# Modern Topics in Solid-State Theory: Topological insulators and superconductors

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# Lecture Four: Classification schemes

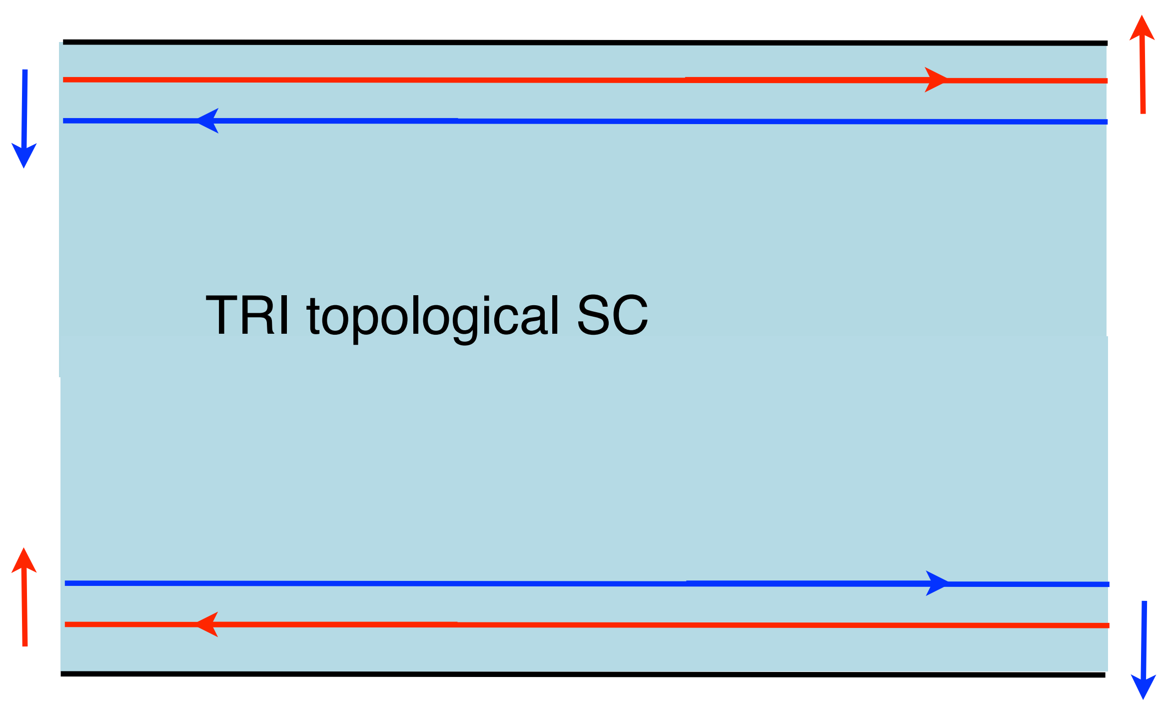
## 1. Topological superconductors

- Topological superconductors w/ TRS in 2D
- Topological superconductors w/ TRS in 3D

## 2. Symmetries & ten-fold classification

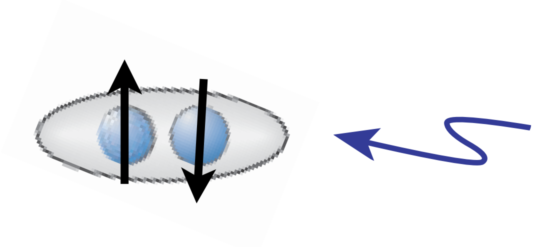
- Symmetry classes of ten-fold way
- Dirac Hamiltonians and Dirac mass gaps
- Periodic table of topological insulators and superconductors

# Helical superconductors
## (w/ time-reversal symmetry)



# Time-reversal-invariant topological superconductor

## Superconducting pairing with spin:




$$H_{\rm MF} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\sigma\sigma'} \left[ \Delta_{\sigma\sigma'}(\mathbf{k}) c^\dagger_{\mathbf{k},\sigma} c^\dagger_{-\mathbf{k},\sigma'} + \Delta^*_{\sigma\sigma'}(\mathbf{k}) c_{-\mathbf{k},\sigma'} c_{\mathbf{k},\sigma} \right]$$

2 x 2 Gap matrix: $\Delta(\mathbf{k}) = [\Delta_s(\mathbf{k})\sigma_0 + \mathbf{d}(\mathbf{k})\cdot\vec{\sigma}] i\sigma_y$

Time-reversal symmetry: $\sigma_y \Delta^\dagger(\mathbf{k}) \sigma_y = \Delta^{\rm T}(-\mathbf{k})$

Different spin-pairing symmetries: (anti-symmetry of wavefunction)

spin-singlet: $\Delta_s(\mathbf{k}) : \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ even parity: $\Delta_s(\mathbf{k}) = \Delta_s(-\mathbf{k})$

spin-triplet:
$$\begin{cases} d_x(\mathbf{k}) - i d_y(\mathbf{k}) : |\uparrow\uparrow\rangle \\ d_x(\mathbf{k}) + i d_y(\mathbf{k}) : |\downarrow\downarrow\rangle \\ d_z(\mathbf{k}) : \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \end{cases}$$
odd parity: $\mathbf{d}(\mathbf{k}) = -\mathbf{d}(-\mathbf{k})$

# 2D time-reversal-invariant topological superconductor

(also known as "helical superconductor")

Square lattice BdG Hamiltonian in the presence of time-reversal symmetry:

Simplest model:
(spinless chiral p-wave SC)²

$$\mathcal{H}_{\rm BdG}(\mathbf{k}) = \begin{pmatrix} \mathcal{H}_{p+ip}(\mathbf{k}) & 0 \\ 0 & \mathcal{H}_{p-ip}(\mathbf{k}) \end{pmatrix}$$

$$\varepsilon(\mathbf{k}) = 2t(\cos k_x + \cos k_y) - \mu \qquad d_x(\mathbf{k}) = \sin k_x \quad d_y(\mathbf{k}) = \sin k_y \quad d_z(\mathbf{k}) = 0$$

TRS: $T\mathcal{H}_{\rm BdG}(\mathbf{k})T^{-1} = +\mathcal{H}_{\rm BdG}(-\mathbf{k})$ $\quad T = i\sigma_y \otimes \tau_0 \mathcal{K} \quad T^2 = -1$

PHS: $C\mathcal{H}_{\rm BdG}(\mathbf{k})C^{-1} = -\mathcal{H}_{\rm BdG}(-\mathbf{k})$ $\quad C = \sigma_0 \otimes \tau_x \mathcal{K} \quad C^2 = +1$

} class DIII

- Combination of time-reversal and particle-hole symmetry:

(chiral symmetry) $\quad U_S = (i\sigma_y \otimes \tau_0)(\sigma_0 \otimes \tau_x) \qquad U_S \mathcal{H}_{\rm BdG}(\mathbf{k}) + \mathcal{H}_{\rm BdG}(\mathbf{k}) U_S = 0$

- $\mathcal{H}_{\rm BdG}$ can be brought into block-off diagonal form: (transform to basis in which S is diagonal)

$$\widetilde{\mathcal{H}}_{\rm BdG}(\mathbf{k}) = \begin{pmatrix} 0 & D(\mathbf{k}) \\ D^\dagger(\mathbf{k}) & 0 \end{pmatrix} \qquad D(\mathbf{k}) = (i\sigma_y)\left\{\varepsilon_{\mathbf{k}}\sigma_0 + i\Delta_t[\mathbf{d}_{\mathbf{k}} \cdot \vec{\sigma}]\right\}$$

- TRS acts on $D(\mathbf{k})$ as follows: $D^T(-\mathbf{k}) = -D(\mathbf{k})$

# 2D time-reversal-invariant topological superconductor

(also known as "helical superconductor")

Square lattice BdG Hamiltonian in the presence of time-reversal symmetry:

Simplest model:
(spinless chiral p-wave SC)$^2$

$$\mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) = \begin{pmatrix} \varepsilon(\mathbf{k})\sigma_0 & \Delta_t[\mathbf{d}(\mathbf{k})\cdot\vec{\sigma}](i\sigma_y) \\ \Delta_t(-i\sigma_y)[\mathbf{d}(\mathbf{k})\cdot\vec{\sigma}] & -\varepsilon(\mathbf{k})\sigma_0 \end{pmatrix}$$

$$\varepsilon(\mathbf{k}) = 2t(\cos k_x + \cos k_y) - \mu \qquad d_x(\mathbf{k}) = \sin k_x \quad d_y(\mathbf{k}) = \sin k_y \quad d_z(\mathbf{k}) = 0$$

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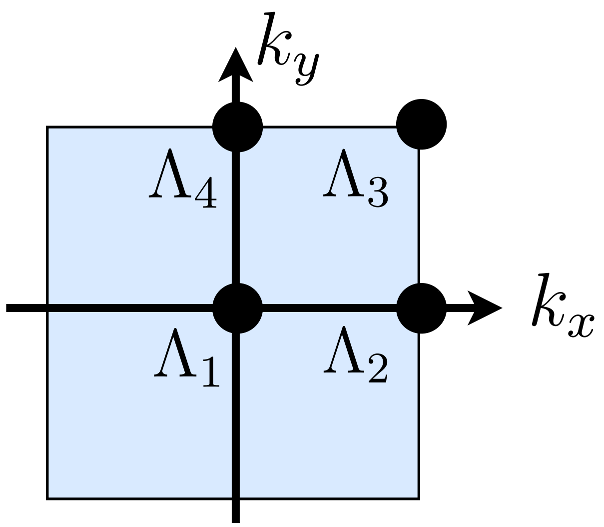
- TRS acts on $D(\mathbf{k})$ as follows: $D^T(-\mathbf{k}) = -D(\mathbf{k})$

# 2D time-reversal-invariant topological superconductor

$$\widetilde{\mathcal{H}}_{\mathrm{BdG}}(\mathbf{k}) = \begin{pmatrix} 0 & D(\mathbf{k}) \\ D^\dagger(\mathbf{k}) & 0 \end{pmatrix} \quad \text{where:} \quad D(\mathbf{k}) = (i\sigma_y)\{\varepsilon_{\mathbf{k}}\sigma_0 + i\Delta_t[\mathbf{d}_{\mathbf{k}}\cdot\vec{\sigma}]\}$$




- Spectrum flattening:

Projector onto filled Bloch bands

$$Q = \mathbb{1}_{4N} - 2P \qquad Q(\mathbf{k}) = \begin{pmatrix} 0 & q(\mathbf{k}) \\ q^\dagger(\mathbf{k}) & 0 \end{pmatrix}$$

- TRS acts on $q(\mathbf{k})$ as follows: $\quad q(\mathbf{k}) = -q^T(-\mathbf{k})$

- The eigenfunctions of $Q(\mathbf{k})$ are:

$$|u_a^\pm(\boldsymbol{k})\rangle_{\mathrm{N}} = \frac{1}{\sqrt{2}}\begin{pmatrix} n_a \\ \pm q^\dagger(\boldsymbol{k}) n_a \end{pmatrix} \quad \text{where:} \quad (n_a)_b = \delta_{ab}$$

are globally defined.

**$Z_2$ topological invariant**:

sewing matrix

$$(-1)^\nu = \prod_{a=1}^{4} \frac{\mathrm{Pf}\,[\omega(\Lambda_a)]}{\sqrt{\det\,[\omega(\Lambda_a)]}} = \pm 1 \qquad \omega(\mathbf{k}) = {}_{\mathrm{N}}\langle u_a^-(-\mathbf{k})|\Theta u_b^-(\mathbf{k})\rangle_{\mathrm{N}}$$

$$\Rightarrow \boxed{(-1)^\nu = \prod_{a=1}^{4} \frac{\mathrm{Pf}\,[q^T(\Lambda_a)]}{\sqrt{\det\,[q(\Lambda_a)]}} = \pm 1}$$

$$q(\mathbf{k}) = -q^T(-\mathbf{k})$$

$$q^\dagger(\mathbf{k}) = q^{-1}(\mathbf{k})$$

(same symmetries as sewing matrix)

# 2D time-reversal-invariant topological superconductor

Effective low-energy continuum theory:
(expand around $\mathbf{k} = 0$)

$$\mathcal{H}_{\rm BdG}(\mathbf{k}) = \begin{pmatrix} \varepsilon(\mathbf{k})\sigma_0 & \Delta_t[\mathbf{d}(\mathbf{k})\cdot\vec{\sigma}](i\sigma_y) \\ \Delta_t(-i\sigma_y)[\mathbf{d}(\mathbf{k})\cdot\vec{\sigma}] & -\varepsilon(\mathbf{k})\sigma_0 \end{pmatrix}$$

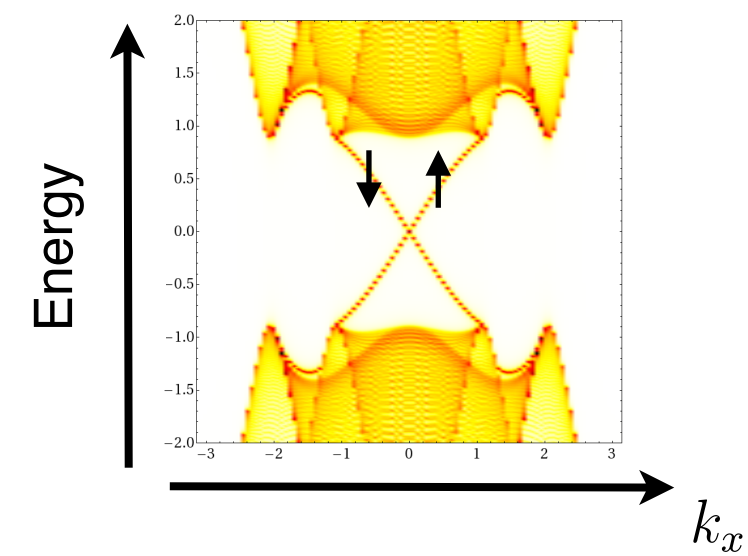
$$\varepsilon(\mathbf{k}) = -tk^2 + 4t - \mu \qquad d_x(\mathbf{k}) = k_x \qquad d_y(\mathbf{k}) = k_y \qquad d_z(\mathbf{k}) = 0$$

Energy spectrum: $E_\pm = \pm\lambda(\mathbf{k}) = \pm\sqrt{\varepsilon^2(\mathbf{k}) + \Delta_t(k_x^2 + k_y^2)}$ TRIM: $k = 0, \quad k = +\infty$

**$Z_2$ topological invariant :** $(-1)^\nu = -\mathrm{sgn}(4t-\mu)\mathrm{sgn}(t)$

$\mu > 4t :$ trivial superconductor

$\mu < 4t :$ TRI topological superconductor



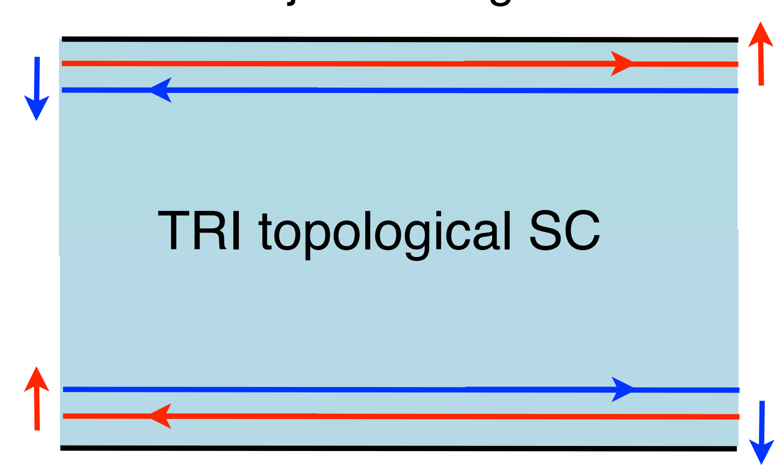

**Bulk-boundary correspondence:**

By analogy to chiral p-wave SC: (for $|\mu| < 4t$)
two counter-propagating Majorana edge modes

— protected by TRS and PHS

— possible condensed matter realization:
thin film of $CePt_3Si$?

helical Majorana edge states:

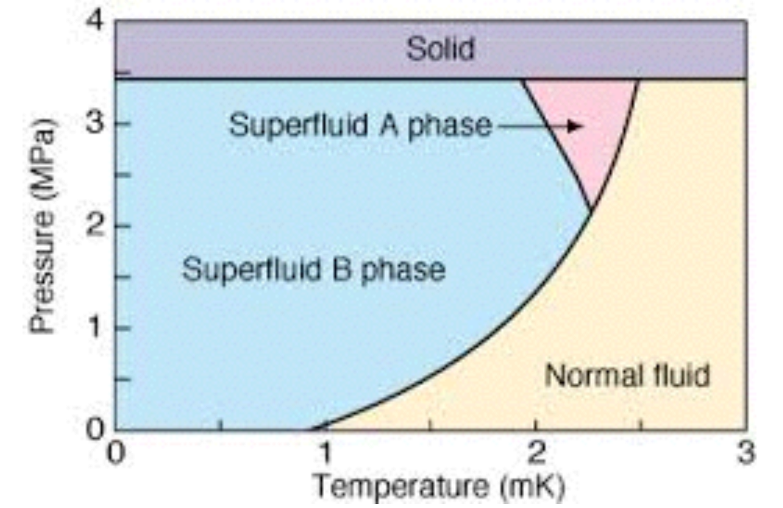


# 3D time-reversal-invariant topological superconductor

Cubic lattice BdG Hamiltonian in the presence of time-reversal symmetry:

$$\mathcal{H}_{\rm BdG}(\mathbf{k}) = \begin{pmatrix} \varepsilon(\mathbf{k})\sigma_0 & \Delta_t[\mathbf{d}(\mathbf{k})\cdot\vec{\sigma}](i\sigma_y) \\ \Delta_t(-i\sigma_y)[\mathbf{d}(\mathbf{k})\cdot\vec{\sigma}] & -\varepsilon(\mathbf{k})\sigma_0 \end{pmatrix}$$

equivalent to B phase of $^3$He




$$\varepsilon(\mathbf{k}) = 2t(\cos k_x + \cos k_y + \cos k_z) - \mu$$

$$d_x(\mathbf{k}) = \sin k_x \quad d_y(\mathbf{k}) = \sin k_y \quad d_z(\mathbf{k}) = \sin k_z$$

TRS $\boxed{T\mathcal{H}_{\rm BdG}(\mathbf{k})T^{-1} = +\mathcal{H}_{\rm BdG}(-\mathbf{k})}$ $\quad T = i\sigma_y \otimes \tau_0 \mathcal{K} \quad T^2 = -1$

PHS $\boxed{C\mathcal{H}_{\rm BdG}(\mathbf{k})C^{-1} = -\mathcal{H}_{\rm BdG}(-\mathbf{k})}$ $\quad C = \sigma_0 \otimes \tau_x \mathcal{K} \quad C^2 = +1$

} class DIII

Chiral symmetry (TRS x PHS): $U_S\mathcal{H}_{\rm BdG}(\mathbf{k}) + \mathcal{H}_{\rm BdG}(\mathbf{k})U_S = 0$

▶ $\mathcal{H}_{\rm BdG}$ can be brought into block-off diagonal form: (transform to basis in which S is diagonal)

$$\widetilde{\mathcal{H}}_{\rm BdG}(\mathbf{k}) = \begin{pmatrix} 0 & D(\mathbf{k}) \\ D^\dagger(\mathbf{k}) & 0 \end{pmatrix} \qquad D(\mathbf{k}) = (i\sigma_y)\left\{\varepsilon_{\mathbf{k}}\sigma_0 + i\Delta_t[\mathbf{d}_{\mathbf{k}}\cdot\vec{\sigma}]\right\}$$

▶ TRS acts on $D(\mathbf{k})$ as follows: $D^T(-\mathbf{k}) = -D(\mathbf{k})$

# 3D time-reversal-invariant topological superconductor

Lattice BdG Hamiltonian: $\widetilde{\mathcal{H}}_{\mathrm{BdG}}(\mathbf{k}) = \begin{pmatrix} 0 & D(\mathbf{k}) \\ D^\dagger(\mathbf{k}) & 0 \end{pmatrix}$

▶ Off-diagonal block: $D(\mathbf{k}) = (i\sigma_y)\left\{\varepsilon_{\mathbf{k}}\sigma_0 + i\Delta_t[\mathbf{d}_{\mathbf{k}} \cdot \vec{\sigma}]\right\}$

Mapping $D(\mathbf{k})$ : Brillouin zone $\longmapsto$ $D(\mathbf{k})$ TRS: $D(\mathbf{k}) = -D^T(\mathbf{k})$

▶ Spectrum flattening: $q(\mathbf{k}) = \sum_a \frac{1}{\lambda_a(\mathbf{k})} u_a(\mathbf{k}) u_a^\dagger(\mathbf{k}) D(\mathbf{k})$ $u_a(\boldsymbol{k})$ : eigenvectors of $DD^\dagger$

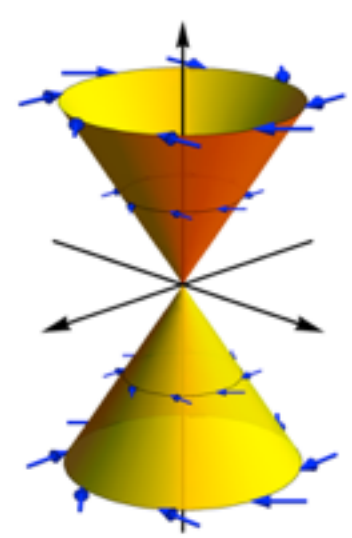
Mapping $q(\boldsymbol{k})$ : Brillouin zone $\longmapsto$ $q(\boldsymbol{k}) \in U(2)$ $\pi_2[U(2)] = 0$

TRS: $q(\mathbf{k}) = -q^T(-\mathbf{k})$ $\pi_3[U(2)] = \mathbb{Z}$

$\Longrightarrow$ classified by winding number: $W = \frac{1}{24\pi^2} \int_{\mathrm{BZ}} d^3k\, \varepsilon^{\mu\nu\rho}\, \mathrm{Tr}\left[(q^{-1}\partial_\mu q)(q^{-1}\partial_\nu q)(q^{-1}\partial_\rho q)\right]$

▶ Bulk-boundary correspondence:

$|W| = \#$ Kramers-degenerate Majorana states



Possible condensed matter realization:
$CePt_3Si$, $Li_2Pt_3B$, $CeRhSi_3$, $CeIrSi_3$, etc.

# Classification schemes

| Symmetry | | | | dim | | |
|---|---|---|---|---|---|---|
| Class | $T$ | $P$ | $S$ | 1 | 2 | 3 |
| A | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | 0 |
| AIII | 0 | 0 | 1 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| AI | 1 | 0 | 0 | 0 | 0 | 0 |
| BDI | 1 | 1 | 1 | $\mathbb{Z}$ | 0 | 0 |
| D | 0 | 1 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 |
| DIII | -1 | 1 | 1 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| AII | -1 | 0 | 0 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| CII | -1 | -1 | 1 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ |
| C | 0 | -1 | 0 | 0 | $\mathbb{Z}$ | 0 |
| CI | 1 | -1 | 1 | 0 | 0 | $\mathbb{Z}$ |

# Symmetry classes: "Ten-fold way"

(originally introduced in the context of random Hamiltonians / matrices)

time-reversal invariance: $T = U_T \mathcal{K}$ (is antiunitary)

$$T^{-1}\mathcal{H}(-\mathbf{k})T = +\mathcal{H}(\mathbf{k})$$

$$T : \begin{cases} 0 & \text{no time reversal invariance} \\ +1 & \text{time reversal invariance and } T^2 = +1 \\ -1 & \text{time reversal invariance and } T^2 = -1 \end{cases}$$

complex conjugation

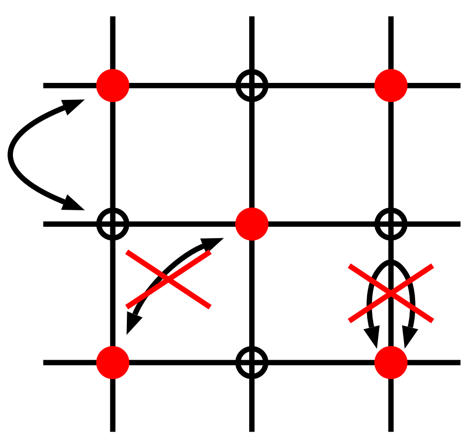
particle-hole symmetry ($\Xi$): $C = U_C \mathcal{K}$

$$C^{-1}\mathcal{H}(-\mathbf{k})C = -\mathcal{H}(\mathbf{k})$$

$$C : \begin{cases} 0 & \text{no particle-hole symmetry} \\ +1 & \text{particle-hole symmetry and } C^2 = +1 \\ -1 & \text{particle-hole symmetry and } C^2 = -1 \end{cases}$$

In addition we can also consider the "sublattice symmetry" $S \propto TC$

S : $S\mathcal{H}(\mathbf{k}) + \mathcal{H}(\mathbf{k})S = 0$



Note: SLS is often also called "chiral symmetry"

# Ten-fold classification of topological insulators and superconductors

## Ten-fold classification:

— classifies fully gapped topological materials in terms of *non-spatial symmetries* (i.e., symmetries that act *locally* in space)

— non-spatial symmetries:

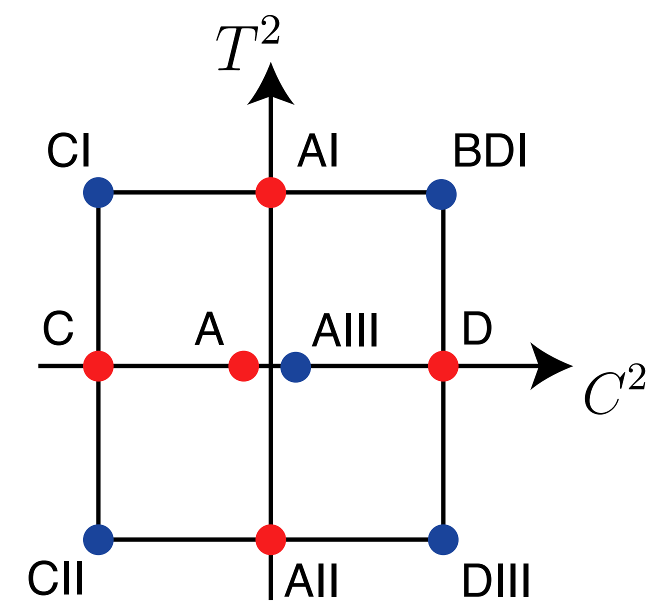
- time-reversal: $T\mathcal{H}(\mathbf{k})T^{-1} = +\mathcal{H}(-\mathbf{k});\qquad T^2 = \pm 1$
- particle-hole: $C\mathcal{H}(\mathbf{k})C^{-1} = -\mathcal{H}(-\mathbf{k});\qquad C^2 = \pm 1$
- sublattice: $S\mathcal{H}(\mathbf{k})S^{-1} = -\mathcal{H}(\mathbf{k});\qquad S \propto TP$

ten symmetry classes

Altland-Zirnbauer Random Matrix Classes

| Symmetry Class | $T$ | $C$ | $S$ |
|---|---|---|---|
| A | 0 | 0 | 0 |
| AIII | 0 | 0 | 1 |
| AI | 1 | 0 | 0 |
| BDI | 1 | 1 | 1 |
| D | 0 | 1 | 0 |
| DIII | -1 | 1 | 1 |
| AII | -1 | 0 | 0 |
| CII | -1 | -1 | 1 |
| C | 0 | -1 | 0 |
| CI | 1 | -1 | 1 |

“Bott clock”



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- sublattice: $S\mathcal{H}(\mathbf{k})S^{-1} = -\mathcal{H}(\mathbf{k});$ $S \propto TP$

ten symmetry classes

Altland-Zirnbauer Random Matrix Classes

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| A | 0 | 0 | 0 |
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**?** For which symmetry class and dimension is there a topological insulator/superconductor?

# Symmetries and Dirac Hamiltonians

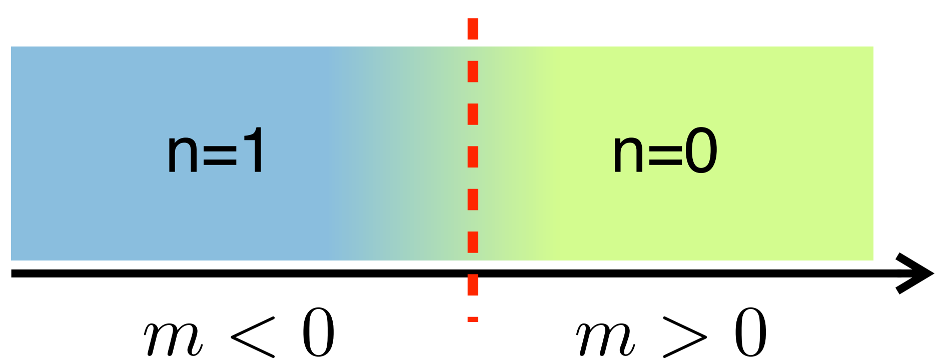
Dirac Hamiltonian in spatial dimension $d$ : $\mathcal{H}(k) = \sum_{i=1}^{d} k_i \gamma_i + m\gamma_0$ $\quad E_\pm = \pm\sqrt{m^2 + \sum_{i=1}^{d} k_i^d}$

- Gamma matrices $\gamma_i$ obey: $\{\gamma_i, \gamma_j\} = 2\delta_{ij} \qquad i = 0, 1, \ldots, d$

- TRS, PHS and chiral symmetry lead to the conditions:

$$[\gamma_0, T] = 0 \qquad \{\gamma_{i\neq 0}, T\} = 0$$
$$\{\gamma_0, C\} = 0 \qquad [\gamma_{i\neq 0}, C] = 0$$
$$\{\gamma_i, S\} = 0$$

- Topological phase transition as a function of mass term $m\gamma_0$




**?** are there extra symmetry preserving mass terms $M\gamma_{d+1}$ that connect the two phases without gap closing?

$$\{\gamma_{d+1}, \gamma_i\} = 0 \quad i = 0, 1, \cdots 2 \qquad E_\pm = \pm\sqrt{m^2 + M^2 + \sum_{i=1}^{d} k_i^d}$$

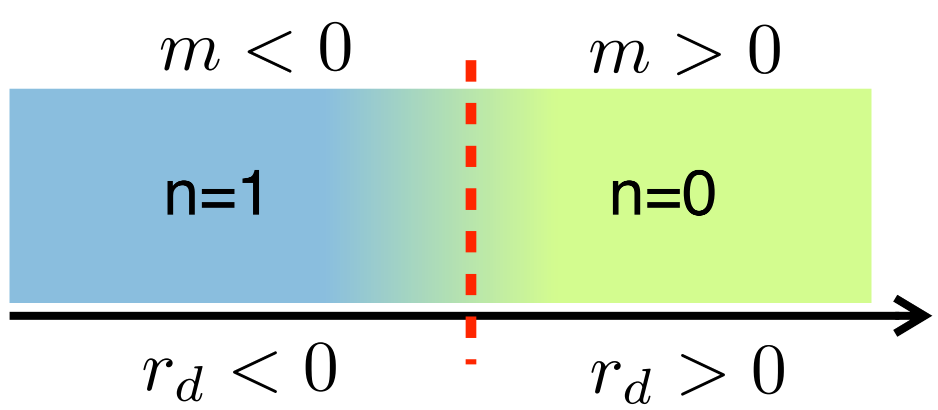
**NO:** topological non-trivial **YES:** topologically trivial

# Symmetries and Dirac Hamiltonians

Dirac Hamiltonian in spatial dimension $d$ : $\mathcal{H}(k) = \sum_{i=1}^{d} k_i \gamma_i + m\gamma_0$ $\quad E_\pm = \pm\sqrt{m^2 + \sum_{i=1}^{d} k_i^d}$

- Gapless surface states (interface states): $k_d \to i\partial/\partial r_d$

$$\mathcal{H} = \gamma_0\left(\widetilde{m}\mathbb{I} - i\gamma_0\gamma_d \frac{\partial}{\partial r_d}\right) + \sum_{i=1}^{d-1} k_i \gamma_i$$




surface state $\phi$: $\quad i\gamma_0\gamma_d\Phi = \pm\Phi$

surface Hamiltonian: $\quad \mathcal{H}_{\text{surf}} = \sum_{i=1}^{d-1} k_i\, \mathbf{P}\gamma_i\mathbf{P}$

$$\mathbf{P} = (\mathbb{I} - i\gamma_0\gamma_d)/2$$

gapless surface spectrum: $\quad E_{\text{surf}}^{\pm} = \pm\sqrt{\sum_{i=1}^{d-1} k_i^2}$

- Presence of extra symmetry preserving mass term implies gapped surface states

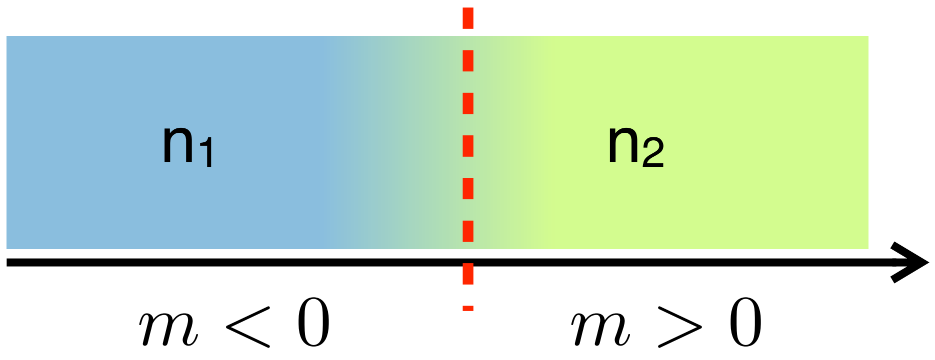
— extra mass term projected onto surface is non-vanishing

$M\mathbf{P}\gamma_{d+1}\mathbf{P}$ anti-commutes with $\mathbf{P}\gamma_i\mathbf{P}$ $\quad i = 1, \ldots, d-1$

$\Longrightarrow$ gapped surface spectrum

# Dirac Hamiltonian in symmetry class AIII

- Topological phase transition as a function of mass term $m\gamma_0$




$$S = \sigma_1 \qquad S\mathcal{H}(\mathbf{k}) + \mathcal{H}(\mathbf{k})S = 0$$

- *One-dimensional* Dirac Hamiltonian with rank 2:

$$\mathcal{H}(k) = k\sigma_3 + m\sigma_2$$

— no extra symmetry preserving mass term exists

$\Longrightarrow$ class AIII in 1D is topologically non-trivial

— space of normalized mass matrices $V^{\mathrm{AIII}}_{d=1,r=2} = \{\pm\sigma_2\}$

# One-dimensional Dirac Hamiltonian in symmetry class AII

$$T^{-1}\mathcal{H}(-\mathbf{k})T = +\mathcal{H}(\mathbf{k}) \qquad T^2 = -1$$

✳Dirac matrices with rank 2:

$$\mathcal{H}(k) = k\sigma_3 \qquad T = i\sigma_2\mathcal{K}$$

— no symmetry-allowed mass term exists $\Rightarrow$ impossible to localize
$(\sigma_1 \text{ and } \sigma_2 \text{ violate TRS})$

— describes edge state of 2D topological insulator in class AII

✳Dirac matrices with rank 4:

$$\mathcal{H}(k) = k\sigma_3 \otimes \tau_1 + m\sigma_0 \otimes \tau_3 \qquad T = i\sigma_2 \otimes \tau_0\mathcal{K}$$

— extra symmetry preserving mass term: $M\sigma_3 \otimes \tau_2$

$\Longrightarrow$ class AII in 1D is topologically trivial

— space of normalized mass matrices

$$V^{\mathrm{AII}}_{d=1,r=4} = \{\mathbf{M}\cdot\mathbf{X}|\ \mathbf{M}^2 = 1\} = S^1 \qquad R_3 : U(2N)/Sp(N)$$

$$\mathbf{M} = (m, M), \qquad \mathbf{X} = (\sigma_0 \otimes \tau_3, \sigma_3 \otimes \tau_2)$$

• connectedness of space of normalized Dirac masses: $\pi_0(R_3) = 0$

# Two-dimensional Dirac Hamiltonian in symmetry class AII

$$T^{-1}\mathcal{H}(-\mathbf{k})T = +\mathcal{H}(\mathbf{k}) \qquad T^2 = -1$$

- Dirac matrices with rank 4: $T = i\sigma_2 \otimes \tau_0 \mathcal{K}$

$$\mathcal{H}(\mathbf{k}) = k_1 \sigma_3 \otimes \tau_1 + k_2 \sigma_0 \otimes \tau_2 + m \sigma_0 \otimes \tau_3$$

— no symmetry-allowed mass term exists $\Rightarrow$ topologically non-trivial

$(\sigma_1 \otimes \tau_1,\ \sigma_2 \otimes \tau_1$ violate TRS$)$

- "Doubled" Dirac Hamiltonian:

$$\mathcal{H}_2(\mathbf{k}) = \begin{pmatrix} \mathcal{H}(\mathbf{k}) & 0 \\ 0 & \hat{\mathcal{H}}_{\mu\nu\lambda}(\mathbf{k}) \end{pmatrix} \qquad \mu, \nu, \lambda \in \{+1, -1\}$$

$$\hat{\mathcal{H}}_{\mu\nu\lambda}(\mathbf{k}) = \mu k_1 \sigma_3 \otimes \tau_1 + \nu k_2 \sigma_0 \otimes \tau_2 + \lambda m \sigma_0 \otimes \tau_3$$

— extra symmetry preserving mass terms:

e.g. for $\mu = +,\ \nu = +,\ \lambda = +$: $\sigma_2 \otimes \tau_1 \otimes s_1,\ \sigma_1 \otimes \tau_2 \otimes s_2$

$\Longrightarrow$ gapped surface spectrum

$\Longrightarrow$ class AII in 2D has $Z_2$ classification

— space of normalized mass matrices: $R_2 = O(2N)/U(N)$ $\qquad \pi_0(R_2) = \mathbb{Z}_2$

# Dirac Hamiltonian in symmetry class A

- *One-dimensional* Dirac Hamiltonian with rank 2:

$$\mathcal{H}(k) = k\sigma_1 + m\sigma_2 + \mu\sigma_0$$

  — extra symmetry preserving mass term: $M\sigma_3$

  $\Longrightarrow$ class A in 1D is topologically trivial

  — space of normalized mass matrices

$$V^{\mathrm{A}}_{d=1,r=2} = \{\tau_2 \cos\theta + \tau_3 \sin\theta | 0 \leq \theta < 2\pi\} = S^1 \qquad C_1 : \; U(N)$$

  — connectedness of space of normalized Dirac masses: $\pi_0(C_1) = 0$

- *Two-dimensional* Dirac Hamiltonian with rank 2:

$$\mathcal{H}(\mathbf{k}) = k_x\sigma_x + k_y\sigma_y + m\sigma_z + \mu\sigma_0$$

  — no extra mass term exists $\Longrightarrow$ class A in 2D is topologically non-trivial

  — describes two-dimensional Chern insulator

- *Two-dimensional* “doubled” Dirac Hamiltonian:

$$\mathcal{H}_2(\mathbf{k}) = \mathcal{H}(\mathbf{k}) \otimes \tau_0$$

  — no extra gap opening mass term exists $\Longrightarrow$ topologically non-trivial

  $\Rightarrow$ indicates $\mathbb{Z}$ classification

# Homotopy classification of Dirac mass gaps

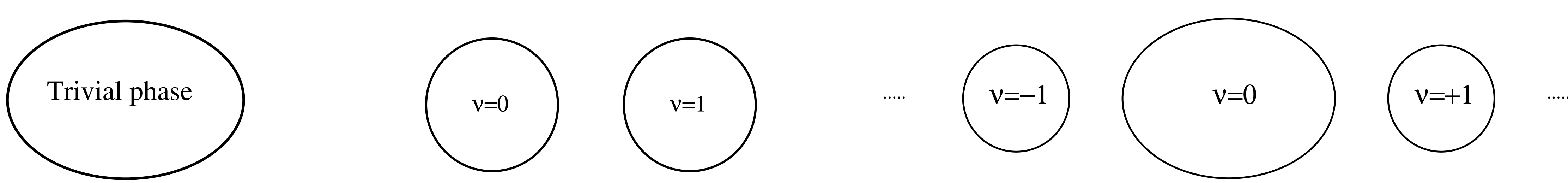
✳ The space of mass matrices $V^{s}_{d,r=N}$ belongs to different classifying spaces $C_{s-d}$ (for "complex class") or $R_{s-d}$ (for "real class")

— the relation between AZ symmetry class and classifying space is as follows:

| | classifying space | $\pi_0(*)$ | 1D AZ class | 2D AZ class |
|---|---|---|---|---|
| $\mathcal{C}_0$ | $\cup_{n=0}^{N}\{U(N)/[U(n)\times U(N-n)]\}$ | $\mathbb{Z}$ | AIII | A |
| $\mathcal{C}_1$ | $U(N)$ | 0 | A | AIII |
| $\mathcal{R}_0$ | $\cup_{n=0}^{N}\{O(N)/[O(n)\times O(N-n)]\}$ | $\mathbb{Z}$ | BDI | D |
| $\mathcal{R}_1$ | $O(N)$ | $\mathbb{Z}_2$ | D | DIII |
| $\mathcal{R}_2$ | $O(2N)/U(N)$ | $\mathbb{Z}_2$ | DIII | AII |
| $\mathcal{R}_3$ | $U(N)/Sp(N)$ | 0 | AII | CII |
| $\mathcal{R}_4$ | $\cup_{n=0}^{N}\{Sp(N)/[Sp(n)\times Sp(N-n)]\}$ | $\mathbb{Z}$ | CII | C |
| $\mathcal{R}_5$ | $Sp(N)$ | 0 | C | CI |
| $\mathcal{R}_6$ | $Sp(2N)/U(N)$ | 0 | CI | AI |
| $\mathcal{R}_7$ | $U(N)/O(N)$ | 0 | AI | BDI |

✳ The $0th$ homotopy group indexes the disconnected parts of the space of normalized mass matrices

$\pi_0(V)=0$ $\qquad$ $\pi_0(V)=\mathbb{Z}_2$ $\qquad$ $\pi_0(V)=\mathbb{Z}$



# Ten-fold classification of topological insulators and superconductors

## Ten-fold classification:

— classifies fully gapped topological materials in terms of *non-spatial symmetries* (i.e., symmetries that act *locally* in space)

— non-spatial symmetries:

- time-reversal: $T\mathcal{H}(\mathbf{k})T^{-1} = +\mathcal{H}(-\mathbf{k});\qquad T^2 = \pm 1$
- particle-hole: $C\mathcal{H}(\mathbf{k})C^{-1} = -\mathcal{H}(-\mathbf{k});\qquad C^2 = \pm 1$
- sublattice: $S\mathcal{H}(\mathbf{k})S^{-1} = -\mathcal{H}(\mathbf{k});\qquad S \propto TP$

} ten symmetry classes

Altland-Zirnbauer Random Matrix Classes

| | Symmetry | | | dim | | |
|---|---|---|---|---|---|---|
| Class | $T$ | $C$ | $S$ | 1 | 2 | 3 |
| A | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | 0 |
| AIII | 0 | 0 | 1 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| AI | 1 | 0 | 0 | 0 | 0 | 0 |
| BDI | 1 | 1 | 1 | $\mathbb{Z}$ | 0 | 0 |
| D | 0 | 1 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 |
| DIII | -1 | 1 | 1 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| AII | -1 | 0 | 0 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| CII | -1 | -1 | 1 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ |
| C | 0 | -1 | 0 | 0 | $\mathbb{Z}$ | 0 |
| CI | 1 | -1 | 1 | 0 | 0 | $\mathbb{Z}$ |

Chern insulator (A, dim 2)

polyacetylene (BDI, dim 1)

2D topological insulator w/ SOC (AII, dim 2)

3D topological insulator w/ SOC (AII, dim 3)

$\mathbb{Z}$ : integer classification
$\mathbb{Z}_2$: binary classification
0 : no topological state

Schnyder, Ryu, Furusaki, Ludwig, PRB (2008)

A. Kitaev, AIP (2009)

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Altland-Zirnbauer Random Matrix Classes

| Symmetry Class | $T$ | $C$ | $S$ | dim 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| A | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | 0 |
| AIII | 0 | 0 | 1 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| AI | 1 | 0 | 0 | 0 | 0 | 0 |
| BDI | 1 | 1 | 1 | $\mathbb{Z}$ | 0 | 0 |
| D | 0 | 1 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 |
| DIII | -1 | 1 | 1 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| AII | -1 | 0 | 0 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| CII | -1 | -1 | 1 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ |
| C | 0 | -1 | 0 | 0 | $\mathbb{Z}$ | 0 |
| CI | 1 | -1 | 1 | 0 | 0 | $\mathbb{Z}$ |

$\mathbb{Z}$ : integer classification
$\mathbb{Z}_2$: binary classification
0 : no topological state

chiral p-wave superconductor ($Sr_2RuO_4$) (→ D, dim 2)

TRI topological triplet SC ($^3$He B) (→ DIII, dim 3)

chiral d-wave superconductor (→ C, dim 2)

# Ten-fold classification of topological insulators and superconductors

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- sublattice: $S\mathcal{H}(\mathbf{k})S^{-1} = -\mathcal{H}(\mathbf{k});\qquad S \propto TP$

} ten symmetry classes

Altland-Zirnbauer Random Matrix Classes

| Symmetry | | | | Spatial Dimension $d$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Class | $T$ | $C$ | $S$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\cdots$ |
| A | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | $\cdots$ |
| AIII | 0 | 0 | 1 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\cdots$ |
| AI | 1 | 0 | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | $\cdots$ |
| BDI | 1 | 1 | 1 | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\cdots$ |
| D | 0 | 1 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\cdots$ |
| DIII | -1 | 1 | 1 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\cdots$ |
| AII | -1 | 0 | 0 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ | $\cdots$ |
| CII | -1 | -1 | 1 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\cdots$ |
| C | 0 | -1 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | $\cdots$ |
| CI | 1 | -1 | 1 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | $\cdots$ |

# Ten-fold classification of topological insulators and superconductors

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— classifies fully gapped topological materials in terms of *non-spatial symmetries* (i.e., symmetries that act *locally* in space)

Altland-Zirnbauer Random Matrix Classes

| Symmetry | | | | Spatial Dimension $d$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Class | $T$ | $C$ | $S$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\cdots$ |
| A | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | $\cdots$ |
| AIII | 0 | 0 | 1 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\cdots$ |
| AI | 1 | 0 | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | $\cdots$ |
| BDI | 1 | 1 | 1 | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\cdots$ |
| D | 0 | 1 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\cdots$ |
| DIII | -1 | 1 | 1 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\cdots$ |
| AII | -1 | 0 | 0 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ | $\cdots$ |
| CII | -1 | -1 | 1 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\cdots$ |
| C | 0 | -1 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | $\cdots$ |
| CI | 1 | -1 | 1 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | $\cdots$ |

- Topological invariants: Chern numbers and winding numbers

$$Ch_{n+1}[\mathcal{F}] = \frac{1}{(n+1)!} \int_{\mathrm{BZ}^{d=2n+2}} \mathrm{tr} \left( \frac{i\mathcal{F}}{2\pi} \right)^{n+1}$$

$$\nu_{2n+1}[q] = \frac{(-1)^n n!}{(2n+1)!} \left( \frac{i}{2\pi} \right)^{n+1} \int_{\mathrm{BZ}} \epsilon^{\alpha_1 \alpha_2 \cdots} \mathrm{tr} \left[ q^{-1} \partial_{\alpha_1} q \cdot q^{-1} \partial_{\alpha_2} q \cdots \right] d^{2n+1} k$$

# Extension I: Weak topological insulators and supercondutors

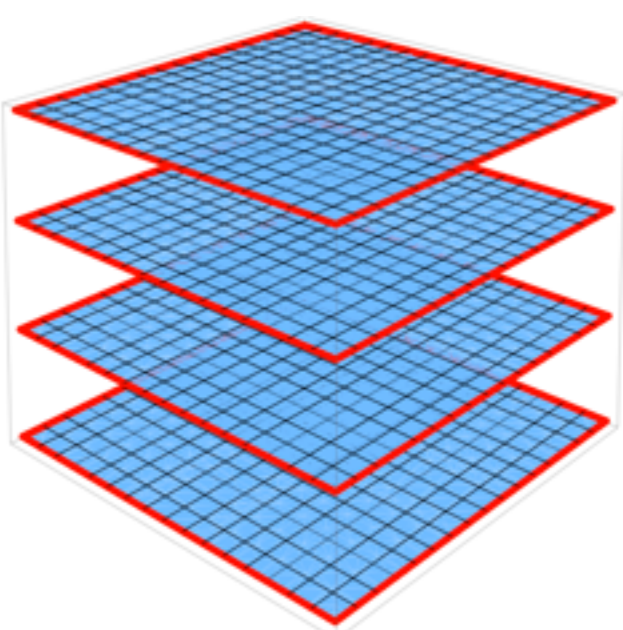
- strong topological insulators (superconductors):
  not destroyed by positional disorder

- weak topological insulators (superconductors):
  only possess topological features
  when translational symmetry is present

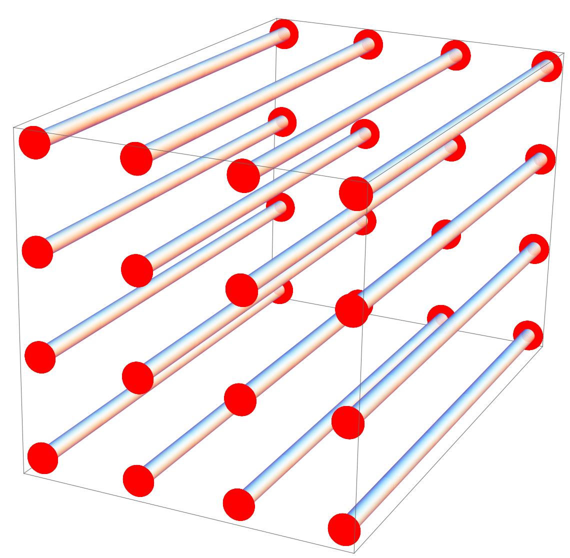
→ weak topological insulators (superconductors) are topologically equivalent to parallel stacks of lower-dimensional strong topological insulator (SCs).

| | Symmetry | | | Dimension | | | |
|---|---|---|---|---|---|---|---|
| AZ | T | C | S | 1 | 2 | 3 | 4 |
| A | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| AIII | 0 | 0 | 1 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 |
| AI | 1 | 0 | 0 | 0 | 0 | 0 | $\mathbb{Z}$ |
| BDI | 1 | 1 | 1 | $\mathbb{Z}$ | 0 | 0 | 0 |
| D | 0 | 1 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 |
| DIII | -1 | 1 | 1 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 |
| AII | -1 | 0 | 0 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| CII | -1 | -1 | 1 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| C | 0 | -1 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ |
| CI | 1 | -1 | 1 | 0 | 0 | $\mathbb{Z}$ | 0 |

co-dimension k=1



co-dimension k=2



→ d-dim.weak topological insulators (SCs) of co-dimension k can occur whenever there exists a strong topological state in same symmetry class but in (d-k) dimensions.

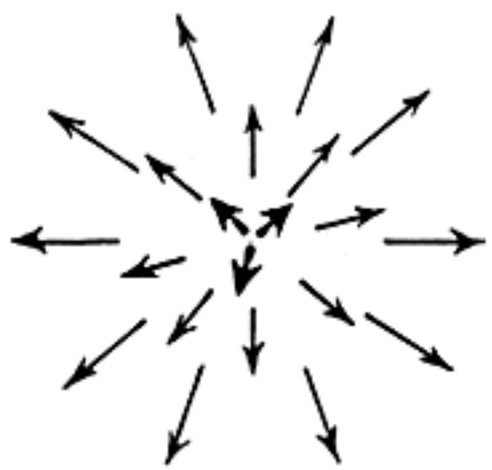
$\binom{d}{k}$ top. invariants $\quad 0 < k \leq d$

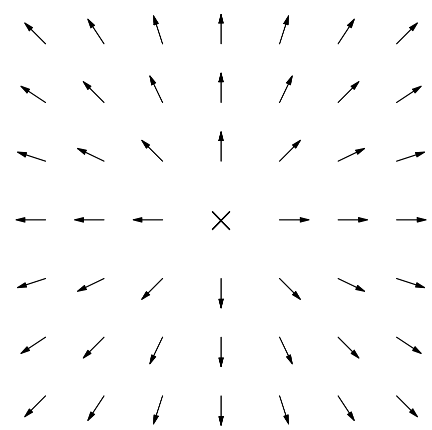
*cf. Kitaev, AIP Conf Proc. 1134, 22 (2009)*

# Extension II: Zero mode localized on topological defect

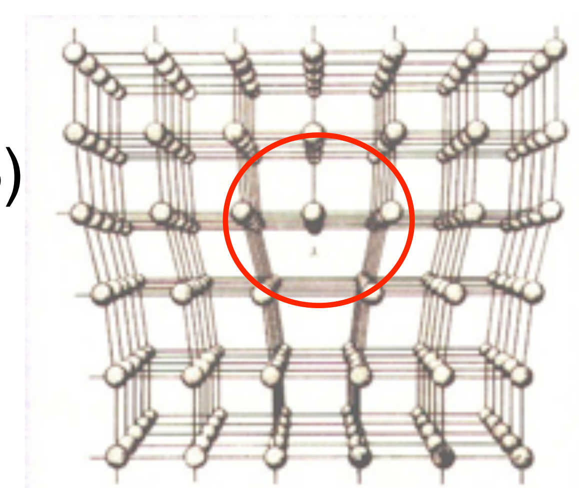
Protected zero modes can also occur at topological defects in D-dim systems

- Point defect (r=0): Hedgehog (D=3), vortex (D=2), domain wall (D=1)





- Line defect (r=1): dislocation line (D=3) domain wall (D=2)



- Two-dim defects (r=2): domain wall (D=3)

***Freedman, et. al., PRB (2010)***

***Teo & Kane, PRB (2010)***

***Ryu, et al. NJP (2010)***

| | Symmetry | | | Dimension | | | |
|---|---|---|---|---|---|---|---|
| AZ | T | C | S | 1 | 2 | 3 | 4 |
| A | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| AIII | 0 | 0 | 1 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 |
| AI | 1 | 0 | 0 | 0 | 0 | 0 | $\mathbb{Z}$ |
| BDI | 1 | 1 | 1 | $\mathbb{Z}$ | 0 | 0 | 0 |
| D | 0 | 1 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 |
| DIII | -1 | 1 | 1 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 |
| AII | -1 | 0 | 0 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| CII | -1 | -1 | 1 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| C | 0 | -1 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ |
| CI | 1 | -1 | 1 | 0 | 0 | $\mathbb{Z}$ | 0 |

Can an r-dimensional topological defect of a given symmetry class bind gapless states or not?

→ look at column d=(r+1)

(answer does not depend on D!)

line defect in class A:

$$n = \frac{1}{8\pi^2}\int_{T^3\times S^1} \mathrm{Tr}[\mathcal{F}\wedge\mathcal{F}]$$

(second Chern no = no of zero modes)