network Control Systems

Consider NN control systems comprising an uncertain system and a feedforward NN closing the feedback loop as shown in Figure 1. The controlled system is represented by an n-order state-space model

$$\dot{x} = f(x,u,\delta) \tag{1a}$$

where $x \in \Re^n$ is the state vector, $u \in \Re^m$ is the input vector, $\delta \in \Re^p$ is an uncertain parameter, and $0 = f(0,0,\delta)$. It is also assumed that the system can be modeled as the linear uncertain system

$$\dot{x} = [A_0 + \alpha_1 A_1]x + B_0 u \ . \tag{1b}$$

Here $A_0 \in \Re^{n\times n}$ and $B_0 \in \Re^{n\times m}$ are nominal constant system and input matrices, respectively. The system's uncertainties are represented by $A_1 \in \Re^{n\times n}$ and $\alpha_1 \in \Re$ where $|\alpha_1| \le \mu \in \Re^+$. The pair (A_0, B_0) is assumed to be controllable.

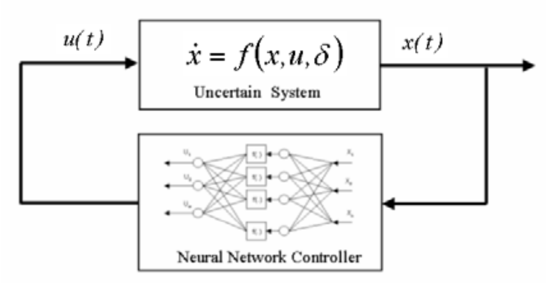

Fig. 1: Neural network control systems.

The controller is a full state regulator implemented as a single hidden layer feedforward NN with a linear output layer. The hidden layer consists of p nonlinear neurons whose activation functions are hyperbolic tangent. Let W_1 and W_2 be the weight matrices in the hidden layer and the output layer, respectively. The control law can then be written in the form

$$u(t) = W_2 F(W_1 x(t)) = W_2 F(h) \tag{2}$$

where $F(h)$ is a p-vector function whose ith component is $f_i(h_i) = tanh(h_i)$.

3 Stability Condition

Lemma 1 For any $A_1 \in \Re^{nxn}$ and any positive symmetric definite matrix $P \in \Re^{nxn}$ the following matrix inequality holds:

$$A_1^T P + PA_1 \le A_1^T PA_1 + P \tag{3}$$

Proof: Let

$$M = A_1^T PA_1 + P - A_1^T P - PA_1$$

For any nonzero $x \in \Re^n$,

$$\begin{aligned} x^T Mx &= x^T \left[A_1^T PA_1 + P - A_1^T P - PA_1\right]x \\ &= \left\| P^{\frac{1}{2}} A_1 x - P^{\frac{1}{2}} x \right\|^2 \\ &\geq 0 \end{aligned}$$

where $\|\bullet\|$ denotes the Euclidean norm. Thus, the matrix inequality (3) holds. □

Proposition 1 The control system, as shown in Figure 1, consisting of the uncertain system (1) with the NN control law (2) is equilibrium stable in the presence of the parameter uncertainty if there exists a positive symmetric definite matrix $P \in \Re^{n\times n}$ and a matrix $q \in \Re^{n\times p}$ such that

$$\left[A_0 + aB_0W_2W_1\right]^T P + P\left[A_0 + aB_0W_2W_1\right] = -Q - qq^T - \mu\left[A_1^T PA_1 + P\right] \tag{4a}$$

$$PB_0W_2 = -W_1^T - \sqrt{2}qI \tag{4b}$$

where $a \in \Re^+$ is a positive constant having the value less than one, $Q \in \Re^{n\times n}$ is a positive symmetric definite matrix, and I is the identity matrix of dimension p.

Proof: For all $t > t_0$, let the uncertain system be given as (1) with the NNC (2) and assume P and q satisfy (4). Let $V = x(t)^T Px(t)$ be a Lyapunov function candidate. Using (1b) and (2), the time derivative of V along the state trajectory of the control system is

$$\begin{aligned} \frac{dV}{dt} &= \left[(A_0 + \alpha_1 A_1)x + B_0W_2F(W_1x)\right]^T Px + x^T P\left[(A_0 + \alpha_1 A_1)x + B_0W_2F(W_1x)\right] \\ &= \left[(A_0 + aB_0W_2W_1 + \alpha_1 A_1)x + B_0W_2\widetilde{F}(h)\right]^T Px + \\ &\quad x^T P\left[(A_0 + aB_0W_2W_1 + \alpha_1 A_1)x + B_0W_2\widetilde{F}(h)\right] \end{aligned}$$

where $h = W_1x$ and $\widetilde{F}(h) = F(h) - ah$. For convenient, define $\widetilde{A} = A_0 + aB_0W_2W_1$. Thus,

$$\begin{aligned} \frac{dV}{dt} &= \left[(\widetilde{A} + \alpha_1 A_1)x + B_0W_2\widetilde{F}(h)\right]^T Px + x^T P\left[(\widetilde{A} + \alpha_1 A_1)x + B_0W_2\widetilde{F}(h)\right] \\ &= x^T\left[\widetilde{A} + \alpha_1 A_1\right]^T Px + \widetilde{F}^T(h)W_2^T B_0^T Px + x^T P\left[\widetilde{A} + \alpha_1 A_1\right]x + x^T PB_0^T W_2\widetilde{F}(h) \\ &= x^T\left[\widetilde{A}^T P + P\widetilde{A}\right]x + \alpha_1 x^T\left[A_1^T P + PA_1\right]x + 2x^T PB_0W_2\widetilde{F}(h). \end{aligned}$$

By Lemma 1 and $|\alpha_1| \leq \mu$, we obtain

$$\begin{aligned} \frac{dV}{dt} &\leq x^T(\widetilde{A}^T P + P\widetilde{A})x + \alpha_1 x^T\left(A_1^T PA_1 + P\right)x + 2x^T PB_0W_2\widetilde{F}(h) \\ &\leq x^T(\widetilde{A}^T P + P\widetilde{A})x + \mu x^T\left(A_1^T PA_1 + P\right)x + 2x^T PB_0W_2\widetilde{F}(h). \end{aligned}$$

Using (4) yields

$$\begin{aligned}\frac{dV}{dt} &\leq -x^T Qx - x^T qq^T x - 2h^T \tilde{F}(h) - 2\sqrt{2}x^T q\tilde{F}(h) \\ &= -x^T Qx - \left\| x^T q + \sqrt{2}\tilde{F}^T(h) \right\|^2 - 2\left[h^T \tilde{F}(h) - \tilde{F}^T(h)\tilde{F}(h)\right] \\ &= -x^T Qx - \left\| x^T q + \sqrt{2}\tilde{F}^T(h) \right\|^2 - 2\sum_{i=1}^{p} \tilde{f}_i(h_i)\left(h_i - \tilde{f}_i(h_i)\right)\end{aligned}$$

where $\|\bullet\|$ denotes the Euclidean norm. Obviously, the first two terms on the right-hand side of the above equation are less than or equal zero. To complete the proof, we need to show that the last term, $-2\sum_{i=1}^{p} \tilde{f}_i(h_i)\left(h_i - \tilde{f}_i(h_i)\right)$, is also less than or equals zero. Since $h_i \in \Re$, either $h_i \geq 0$ or $h_i \leq 0$. For the case $h_i \geq 0$, we have $h_i - \tilde{f}_i(h_i) = (h_i - tanh(h_i)) + ah_i \geq 0$. Since $0 < a < 1$, $\tilde{f}_i(h_i) = tanh(h_i) - ah_i \geq 0$ for all $0 < h_i < h^*$ where h^* represents the positive root of the equation $tanh(h_i) - ah_i = 0$. Hence, $\tilde{f}_i(h_i)(h_i - \tilde{f}_i(h_i)) \geq 0$ whenever $0 < h_i < h^*$. The argument for the case $h_i \leq 0$ is completely similar. Then, the results yields

$$-\sum_{i=1}^{p} \tilde{f}_i(h_i)\left(h_i - \tilde{f}_i(h_i)\right) \leq 0$$

whenever $\max_i |h_i| = \|W_1 x\|_\infty < h^*$. Thus, there exists $\varepsilon > 0$ such that $\frac{dV}{dt} \leq 0$ whenever $\|x\| < \varepsilon$. Therefore, V is a Lyapunov function of the control system and the control system is equilibrium stable. □

4 Simulation result

Consider Van der Pol's system with parametric uncertainty

$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ -\delta(x_1^2 - 1)x_2 - x_1 + u \end{bmatrix}$$

where $x \in \Re^2$ is the state vector, $u \in \Re$ is the input, and $\delta \in [0.9, 1.1]$ is the uncertain parameter of the system. We first model the system in the form of (1b) as follows:

$$A_0 = \begin{bmatrix} \frac{\partial f_1(x,u)}{\partial x_1} & \frac{\partial f_1(x,u)}{\partial x_2} \\ \frac{\partial f_2(x,u)}{\partial x_1} & \frac{\partial f_2(x,u)}{\partial x_2} \end{bmatrix}_{\substack{x=0 \\ u=0 \\ \delta=1}} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} \frac{\partial f_1(x,u)}{\partial u} \\ \frac{\partial f_1(x,u)}{\partial u} \end{bmatrix}_{\substack{x=0 \\ u=0 \\ \delta=1}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and

$$\alpha_1 A_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\delta\left(x_1^2 - 1\right)x_2 - x_1 \end{bmatrix} - \begin{bmatrix} x_2 \\ -x_1 + x_2 \end{bmatrix}$$
$$= \left(\delta x_1^2 + \delta - 1\right)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

resulting $A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $|\alpha_1| \le 0.1$.

Next, a model reference technique [3] is employed to train the NNC. The reference model is selected to have the damping ratio of 0.8 and the natural frequency of 3 rad/sec. The modified backpropagation algorithm [7] with the stability condition (4) is used to adjust the weights of the NNC. Here, $Q = I$ and $a = 0.5$. The NNC has two input nodes, four nonlinear hidden nodes, and one output node. In training the NNC, a sampling period of 0.001 second is used to gather 2000 data points from the nominal system. The training iteration was stopped when no appreciable change in the errors of the successive states of the control system and the reference model. After training, the trained weights are found to be

$$W_1 = \begin{bmatrix} -1.86309527 & -0.87756231 \\ 0.88423353 & -1.54223837 \\ 0.96402647 & 0.60367174 \\ -4.18941164 & -2.36613537 \end{bmatrix}$$

and $$W_2 = \begin{bmatrix} 1.23605603 & 0.73003566 & -1.09679486 & 1.66658551 \end{bmatrix}.$$

The Lyapunov function is $V = x^T P x$,

$$P = \begin{bmatrix} 4.93735559 & 0.66324957 \\ 0.66324957 & 0.75105172 \end{bmatrix},$$

and the corresponding q is

$$q = \begin{bmatrix} 0.73771 & -0.96762 & -0.16728 & 2.18075 \\ -0.03590 & 0.70282 & 0.15561 & 0.78803 \end{bmatrix}.$$

Note that P and q, along with the weights, are found directly from the training as the by-products of the training process. Figure 2 shows the comparison of responses between the NN control system and the reference model. The simulation shows satisfactory control result and is consistent with the derived stability condition.

5 Conclusion

The stability condition for neural network control of uncertain systems have been derived in this paper. A modified backpropagation algorithm imposed the derived stability condition as the training constraint is used to adjust the weights of the neural network controller. The stability is achieved by showing the existence of a Lyapunov function of the closed loop system.

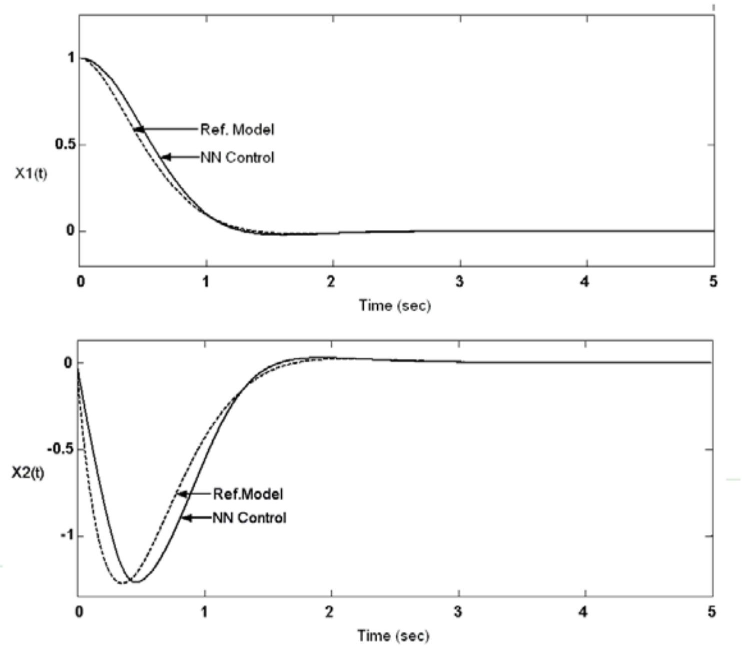

Fig. 2: Comparison of responses.

References

[1] D. H. Nguyen, and B. Widrow, "Neural Networks for Self-learning Control System.," IEEE Contr. Syst. Mag., vol. 10, pp. 18-23, April 1990.

[2] K. S. Narendra and K, Parthasarathy, "Identification and control of dynamical systems using neural networks." *IEEE Trans. Neural Networks*, vol. 1, pp. 4-27, March 1990.

[3] S. Kuntanapreeda, R. W. Gunderson, and R. Fullmer, " Neural-Network Model Reference Control of Non-linear Systems," in Proc. Int. Joint Conf. Neural Networks, Bultimore, 1992, pp. ll94-99.

[4] D. V. Prokhorov and D. C. Wunch, "Adaptive Critic Designs." *IEEE Trans. Neural Networks,* vol. 8, pp. 997-1007, September 1997.

[5] J. A. Suykens, J. Vandewalle, and B. D. Moor, "Lur's Systems with Multilayer Perceptron and Recurrent Neural Networks: Absolute Stability and Dissipativity." *IEEE Trans. Automatic Control*, vol. 44, pp. 770-774, April 1999.

[6] J. A. Suykens, B. D. Moor , and J. Vandewalle "NLq theory: a neural control framework with global asymptotic stability criteria." *INeural Networks*, vol. 10, pp. 615-637, 1997.

[7] S. Kuntanapreeda, and R. Fullmer, "A Training Rule Which Guarantees Finite-Region Stability for a Class of Closed-loop Neural Network Control Systems." *IEEE Trans. Neural Networks*, vol. 7, pp. 745-751, May 1996.

[8] R. Ekachaiworasin and S. Kuntanapreeda, "A Training Rule Which Guarantees Finite-Region Stability of Neural Network Closed-Loop Control: An Extension to Nonhermitian Systems." in *Proc. IEEE-INNS-ENNS International Joint Conference on Neural Networks*, Como, Italy., 2000, pp. 24-27.