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Derivation of nonlinear amplitude equations for the normal modes of a self-organizing system

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Abstract. We here are pointing out a basically well-known pathway to the analysis of self-organizing systems that is now well in reach of numerical methods. Systems of coupled nonlinear differential equations are decomposed into normal modes, are reduced by adiabatic elimination of stable modes to a much smaller system of unstable modes and their nonlinear interaction. In the past, this treatment was accessible only for highly idealized model systems. Guided by an application to retinotopic map formation we discuss the extension to more realistic cases.

1 Introduction

Self-organization is the process by which global order of a system emerges from the interaction of its constituent elements. It is a phenomenon that dominates our universe, with the most striking examples from the living world. In particular, the brain is a self-organizing system at levels of evolution, ontogenesis, learning, and functional organization [1]. Various aspects of brain structure and function have been modelled as self-organizing systems, such as retinotopic projections [2], cortical maps and pattern formation [3, 4], and stereo correspondence in vision [5, 6], to name but a few.

In self-organizing systems, the basic operations are cooperation and competition between elements. These are naturally modelled as dynamical systems, using coupled nonlinear differential equations. As analytical solutions exist only in rare cases, most modelling studies depend on computer simulation to explore the behaviour of the system. Unfortunately, simulations do not help to clarify the connection between the global behaviour of a system on the one hand and the underlying interaction terms and parameters on the other hand. For example, the parameter regime for joint development of orientation and ocular dominance has been revealed only by analytical treatment [7]. Methods to analyze general self-organizing systems are urgently required.

One approach to analyze a dynamical system is to decompose it into the normal modes of a linearization of the system, to reduce it to the amplitudes of a small subset of modes and their nonlinear interactions. Indeed, in self-organizing systems, there usually are only a few modes that are unstable and grow. By elimination of the decaying, stable, modes, e.g., by adiabatic approximation, the system can be reduced to the amplitudes of the unstable modes [8]. This method

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has been applied successfully to the analysis of a model for the ontogenesis of retinotopy [9]. To make this application possible, the system had to be idealized to a highly symmetrical state. Our aim is to point out the possibility of a pathway to similar analysis of less symmetrical and more realistic systems. Our discussion is based on systems with polynomial dynamics, but the method can be applied to general systems with the help of polynomial expansion.

The paper is organized as follows: section 2 reviews the formulation and analysis of the retinotopy system [9], section 3 presents a method to derive the amplitude equations of polynomial equations, and section 4 is conclusion and discussion.

2 Formulation and analysis of a retinotopy system [9]

In this section, we briefly review a dynamical system that is typical for brain organization, and at the same time is simple enough to analyze. This serves as motivation of our formulation of a more general self-organizing system and the method of normal mode analysis. The system is the retinotopic map formation proposed by Häussler and von der Malsburg [9] (here called Häussler system), which concerns the establishment of ordered projections between two brain areas, retina and tectum.

Assume there are N points in retina and tectum, each taken as a one-dimensional chain. The projection between them is represented by a set of links (s, r), where s and r are points in the tectum and retina, respectively. The link weight w_{sr} indicates the strength with which s and r are connected. The larger the value, the stronger the connection. Zero value means there is no connection. All links form a mapping $W = (w_{sr})$. The dynamics are described by the set of $N \times N$ differential equations:

$$\dot{w}_{sr} = f_{sr}(W) - \frac{1}{2N} w_{sr} \left(\sum_{s'} f_{s'r}(W) + \sum_{r'} f_{sr'}(W) \right) \tag{1}$$

where the growth term $f_{sr}(W)$ of link w_{sr} expresses the cooperation from all its neighbors, with positive rate β, plus a non-negative synaptic formation rate α:

$$f_{sr}(W) = \alpha + \beta w_{sr} \sum_{s',r'} C(s, s', r, r') w_{s'r'} \tag{2}$$

The link interaction kernel $C(s, s', r, r')$ describes the mutual cooperative help link (s, r) receives from its neighbor (s', r'), and is assumed to be separable and isotropic.

The rhs of (1) is a third-order polynomial. The goal of analysis is to prove that, starting from a state that is near homogeneous, a W of diagonal form (that is, a retinotopic mapping) is the final configuration of the system.

The analytical treatment of the system is outlined as follows. Details can be found in the original paper [9]. The homogeneous state $W_0 = \mathbf{1}$ (where $\mathbf{1}$ is a matrix with all elements equal 1) is an unstable fixed point of the system.

Linear analysis is performed around W_0, by introducing the deviation V as a new variable:

$$V = W - W_0.$$

By assuming periodic boundary conditions in retina and tectum, the eigenvectors (modes) of the linearized system are the N^2 plane waves in W. There are four modes of maximal eigenvalue, two diagonals in either orientation, each of two different phases. The control parameter α can be set such that only these four modes grow. Then in higher-order analysis the amplitude equations for these four modes are obtained, from which it is shown that the two orientations of the diagonal compete with each other, until the one that is favoured in the initial configuration wins. The winning diagonal mode (of arbitrary phase) then excites harmonics of the same orientation and phase to develop into narrow diagonal. It was shown that by a gradual decrease of α to zero a one-to-one retinotopic projection can be obtained.

In this treatment, the analysis of modes is made possible by several assumptions — periodic boundary conditions in retina and tectum and homogeneity and isotropy of the interaction kernel in (2). These assumptions are, however, not valid in more realistic systems, as can be seen in our generalisation of this system to the visual correspondence problem for invariant object recognition [10, 11]. In the more general system the brain areas are not periodic, the link interaction kernel is learned through activities rather than defined by geometry, the initial state is a pattern of feature similarity rather than homogeneous, and so on. Therefore, new methods of analysis, most likely numerical, need to be developed for the understanding of general self-organizing systems.

3 Deriving amplitude equations for a general system

System (1) is a special case of the more general nonlinear system of third degree:

$$\dot{V} = \mathbf{L}V + Q(V) + K(V), \tag{3}$$

where $V = (v_1, \cdots, v_n)^T$ is the n dimensional state vector. ($n = N^2$ in the Häussler system). $\mathbf{L}$ is the matrix representing the linear term, and $Q(V)$ and $K(V)$ are quadratic and cubic functions, respectively. Our goal in this section is to develop a general method to derive the amplitude equations for this system.

The normal modes of this system are the eigenvectors of $\mathbf{L}$, $u^i, i = 1, \cdots, n$: $\mathbf{L}u^i = \lambda_i u^i$, which are conveniently obtained by numerical methods in the general case. Representing V as superposition of the modes:

$$V(t) = \sum_i \xi_i(t) u^i. \tag{4}$$

The temporal dynamics of ξ_i will be the amplitude equations we are to derive.

By inserting equation 4 into equation 3 we get

$$\begin{aligned}\sum_j \dot{\xi}_j u^j &= \mathbf{L}\sum_j \xi_j u^j + Q(V) + K(V)\\ &= \sum_j \xi_j \lambda_j u^j + Q(V) + K(V).\end{aligned} \tag{5}$$

$\mathbf{L}$ has a set of left eigenvectors $\tilde{u}^i, i = 1, \cdots, n$: $\tilde{u}^i\mathbf{L} = \lambda_i \tilde{u}^i$. They are orthonormal to the normal modes: $\tilde{u}^i u^j = \delta_{ij}$. By multiplying both sides of equation 5 by $\tilde{u}^i$ from the left, we obtain

$$\dot{\xi}_i = \lambda_i \xi_i + \tilde{u}^i Q(V) + \tilde{u}^i K(V). \tag{6}$$

We now have to obtain the nonlinear terms $Q(V)$ and $K(V)$ in explicit numerical form. Each element of $Q(V)$ is a linear combination of monomials of degree 2 on the n variables of V: $v_1v_1, v_1v_2, \cdots, v_nv_n$. Let $V^{(2)}$ be a vector of all different monomials of degree 2 in a fixed order: $V^{(2)} = \begin{pmatrix} v_1v_1 \\ v_1v_2 \\ \vdots \\ v_nv_n \end{pmatrix}$. Then for each $Q(V)$ there exists a matrix $\mathbf{Q}$ (of size $n \times \frac{n(n+1)}{2}$) such that

$$Q(V) = \mathbf{Q}V^{(2)}$$

Expand the elements of $V^{(2)}$, showing only a general term v_iv_j:

$$\begin{pmatrix} \vdots \\ v_iv_j \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \left(\sum_l \xi_l u_i^l\right)\left(\sum_{l'} \xi_{l'} u_j^{l'}\right) \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \sum_{ll'} \xi_l \xi_{l'} u_i^l u_j^{l'} \\ \vdots \end{pmatrix} = \sum_{ll'} \xi_l \xi_{l'} \begin{pmatrix} \vdots \\ u_i^l u_j^{l'} \\ \vdots \end{pmatrix}.$$

Note that the form of the last column vector in the above equation is the same as that of $V^{(2)}$, except that the two variables in each monomial are now elements of two different vectors u^l and $u^{l'}$. We denote this operator using the $\star$ symbol:

$$V \star V' \equiv \begin{pmatrix} v_1v_1' \\ v_1v_2' \\ \vdots \\ v_nv_n' \end{pmatrix}.$$

It follows that $V^{(2)} = V \star V$, and we have

$$V^{(2)} = \sum_{ll'} \xi_l \xi_{l'} (u^l \star u^{l'})$$

Multiply both sides by $\mathbf{Q}$ from the left:

$$\mathbf{Q}V^{(2)} = \mathbf{Q}\sum_{ll'} \xi_l \xi_{l'} (u^l \star u^{l'}) = \sum_{ll'} \xi_l \xi_{l'} \mathbf{Q}(u^l \star u^{l'}), \tag{7}$$

where $\mathbf{Q}(u^l \star u^{l'}) \equiv Q^{ll'}$ is a vector of length n, that for any given system, can conveniently be calculated numerically. Its multiplication with $\tilde{u}^i$ can also be computed, and we have

$$\tilde{u}^i Q(V) = \tilde{u}^i \mathbf{Q} V^{(2)} = \sum_{ll'} \xi_l \xi_{l'} \tilde{u}^i Q^{ll'}.$$

The exact same procedure applies to polynomials of any order. In particular, for the third order cubic term, we have

$$\tilde{u}^i K(V) = \sum_{ll'l''} \xi_l \xi_{l'} \xi_{l''} \tilde{u}^i K^{ll'l''},$$

where $K^{ll'l''} \equiv \mathbf{K}(u^l \star u^{l'} \star u^{l''})$.

This leads to an explicit amplitude equation set:

$$\dot{\xi}_i = \lambda_i \xi_i + \sum_{ll'} \xi_l \xi_{l'} \tilde{u}^i Q^{ll'} + \sum_{ll'l''} \xi_l \xi_{l'} \xi_{l''} \tilde{u}^i K^{ll'l''} \tag{8}$$

This set still has the size of the original set of equations. As carried through in [9], the system can be simplified by subdividing it into two sets, of principal (unstable) modes whose eigenvalues are positive (or have positive real parts), and ancillary (stable) modes with negative eigenvalues. The latter can be approximated adiabatically to obtain their amplitudes as functions of the principal mode amplitudes, so that the system is reduced just to the amplitudes of the principal modes.

4 Discussion

A central difficulty in understanding the brain is the problem of bridging between local descriptions in terms of neurons, synapses and molecules and global descriptions in terms of the organized states which we experience as thoughts. There are actually very few methods with which science has achieved such bridging of levels in more than verbal form. One instance is statistical mechanics, the other stability analysis. The former is restricted to the idealizations of equilibrium systems, the latter has classically been restricted to highly simplified and symmetric arrangements. It is time now to break out of this cage. The main point of our paper is to outline a way to do just that.

The formulation presented here is not new in principle, following closely the treatment given in [8]. Our point is rather that whereas the approach lay essentially dormant for decades, as due to its complications in detail it was restricted to unrealistically idealized model cases, it now can become a routine tool for realistic systems on the basis of automated numerical implementation. The resulting amplitude equations give a simplified description of a system in terms of global patterns and their interactions. On this basis, one may hope, it will become easier to conceptually connect the microscopic terms and parameters of a system with the structure, relative stability and interactions of patterns on a global scale. This should make it possible to design (or rather back-engineer) those local interactions that lead to desirable global behaviour.

Organization of a complex system with many degrees of freedom must take the form of gradual differentiation. In each moment in time, only very few degrees of freedom should be able to play and decide, with the help of cooperation and competition, and only after a decision has been reached should more degrees of freedom be opened. A perhaps typical example is given by the retinotopic system as formulated in [9]. Initially, with high values of the control parameter α, the system has only two diagonals, very broad retinotopic mappings, as unstable modes, and they compete with each other. Once a decision has been reached, α can be lowered, and more modes are made unstable. However, only a very small subset of these is excited by the winning diagonal: all those that are parallel to it and have the same phase. All these remaining modes, once they are liberated by falling α, conspire to differentiate the mapping into a narrow, one-to-one retinotopic mapping. The numerical analysis as discussed, may help to understand differentiation of other organizing systems in close analogy to this example.

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