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Representation for Multiple Right-Hand Sides

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We are given finitely many polyhedra defined by linear constraints, using the same constraint matrix and different right-hand sides. We consider a simple constraint system and give necessary and sufficient conditions for this system to define the union of the polyhedra.

Key words: formulation, linear inequalities, representation

Running title: Multiple Right-hand Sides

Let A be a $m \times n$ matrix of rank n. For any $b \in R^m$, $\{x|Ax \geq b\}$ is a polyhedron. Suppose we have several right-hand-sides $b^{(1)}, \ldots, b^{(t)}$. These give t polyhedra:

$$P^{(h)} = \{x|Ax \geq b^{(h)}\} \quad 1 \leq h \leq t$$

Define:

$$Q = \operatorname{conv}(\bigcup_{h=1}^{t} P^{(h)})$$

$$T = \{x|Ax \geq \sum \lambda_h b^{(h)}, \text{ for some } \lambda \text{ with } \sum \lambda_h = 1, \ \lambda_h \geq 0\}$$

Jeroslow [1] raises the question of when $Q = T$. The motivation is that T is defined using linear constraints with the auxillary variables λ_h. Thus, when $Q = T$, the problem of maximizing a linear objective over the union of P_h can be done by solving a linear program of modest size. In particular, it is not necessary to make one copy of A for each h.

[1] gives a sufficient condition (Theorem 1 below) for $Q = T$. In this note we give a modification which is simpler and includes more cases (Theorem 2). Then we give a weaker sufficient condition (Theorem 3). If we make a nondegeneracy assumption, this condition is necessary (Theorem 4). The condition of Theorems 3–4 is not easy to verify. We show (Theorem 5) that the problem of deciding whether $Q = T$ for given A, $b^{(h)}$ is NP-Hard, which suggests that no easily verifiable necessary and sufficient condition exists.

Definitions. For I a subset of the rows of A, $1 \leq h \leq t$, we define

$$E_{I,h} = \{x|\ (Ax)_i = b_i^{(h)} \text{ for all } i \in I\}$$

$$F_{I,h} = \{x|\ (Ax)_i \geq b_i^{(h)} \text{ for all } i \in I\}$$

When $E_{I,h}$ consists of a single vector, we define $x_{I,h}$ to be that vector. For h fixed, those $x_{I,h}$ which are in $P^{(h)}$ are the extreme points of $P^{(h)}$.

THEOREM 1 [1, THEOREM 2.2]. *$Q = T$ if, for all $x_{I,h}$, $x_{I,h} \notin P^{(h)}$ implies that for some $1 \le j \le m$, $(Ax_{I,k})_j < b_j^{(k)}$ for all $1 \le k \le t$.*

Example 1. We let $n = t = 2$, $m = 4$ and take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad b^{(1)} = \begin{pmatrix} 4 \\ 4 \\ 10 \\ 5 \end{pmatrix} \qquad b^{(2)} = \begin{pmatrix} 4 \\ 4 \\ 5 \\ 10 \end{pmatrix}$$

It is easy to see that for both right-hand sides, the inequalities from the last two rows are redundant and that $Q = T = \{x | x_i \ge 4\}$. However, Theorem 1 cannot be used. When I consists of the bottom two rows of A, $x_{I,1} = (5,0) \notin P^{(1)}$ because row 2 of A is violated, and this is the only violated row. However, $x_{I,2} = (0,5)$ satisfies row 2 so the conditions of Theorem 1 are not satisfied.

This example suggests that the important thing is that when $x_{I,h} \notin P^{(h)}$ for some h, there must be a reason why $x_{I,k} \notin P^{(k)}$ for all k, but the reason (i. e., the violated row) may be different for different k. In our example, $x_{I,2}$ violates row 1 instead of row 2.

THEOREM 2. *$Q = T$ if, for all $x_{I,h}$, $x_{I,h} \notin P^{(h)}$ implies $x_{I,k} \notin P^{(k)}$ for all $1 \le k \le t$.*

Another way to interpret Theorem 2 is that for all h, the set of I which give extreme points of $P^{(h)}$ must be the same— the $P^{(h)}$ must all have the same shape. Since Theorem 2 follows easily from Theorem 3, we do not give a separate proof.

To develop a necessary and sufficient condition for $Q = T$, it helps to consider two examples in which the condition of Theorem 2 does not hold, with $Q = T$ in one case, $Q \ne T$ in the other.

Example 2. We let $n = t = 2$, $m = 3$ and take

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} \qquad b^{(1)} = \begin{pmatrix} 14 \\ 10 \\ 2 \end{pmatrix} \qquad b^{(2)} = \begin{pmatrix} 18 \\ 11 \\ 6 \end{pmatrix}$$

Example 3. Same as Example 2, except

$$b^{(2)} = \begin{pmatrix} 15 \\ 7 \\ 3 \end{pmatrix}$$

In both examples, when we let I be the first and third rows of A, $x_{I,1} = (6,8) \notin P^{(1)}$, but $x_{I,2} \in P^{(2)}$, so Theorem 2 cannot be used. However, it is easy to show in Example 2 that $Q = T = P^{(1)}$, but that in Example 3, $(6, 8.5) \in T \setminus Q$. (to see that $(6, 8.5) \in T$, let $\lambda = (.5, .5)$)

The crucial distinction between the two examples is that in Example 3, the "problem vector" $x_{I,2} = (6,9)$ was an extreme point of Q, while in Example 2 it was not.

THEOREM 3. *If $Q \neq T$, there is $c \in R^n$, I, h, j with (i) $x_{I,h} \in P^{(h)}$, (ii) $cx_{I,h} = M$, (iii) $cx_{I,j} > M$, where $M = \max\{cx | x \in Q\}$.*

Note that (ii) and (iii) imply $x_{I,j} \notin Q$, hence $x_{I,j} \notin P^{(j)}$.

PROOF: Let $y \in T \setminus Q$. There is $c \in R^n$ with $cy > \max\{cx|x \in Q\} = M$. The maximum of cx over Q is obtained by finding, for each h, the maximum of cx over $P^{(h)}$. Standard linear programming results (with the assumption that A is of rank n) imply that there is I (consisting of n rows), h such that $cx_{I,h} = M = \max\{cx|x \in F_{I,h}\}$. For any j, $\max\{cx|x \in P^{(j)}\} \leq \max\{cx|x \in F_{I,j}\} = cx_{I,j}$. Since $y \in T$, there is λ with $Ay \geq \sum \lambda_j b^{(j)}$. For those $\lambda_j > 0$, let $y^{(j)}$ be the solution to:

$$\left(Ay^{(j)}\right)_i = b_i^{(j)} + \frac{1}{\lambda_j}\left(Ay - \sum_{j=1}^{t} \lambda_j b^{(j)}\right)_i \quad \text{for all } i \in I$$

By considering $(Ay)_i$, it can be shown that $y = \sum \lambda_j y^{(j)}$. Since $cy > M$, $cy^{(j)} > M$ for some j. Since $y^{(j)} \in F_{I,j}$, $cx_{I,j} > M$. Q. E. D.

Thus the nonexistence of c, I, h, j satisfying (i)–(iii) is a sufficient condition for $Q = T$. It is not necessary in some special cases.

Example 4. We let $n = t = 2$, $m = 3$ and take

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{pmatrix} \qquad b^{(1)} = \begin{pmatrix} 4 \\ 8 \\ 4 \end{pmatrix} \qquad b^{(2)} = \begin{pmatrix} 4 \\ 6 \\ 4 \end{pmatrix}$$

It is easy to see that $Q = T = P^{(1)} = P^{(2)}$, but if I is the second and third rows of A, $x_{I,1} = (4,4)$ $x_{I,2} = (3,4)$ can be used to satisfy conditions (i)–(iii) of Theorem 3. To avoid this type of pathology we make a nondegeneracy assumption.

THEOREM 4. *Assume that, whenever $x_{I,h}$, $x_{J,h}$ are both defined, that they are equal only if $I = J$. Then the existence of I, c, h, j satisfying (i)–(iii) of Theorem 3 implies $Q \neq T$.*

PROOF: Our assumption implies that the system $Ax_{I,h} \geq b^{(h)}$ has all rows other than those corresponding to I as strict inequalities. This implies that the solution to the system

$$(Ax)_i = \left((1-\epsilon)b^{(h)} + \epsilon b^{(j)}\right)_i \quad \text{for all } i \in I$$

will be a member of T for small positive ϵ. But such solutions will have $cx > M$, hence not be members of Q. Q. E. D.

THEOREM 5. *The problem of deciding whether $Q = T$ is NP-Hard.*

PROOF: Given natural numbers n_i, N we construct A, $b^{(h)}$ so that $Q \neq T$ if and only if there is some subset of the n_i which adds up to exactly N (this is the knapsack problem, which is NP-Hard). Our problem will have $t = 2$ and one x_i for each n_i. The inequalities defining $P^{(1)}$, $P^{(2)}$ are:

$$
\begin{aligned}
-\sum n_i x_i &\geq -N & \qquad -\sum n_i x_i &\geq -N + \epsilon \\
-x_i &\geq -1 + \epsilon^2 & -x_i &\geq -1 - \epsilon \\
x_i &\geq 0 & x_i &\geq 0
\end{aligned}
$$

where $\epsilon > 0$ is chosen so that $\epsilon(1 + \sum n_i) < 1$.

If there is no subset S which adds up to exactly N then for any S

$$\sum_{i \in S} n_i(1 - \epsilon^2) \leq N \text{ iff } \sum_{i \in S} n_i \leq N - 1 \text{ iff } \sum_{i \in S} n_i(1 + \epsilon) \leq (N - 1) + (1 - \epsilon)$$

Thus the extreme points of $P^{(1)}$, $P^{(2)}$ are the same and Theorem 2 implies that $Q = T$.

If there is S whose members add up to N, then by letting $\lambda = (1 - .5\epsilon, .5\epsilon)$, we can show that $y \in T$, where $y_i = 1 - .5\epsilon^2$ for all $i \in S$, all other components 0. Since

$$\sum_{i \in S} n_i x_i \leq (1 - \epsilon^2)N \text{ for } x \in P^{(1)}, \sum_{i \in S} n_i x_i \leq N - \epsilon \text{ for } x \in P^{(2)}, \sum_{i \in S} n_i y_i = (1 - .5\epsilon^2)N$$

$y \notin Q$, hence $Q \neq T$. Q. E. D.

References

1. R. Jeroslow, "A simplification for some disjunctive formulations," *European Journal of Operations Research*, to appear.