It's Not Whom You Know, It's What You, or Your Friends, Can Do: Coalitional Frameworks for Network Centralities.

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ABSTRACT

We investigate the representation of game-theoretic measures of network centrality using a framework that blends a social network representation with the formalism of cooperative skill games. We discuss the expressiveness of the new framework and highlight some of its advantages, including a fixed-parameter tractability result for computing centrality measures under such representations. As an application we introduce new network centrality measures that capture the extent to which neighbors of a certain node can help it complete relevant tasks.

KEYWORDS

Coalitional skill games; centrality measures; helping

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1 INTRODUCTION

Measures of network centrality have a long and rich history in the social sciences [7] and Artificial Intelligence. Such measures have proved useful for a variety of tasks, such as identifying spreading nodes [32] and gatekeepers for information dissemination [28], advertising in multiagent markets [23], marketing [15], finding important nodes in terrorist networks [22, 25]. Recent work has demonstrated that the use of *(coalitional) game-theoretic* versions of centrality measures is especially beneficial [34, 35], and has motivated the study of other topics, such as the extension of centrality measures to more realistic settings [36], or the study of (frontiers of) tractability of such measures [1, 33, 37].

The starting point of this paper is the observation that, centrality measures are often regarded (e.g. [18]) as *exclusively graph theoretic properties*, the informal "motivating stories" behind many centrality measures involve the capability of "central" nodes to perform certain useful actions, a feature *not* explicitly encoded by a graph theoretic framework. For example: - **degree centrality** quantifies the capability of a given node to *disseminate information*; - **betweenness centrality** attempts to measure the propensity of the node to *control information flow* in the network by possibly disrupting a large percentage of the flows along shortest paths.

For another (famous) example: Granovetter's celebrated paper on the strength of weak ties [14] considers edges adjacent to a given node by their frequency of interaction. It argues that so-called *weak ties* (i.e. to those agents only interacting with the given node occasionally) are especially important. Such nodes *may be capable* to **tell v about a certain job j, that v itself does not know about.** The bolded statement may be seen, of course, as specifying a task $tell[j]$, that weak tie neighbors of v may be able to complete as a consequence of their network position.

The purpose of this paper is **to propose and study representational frameworks for network centrality that explicitly take into account the acting capabilities of various nodes.** We follow [30] in advocating the study of network centrality measures from a coalitional game-theoretic perspective. Our concerns are somewhat different: whereas [30] mostly investigated representations of centrality measures from an axiomatic perspective, we investigate the use of *(succinct) coalitional representation frameworks* [6, 9, 11, 13, 16] for such representations. The particular setting we consider blends a network-based specification $G = (V, E)$ of the agent system with one such framework, the so-called *coalitional skill games* (CSG) [6]. Informally, agents are endowed with *skills* that may prove instrumental in completing certain *tasks*, and in getting profit from completing them. A centrality measure arises as an indicator (often the Shapley value) of the "importance" of the agent in the associated coalitional game, measuring the extent to which the agent helps coalitions profit from completing tasks. A significant motivation for choosing CSG over alternative frameworks is that models similar to CSG are informally used in the Knowledge Discovery literature in the context of the *team formation* problem [20, 21]. This opens the prospect of applying such game-theoretic centrality measures to problems in this direction.

Through our paper we aim to start offering answers to the following questions:

1. *Are CSG representations of centrality "universal" ?* That is, can all centrality measures be ("naturally") represented as the Shapley value of some CSG ? This is a natural question, given that all monotone coalitional games can be simulated by CSG games [6], while all centrality measures can be represented by a (monotone) coalitional game [30].
2. *Are there representational advantages to using such frameworks ?* I.e., are there natural examples of social phenomena and centrality measures inspired by the concepts of skills/tasks from CSG representation ?
3. *Are there computational advantages of such frameworks ?*

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The following is an outline of our answers to the previous questions (and of the main results of the paper):

1) We show that question (1) has a somewhat subtle answer: on one hand we prove (Theorem 3.1) that our representations are universal: *all* centrality measures have an equivalent CSG representation. However, if we insist on "natural" representations the answer is negative: eigenvector centrality cannot be represented in this way. We prove this result by identifying a useful class of centrality measures (those represented as a rational function of vertex/edge indicator functions). We show that this class is quite robust, containing for example degree and betweenness centrality, and subsuming all "natural" CSG representations. However we show (Theorem 3.14) that eigenvector centrality is not rational.

2) We show that the answer to the second question is affirmative: the social phenomenon we showcase is *helping*, i.e. the capability of agents to help completing tasks not by their skills but by *enlisting their neighbors* for this purpose. The importance of this phenomenon has been previously recognized in the literature on social networks: For instance, authors of [10] write: "Given a person with a set of skills and a neighborhood of friends in a social network, the skills of the friends to some extent are accessible through that person, and therefore they should be considered when evaluating her..." However, the results in [10] go in a different directions than ours. We define two associated *helping centrality* measures. These measures are related to some important concepts such as the original game-theoretic network centrality ν_1 [24, 32]. We compute these measures in a setting inspired by [25], that of a 9/11 terrorist network. We then show that our measures, though intractable in general, have efficient explicit formulas for some special CSGs.

3) Finally we highlight a computational benefit of CSG representations, in the form of fixed parameter tractability results for centrality measures induced representable as the Shapley value of a CSG (Thm. 4.1) and for our new centrality measures (Thm. 6.2).

The paper concludes with brief discussions and open problems.

2 PRELIMINARIES AND NOTATIONS

We will use (and review below) notions from several areas:

Probabilistic Methods. We will assume familiarity with discrete probability at the level provided, e.g. by [26]. In particular for a discrete integral random variable X we will denote by $\mathbb{E}[X]$ its expected value, equal to $\sum_i i \cdot Pr[X = i]$.

Theory of multisets. A *multiset* is a generalization of a set in which each element appears with a non-negative multiplicity. The *union of two multisets* A, B, also denoted $A \cup B$, contains those elements that appear in A, B, or both. The multiplicity of such an element in $A \cup B$ is the sum of multiplicities of the element into A, B. Given multisets A, B, we write $A \subset B$ iff every element with positive multiplicity in A has at least as high a multiplicity in B.

Coalitional game theory. We assume familiarity with the basics of Coalitional Game Theory (see [8] for a recent readable introduction). For concreteness we review some definitions:

A *coalitional game* is specified by a pair $\Gamma = (N, v)$ where $N = \{1, 2, \ldots, n\}$ is a set of *players*, and v is a function $v : 2^N \to \mathbb{R}$, called *characteristic function*, which satisfies $v(\emptyset) = 0$. We will often specify a game by the characteristic function only (since N is implicitly assumed in its definition). Also, denote by $\Gamma(N)$ the set of all coalitional games on N. Given integer $0 \leq k \leq n$, we denote by $C_k(\Gamma)$ the set of coalitions of Γ (i.e. subsets of N) having cardinality exactly k, and let $C(\Gamma)$ be the union of all sets $C_k(\Gamma)$. A game is *monotonically increasing* if v is monotonically increasing with respect to inclusion.

We can represent any game on set N as a linear combination of *veto games*: given $\emptyset \neq S \subseteq N$, the *S-veto game on N* is the game with characteristic function

$$v_S(T) = \begin{cases} 1, & \text{if } S \subset T, \\ 0, & \text{if otherwise.} \end{cases}$$

Indeed, it is well-known [17] (and easy to prove) that the set of veto games (v_S) forms a basis for the linear space of coalitional games on N. Coefficients a_S in the decomposition $v = \sum_S a_S v_S$ are called *Harsányi's dividends* and the decomposition itself is called *veto (game) decomposition*.

We will use solution concepts associated to coalitional games, notably the *Shapley value*. This index tallies the fraction of the value $v(N)$ of the grand coalition that a given player $x \in N$ could fairly request. It has the formula [8]

$$Sh[v](x) = \frac{1}{n!} \cdot \sum_{\pi \in S_n} [v(S_\pi^x \cup \{x\}) - v(S_\pi^x)], \tag{1}$$

where $S_\pi^x = \{\pi[i] | \pi[i] \text{ precedes x in } \pi\}$. On the other hand, if $v = \sum_S a_S v_S$ is the veto game decomposition of v then for every $i \in N$ we have

$$Sh[v](i) = \sum_{i \in S} \frac{a_S}{|S|} \tag{2}$$

We also need to review several particular classes of coalitional games. A *(weighted) dummy game* is a triple $\Gamma = (N, w, v)$ where $w : N \to [0, \infty)$ is a *weight function* and the characteristic function has the form $v(S) = \sum_{i \in S} w(i)$.

Definition 2.1. A *coalitional skill game* (CSG) [5, 6] is a 4-tuple $\Gamma = (N, Sk, T, v)$, where N is a set of *players*, Sk is a set of *skills*, T is a set of *tasks*, and v is a characteristic function. We assume that each player $x \in N$ is endowed with a set of skills $Sk_x \subseteq Sk$. We extend this notation from players to coalitions by denoting, for every $S \subset N$, $Sk_S := \cup_{x \in S} Sk_x$. We will denote by $P(s)$ the set of players having a certain skill s. On the other hand, each task $t \in T$ is identified with a set of skills $T_t \subseteq Sk$, the set of skills needed to complete task T_t. Finally, each task T_t has a *profit* $w_t \geq 0$. The *value of a coalition* $S \subset N$ is defined as

$$v(S) = \sum_{t \in T : T_t \subseteq Sk_S} w_t.$$

In other words: the value of a coalition S is the sum of profits of all tasks that require only skills possessed by members of S.

We will actually slightly extend the framework from [5, 6] by requiring that **tasks are multisets** (rather than sets) **of skills.** Skillsets are still required to be ordinary sets, but the condition $T_t \subseteq Sk_S$ is now considered as a multiset inclusion. A justification for this extension is given by the following example:

Example 2.2. We build upon a scenario from [25] based on the 9/11 terrorist network initially reconstructed in [19]. In addition to ordinary nodes (displayed as white circles), some nodes are endowed with one of two skills: M ("martial arts", displayed as yellow squares), P ("pilot", displayed as grey diamonds) (Figure 1). A coalition of nodes could execute a hijacking attack iff it contains at least two agents with capability M and one agent with capability P. This description maps easily onto an (extended) CSG with a single task ("hijacking") specified as the multiset of skills $\{M, M, P\}$, and profit (for the attackers) equal to one. For ease of exposition/computation we leave out from the specification of our example a condition which was crucial in [25], that an attacking coalition be connected, i.e. that the CSG game is a connectivity game [3] or a Myerson game [27]. With additional technical complications along the lines of [31] one can probably incorporate this condition into our example as well.

Semivalues [12] generalize the well-known concepts of Shapley and Banzhaf index. Given coalitional game Γ and $C \in C(\Gamma)$, denote by $MC(C, i) := v(C \cup \{i\}) - v(C)$ the *marginal contribution* of player i to coalition C. Consider a function $\beta : \{0, \ldots, n-1\} \to [0, 1]$ satisfying $\sum_{i=0}^{n-1} \beta(k) = 1$. Given semivalue β, the semivalue $\phi_i(v)$ for player i in cooperative game v is $\phi_i(v) = \sum_{k=0}^{n-1} \beta(k) \cdot \mathbb{E}_{C \in C_k}[MC(C, i)]$. For $\beta^{Sh}(k) = 1/n$ we recover the Shapley value. Another important case is the trivial semivalue $\beta^{triv}(0) = 1$, $\beta^{triv}(i) = 0$ otherwise. Finally, family of semivalues $\beta = (\beta_n)$ is called *polynomial time computable* if the two-argument function $(n, k) \to (\beta_n)_k$ has this complexity.

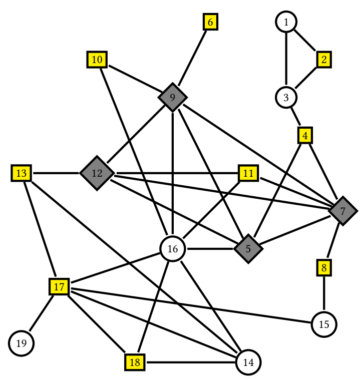

Figure 1: The 9/11 WTC attack social nework (after [19], with skills assigned by [25]).

We note the following very simple result:

Lemma 2.3. *If $\Gamma = (N, w, v)$ is a weighted dummy game and β is a semivalue then for every $i \in N$, $\phi_i(v) = w(i)$.*

Graph Theory and Network Centralities. A *graph* is a pair $G = (V, E)$ with V a set of *vertices* and E a set of *edges*. The degree of v, $deg(v)$ is the number of nodes v is connected to by edges. We will use Δ to denote the maximum degree of a node in V. If $v \in V$ is a vertex we will denote by $N(v)$ the set of neighbors of v in G and by $\widehat{N(v)} = N(v) \cup \{v\}$. We extend these definitions to sets $S \subseteq V$ by $N(S) = \{z \mid \exists w \in S, (z, w) \in E\}$.

We will denote by $\mathcal{G}^V$ the set of all graphs on the vertex set V. A *centrality index* is a function, $c : \mathcal{G}^V \to \mathbb{R}^V$ that assigns to every node $v \in V$ a real number, called *the centrality of* v quantifying the importance of node v in G. We will denote by C^V the set of all centrality measures on the set V. We will usually drop V from our notation and write $\mathcal{G}, C, \ldots$ instead of $\mathcal{G}^V, C^V$ and so forth.

We need several concrete measures of centrality. The following is a listing of some of them:

- *degree centrality* of node v in graph G is defined by $c^D(v, G) = |\{(v, u) \in E | u \in E\}|$.
- *betweenness centrality* of node v in graph G is defined as follows: given two distinct nodes $z_1, z_2 \in V$, denote by $p(z_1, z_2)$ the number of shortest paths in G between z_1 and z_2, and by $p(z_1, z_2, v)$ the number of shortest paths *passing through* v. Now we can define betweenness centrality as $c^{close}(v, G) = \sum_{z_1 \neq z_2 \in V} \frac{p(z_1, z_2, v)}{p(z_1, z_2)}$.
- *game-theoretic network centrality* v_1 of node x in graph G is defined as the Shapley value of x in game Γ with characteristic function $v_*(S) = |S \cup N(S)|$. This measure is explicitly computed in [24], but versions of it were considered earlier e.g. in [32, 38, 39].
- the *eigenvector centrality* of node v in graph G is defined as the v'th component of the eigenvector associated to the largest eigenvalue of the adjacency matrix of G.

Coalitional Network Centralities. Following [30], a *representation function* is a function ψ mapping every graph $G = (V, E)$ onto a cooperative game Γ_G whose players are the vertices of G, $\Gamma_G = (V, v_G)$. We will call a representation w *skill-based* if for every graph G, the associated game Γ_G is a CSG. A *coalitional centrality measure* is a pair (ψ, ϕ), where ψ is a representing function and ϕ is a solution concept. A *skill-based centrality measure* is one for which representation ψ is skill-based. Given semivalue β, a *β-skill-based centrality measure* is a pair (ψ, ϕ) where ψ is a skill-based and ϕ is the semivalue induced by β. A skill-based centrality measure is *trivial* iff the solution concept ϕ is simply the value function of the CSG associated to graph G by ϕ. Note that for weighted dummy games this is equivalent to requiring that ϕ is the semivalue induced by the trivial semivalue β^{triv}.

Parameterized Complexity. A *parameterized problem* is specified by a set of pairs $W \subseteq \Sigma^* \times \mathbb{N}$ and a function $f : A \to \mathbb{N}$. Problem (W, f) is *fixed-parameter tractable* if there exists a computable function $g : \mathbb{N} \to \mathbb{N}$, an integer $r > 0$ and an algorithm A that computes $f(z)$ on inputs $z = (x, k)$ from W in time $O(g(k) \cdot |x|^r)$.

3 UNIVERSALITY OF SKILL-BASED CENTRALITIES.

[30] have shown that any centrality measure is equivalent to a coalitional centrality measure. We make this result slightly more precise: the cooperative game can be taken to be a CSG and the solution concept can be induced by any arbitrary semivalue:

Theorem 3.1. *For every semivalue β and every centrality $c \in C$ there exists an equivalent β-skill-based representation.*

Proof. Let $c \in C$ be a centrality measure on graph G. Consider the *dummy game* in which $v(S) = \sum_{v \in S} c(v)$.

One can represent this dummy game by associating to G the CSG game Γ as follows: $Sk = V$, i.e. skills correspond to agents. For every $v \in V$ define $w(v) = c(v)$. Finally $T = V$. This yields a skill-based representation ψ_C. Completing this representation by the trivial semivalue induced by β in Γ yields a β-skill-based centrality measure which (by Lemma 2.3) is equivalent to c. □

Example 3.2. **[Degree centrality:]** Consider a graph $G = (V, E)$. We associate to G a game Γ as follows: skills correspond to vertices of G. Tasks correspond to edges of G, seen as pair of vertices/skills. Each task has profit 2.

Then the Shapley value of each node is equal to its degree (centrality). Indeed, Γ is essentially a *induced subgraph game* [11], whose Shapley value divides the profit of each edge equally between its endpoints.

Sometimes, as the following example shows, the "natural" representation of centralities using CSG is non-effective, as it requires exponential size games:

Example 3.3. **[Betweenness centrality]:** Consider a graph $G = (V, E)$. Associate to G a game Γ as follows:

- skills correspond to nodes of G.
- Tasks, seen as the set of the corresponding skills/nodes, correspond to shortest paths connecting two nodes in G.
- A task corresponding to a shortest path P between two nodes, say z_1, z_2, in G has a weight equal to the number of nodes on P divided by the number of shortest paths between z_1, z_2.

The Shapley value divides the payoff of each task among the nodes of the path. Hence the Shapley value of each node is nothing but its betweenness centrality.

3.1 Limitations of (Rational) Centralities.

In the proof of Theorem 3.1 we were, in some sense, "cheating", as the values of network centrality were "pre-computed" and built-in as the weights of the dummy game representing the measure. In particular the game depended in a complicated manner on the structure of the graph G. It is natural to ask whether this universality result fails to hold once we impose some further restrictions on the framework that preclude such "pathological" representations.

In this section we show that the answer is yes: an important centrality measure, eigenvector centrality, cannot be represented as the Shapley value of a CSG with rational payoffs.

This result is proved by first identifying an interesting and natural restriction on characteristic functions and centrality measures that formalizes the idea of "simple dependence (of these functions) on the structure of the graph": we will require that they are "rational functions of the graph topology", i.e. a quotient of two polynomials with rational coefficients. We show that this class is large enough to include all Shapley values of CSG with constant, rational, coefficients. Then we show that eigenvector centrality cannot be computed by rational functions.

We formalize the idea of "rational functions of the graph topology" as follows: Given a set of vertices V, denote by $E_n(V)$ the set of subsets $w = \{v_1, v_2\}$ of distinct vertices in V. Associate to every $w \in E_n(V)$ a boolean variable X_w. We can interpret the set E of edges of any graph G on V as a 0/1 assignment $(X_w)_{w\in E_n(V)} \in \{0, 1\}^{E_n(V)}$, $X_w = 1$ if $w \in E$, $X_w = 0$, otherwise. By forcing notation we will write E instead of $(X_w)_{w\in E_n(V)} \in \{0, 1\}^{E_n(V)}$. We do similarly for vertices, identifying a vertex i with a boolean variable Y_v. This way we can specify a set of vertices S by a boolean vector of size n, with ones corresponding to those vertices v with $Y_v = 1$.

Definition 3.4. Family of characteristic functions $(v_n)_{n\geq 1}$ is called `rational` if there exists two families of polynomials $P_n(X_e, Y_v)$ and $Q_n(X_e, Y_v) \in \mathbb{Q}[X, Y]$ such that for every $n \geq 1$ and $S \subseteq [n]$

$$v_n(S) = \frac{P_n[E, S]}{Q_n[E, S]} \quad (3)$$

Example 3.5. For every $S \subseteq [n]$ characteristic functions v_S are rational. Indeed $v_S(\cdot) = \prod_{i\in S} Y_i$.

Example 3.6. Characteristic function v_* in the definition of game-theoretic network centrality ν_1 is rational. Indeed,

$$v_*(S) = \sum_{i\in[n]} Y_i + \sum_{i\in[n]} (1 - Y_i)[1 - \prod_{j\neq i}(1 - Y_j X_{i,j})]$$

To see that this equality is true: each $i \in S$ contributes 1 to the first sum. Only $i \notin S$ may contribute to the second sum, but only when some term $1 - Y_i X_{i,j}$ is equal to zero, that is when there is some $j \neq i$ with $Y_j = 1$ (i.e. $j \in S$) and $X_{i,j} = 1$ (i.e. $(i, j) \in E$).

Definition 3.7. Let V be a set of vertices. A centrality measure $c = (c_n)_{n\geq 1}$ is called `rational` iff there exist multivariate polynomials $P_{n,v}, Q_n \in \mathbb{Q}[X]$ such that, for every $n \geq 1$, and every graph $G = (V, E)$ on V we have

$$c_n(v, G) = \frac{P_{n,v}[E]}{Q_n[E]} \quad (4)$$

Example 3.8. Degree centrality is rational. Indeed, one may take $P_{n,v}[X] = \sum_{e\ni v} X_e$ and $Q_n[X] = 1$.

The case of betweenness centrality is more interesting:

THEOREM 3.9. *Betweenness centrality is rational.*

PROOF. Given vertices $z_1 \neq z_2$, define $\mathcal{D}_{z_1,z_2}$ to be the family of simple paths from z_1 to z_2 in G. Given $P \in \mathcal{D}_{z_1,z_2}$, define monomials

$$X_P = \prod_{e\in P} X_e$$

and

$$\tilde{X}_P = X_P \cdot \prod_{\substack{Q\in\mathcal{D}_{z_1,z_2} \\ l(Q)<l(P)}} (1 - X_Q).$$

Also, for vertices $z_1 \neq z_2$ define $P_{n,v,z_1,z_2}[X] = \sum_{P:z_1\to v\to z_2} \tilde{X}_P$, $Q_{z_1,z_2}[X] = \sum_{P:z_1\to z_2} \tilde{X}_P$. With these notations, we claim that we have the following formula

$$BC_n[v] = \sum_{v_1\neq v_2\in V} \frac{P_{n,v,z_1,z_2}[X]}{Q_{z_1,z_2}[X]} \quad (5)$$

To prove equation (5) we first show that

CLAIM 1. *Given graph $G = (V, E)$, $\tilde{X}_P = 1$ iff P is a shortest path in G between its extremities z_1, z_2.*

PROOF. $\tilde{X}_P = 1$ iff $X_P = 1$ and for all $Q \in \mathcal{D}_{z_1,z_2}$, $l(Q) < l(P)$, $X_Q = 0$, that is Q is **not** a path in G. □

Applying Claim 1 we infer that P_{n,v,z_1,z_2} count shortest paths between z_1, z_2 passing through v and Q_{z_1,z_2} counts all shortest paths between z_1, z_2. □

The following two theorems show that the family of rational centrality measures is reasonably comprehensive:

Theorem 3.10. *Every centrality measure induced by a rational family of characteristic functions is rational.*

Proof. From the marginal contribution formula for the Shapley value. □

Corollary 3.11. *Game-theoretic network centrality v_1 [24] is a rational centrality measure.*

Corollary 3.12. *If v_n is a family of characteristic functions whose Harsanyi dividends are rational numbers then the family of centrality measures induced by v_n is rational.*

Proof. From Exp. 3.5 and the fact that linear combinations of rational functions with coefficients in $\mathbb{Q}$ are rational. □

Theorem 3.13. *Every centrality measure induced by a family of CSG with constant coefficients in $\mathbb{Q}$ is rational.*

Proof. We will use the decomposition of coalitional skill games by linearity to reduce the problem to reasoning about *simple skill games*, i.e. CSG consisting of a single task t of unit profit. In this setting a coalition S is called *winning* if it can accomplish task t and *losing* otherwise. Define by $W(t)$ the set of minimal winning coalitions for task t, i.e. of subsets A such that $v(A) = 1$ and $v(B) = 0$ for any strict subset B of A.

Then the polynomial $1 - \prod_{A \in W(t)}(1 - \prod_{r \in A} Y_r)$ is equal to 1 precisely when for some $A \in W(t)$ we have $Y_r = 1$ for all $r \in A$. Hence, denoting $S = \{r : Y_r = 1\}$, we have

$$v(S) = 1 - \prod_{A \in W(t)} (1 - \prod_{r \in A} Y_r).$$

□

Theorem 3.13 connects rationality to the representation of centrality by CSG, essentially showing that when we disallow adaptive representations like those used in the proof of Theorem 3.1 induced centrality measures are indeed rational.

Despite these two results, rational measures are **not** universal; they fail to capture a natural centrality measure:

Theorem 3.14. *Eigenvector centrality is not rational, hence not representable by CSG with constant, rational coefficients.*

Proof. Consider the graph G from Figure 2 (a). Simple computations show that eigenvector centrality values of node 1, 3 are equal to $\frac{1}{\sqrt{2}+1}$ and that of 2 is equal to $\frac{\sqrt{2}}{\sqrt{2}+1}$. In particular the eigenvector centrality of 1, 3 is an irrational number. But this would be impossible if the eigenvector centrality were a rational centrality measure. □

It would be interesting to define an extension of the family of rational centralities that captures all "natural" centralities. Another problem is to characterize (perhaps axiomatically) all centrality measures that have CSG representations of polynomial size.

4 PARAMETERIZED COMPLEXITY OF SKILL-BASED CENTRALITIES.

Since computing the Shapley value of CSG is #P-complete [4], it follows that computing skill-based centralities is intractable in general. On the other hand, by imposing a natural restriction on the family of CSG games under consideration, that of an existence of an upper bound on the largest set of skills needed for a task, we get a fixed-parameter tractable class of algorithms:

Theorem 4.1. *Let β be a poly-time computable family of semivalues. The following problem*

- *[INPUT:] A CSG $\Gamma = (N, v)$ and a player $i \in N$.*
- *[TO COMPUTE:] Semivalue $\Phi_i^{\beta}(v)$.*

parameterized by k, the cardinality of the largest skill set required by any task, is fixed parameter tractable.

Proof. We will again use the decomposition of CSG by linearity to reduce the problem to reasoning about *simple skill games.*

Clearly Φ_i^{β} can be computed by estimating, for $k = 0, \ldots, n-1$ quantities $\mathbb{E}_{C \in C_k}[MC(C, i)]$. Nonzero marginal contributions arise from ordered coalitions $C \in C_k$ such that (1). i is the last element of C. (2). C is winning. (3). $C \setminus \{i\}$ is **not** winning. For every task T_j (identified with a multiset of skills) denote by $\mathcal{T}_j^*$ the set of submultisets of T_j. For $T \in \mathcal{T}_j^*$ and $1 \leq r \leq n-1$ denote by $n_{T,r}^i$ the number of *ordered coalitions X of size exactly r and not containing i* such that $S_X \cap T_j = T$. Denote $W_{i,j} = C_i \cap T_j$ the set of skills of agent i that can contribute towards completing task j. An ordered coalition satisfies properties (1)-(3) above **if and only if** $T_j \setminus W_{i,j} \subset S_{C \setminus \{i\}} \subsetneq T_j$. Thus

$$\Phi_i^{\beta} = \sum_{r=0}^{n-1} \beta(r) \Big[\sum_{T_j \setminus W_{i,j} \subseteq T \subsetneq T_j} \frac{(n-r-1)!}{n!} \cdot n_{T,r}^i \Big]. \quad (6)$$

We use a dynamic programming approach to compute parameters $n_{T,r}^i$. The table has at most 2^k columns, each corresponding to a submultiset of T_j. Rows of the table correspond to pairs (r, s), where $r \leq s \leq n$. The element on row (r, s) and T, denoted by $n_{T,r,s}^i$, counts ordered coalitions X of size r not containing element i, formed with elements from $a_1, a_2, \ldots, a_s$, such that $S_X \cap T_j = T$. Clearly $n_{T,r}^i = n_{T,r,n}^i$.

We start the table by filling in rows $(r, s) = (0, 0)$ and $(r, s) = (1, 1)$. Clearly, $n_{T,0,k}^i = 0$ for all $k \geq 1$ and $T \subseteq T_j$, and $n_{T,1,1}^i$ equals the number of players $k \neq i$ such that $S_k \cap T_j = T$. Thus rows $(0, 0)$ and $(1, 1)$ can be completed by simple player inspection.

Coalitions X of size r not containing element i with elements from the set $a_1, a_2, \ldots a_s$ such that $S_X \cap T_j = T$ decompose into two types:

- Coalitions *not containing* a_s. $n_{T,r,s-1}^i$ counts them.
- Coalitions *containing* a_s. Let $Y = X \setminus a_s$, $D = S_Y \cap T_j$, $D \subseteq T$,$E = S_{a_s} \cap T_j$. It follows that $D \cup E = T$. Thus we get recurrence $n_{T,r,s}^i = n_{T,r,s-1}^i + \sum_{D \cup (S_s \cap T_j) = T} n_{D,r-1,s-1}^i$. This equation allows us to fill row (r, s) from rows $(r, s-1)$ and $(r-1, s-1)$, ultimately allowing to compute $n_{T,r}^i$ for all i.

It is easily seen that the complexity of the provided algorithm is $O(2^k \cdot poly(|\Gamma|))$. Thus the problem is fixed parameter tractable. □

5 MEASURES OF HELPING CENTRALITY.

In this section we give an application inspired by the idea of representing network centralities by CSG. [2] have recently defined (using different ideas) a centrality measure that quantifies the extent to which a given agent adds value to a group. On the other hand, an agent may be valuable to a group even when it lacks the skills to contribute to completing a given task, provided it is capable to *help* by *enlisting neighbors with such skills.*

Example 5.1. Members of the PC of computer science conferences often use subreviewers to referee papers. Each paper needs to receive a minimal number (say three) of reviews. A PC member may lack the **skill** to competently review the paper itself. But the ability it may have to help the reviewing process by enlisting subreviewers with the required reviewing skills, in order to complete the **task** of getting three reviews for the given paper, is highly valuable.

Example 5.2. Consider again the coalitional game-theoretic framework for the WTC 9/11 terrorist network in Example 2.2. Nodes 4, 7, 9, 11, 12, 13, 16 could have assembled an attacking team consisting of (some of) their neighbors. In the case of node 16 (N. Alhazmi) this happens despite not being known to have had any of the two required skills P, M. Because of this fact, node 16 intuitively can always "help" forming attacking teams by enlisting its neighbors. This is intuitively, not true for nodes 4, 7, 9, 11, 12, 13 : they do not "help" those coalitions that were already turned into attacking teams by their mere joining. Neither do any other nodes in the network. So, intuitively, node 16 should be the "most helping" node.

It turns out that properly defining a centrality notion for helping is somewhat subtle and may not have a single, always best solution. A natural first idea is that such a measure may involve the Shapley value of a certain *helping extension* of the original game, in which the value of a coalition involves "getting help from all their neighbors":

Definition 5.3. Given coalitional game Γ on graph G, we define *the helping extension of* Γ as the game with value function $v_{help}(S) = v(S \cup N(S))$.

Example 5.4. Consider the weighted dummy game with $v(S) = \sum_{i \in S} w(i)$. Then for every graph G the Shapley value of the helping extension $v_{help}(S)$ of v on graph G has the formula (which trivially generalizes the one for *game-theoretic centrality* v_1 of G

$$Sh[v_{help}](z) = \frac{w(z)}{deg(z)+1} + \sum_{u \in N(z)} \frac{w(u)}{deg(u)+1} \tag{7}$$

To quantify helping we want to disentangle a given player's capability to *help solve tasks using its own skills* from its capability to get help from its neighbors. Equation (7) suggests that in the helping extension the contribution of each node's skills "diffuses" from the given node equally to itself and its neighbors. Indeed, in the absence of helping each node z would have a Shapley value of $w(z)$ (corresponding to its unique skill/task). In the presence of helping, node z only retains a fraction of $\frac{1}{deg(z)+1}$ of its original Shapley value (since its neighbors can also "help" by enlisting z) but also acquires a fraction of $\frac{1}{deg(u)+1}$ of the Shapley values of all neighbors u of z. The above argument seems to suggest that an

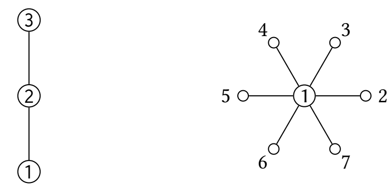

Figure 2: (a). Graph P_2 (b). Star graph S_7.

appropriate measure of the helping centrality of z would be the quantity

$$Sh[v_{help}](z) - \frac{1}{deg(z)+1} Sh[v](z). \tag{8}$$

which subtracts from $Sh[v_{help}]$ the quantity traceable to the original Shapley value of z, only retaining the "help" of its neighbors. Unfortunately, this heuristic is problematic: for some games v and players z the helping centrality of z would be (under this definition) negative !

Example 5.5. In the setting of Example 5.2 a computer program shows that the quantity in equation (8) is negative (equal to -0.012...) for $z = 2$.

To give a measure of helping centrality with nonnegative values we depart from the idea on basing it on a combination of the Shapley values $Sh[v_{help}]$ and $Sh[v]$. Instead, we give a definition close in spirit to the formula (1) for the Shapley value (and which directly extends it when G is the empty graph):

Definition 5.6. Given coalitional game Γ and graph G, we define *the Helping Shapley value of a player x* by

$$HSh[v](x) = \frac{1}{n!} \cdot \sum_{\pi \in S_n} [v(S_\pi^x \cup \{x\} \cup N(x)) - v(S_\pi^x)]. \tag{9}$$

The *helping centrality* of player x is defined as

$$Help(x) = HSh[v](x) - Sh[v](x) =$$
$$= \frac{1}{n!} \sum_{\pi \in S_n} [v(S_\pi^x \cup \{x\} \cup N(x)) - v(S_\pi^x \cup \{x\})]. \tag{10}$$

Note that if game Γ is monotonically increasing then $v(S_\pi^x \cup \{x\} \cup N(x)) - v(S_\pi^x \cup \{x\}) \geq 0$. When the sign is strictly positive say that *x helps ordered coalition* S_π^x.

Example 5.7. Let $G = P_2$ be the graph in Figure 2, and Γ be the *T-veto game* corresponding to coalition $T = \{1, 3\}$. That is $v_{\{1,3\}}(S) = \begin{cases} 1, & \text{if } \{1,3\} \subset S, \\ 0, & \text{if otherwise.} \end{cases}$ Considering the permutations in S_3 in the order $(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$, simple computations show that $HSh[v_{\{1,3\}}](1) = \frac{1}{6}(0+0+0+1+1+1) = \frac{1}{2}$, $HSh[v_{\{1,3\}}](2) = \frac{1}{6}(1+0+1+1+0+1) = \frac{2}{3}$, $HSh[v_{\{1,3\}}](3) = \frac{1}{6}(1+1+1+0+0+0) = \frac{1}{2}$. As for helping centralities, we have $Sh[v_{\{1,3\}}](1) = Sh[v_{\{1,3\}}](3) = \frac{1}{2}$ and $Sh[v_{\{1,3\}}](2) = 0$, so $Help(1) = Help(3) = 0$, and $Help(2) = 2/3$. Node 2 is the only one that has positive helping centrality, despite being a null player !

Observation 1. *A plausible objection to our formula (10) for the Helping Centrality is that elements of set S_π^x should **also** be allowed to enlist their neighbors. That is, the right hand side should be*

$$\frac{1}{n!}\sum_{\pi\in S_n}[v(S_\pi^x\cup\{x\}\cup N(S_\pi^x\cup\{x\}))-v(S_\pi^x\cup N(S_\pi^x)\cup\{x\})]. \quad (11)$$

There are multiple answers to this objection. We give two arguments, a principled one and a pragmatic one:

- First of all, our formula (10) implicitly assumes a very specific process of getting help, with three distinct phases:

1. *First, coalition S forms. It may not be able to solve task t, therefore*
2. *x joins S. $S\cup\{x\}$ may still not be able to solve task t, therefore*
3. *x enlists the elements of $N(x)$. Now the enlarged coalition $S\cup\{x\}\cup N(x)$ may be able to solve t.*

Only step 3 refers to the action taken by x that we want to quantify**, that of helping the coalition S. Otherwise, the assumptions on the actions of coalition S are minimal: we only assume that S forms, but not that it takes any further actions. By contrast, the process embodied by formula (11) assumes more about the actions taken by coalition S. Not only S forms, but that it is highly proactive in dealing with the prospect of not completing task t: it seeks to enlist the help of every neighbor it possibly can; such a course of action may be infeasible for a variety of reasons, e.g. because of communication costs that should be paid by S. **Our measure thus quantifies helping in a "minimal information scenario".

- On the other hand formula (11) has less "nice" mathematical properties. For instance there appears to be no closed-form formula similar to that of Theorem 5.8 for the alternate definition.

Measures HSh and $Help$ do not have easy interpretations in terms of classical notions of coalitional game theory. On the other hand, they have exact formulas somewhat reminiscent of the formula (2) for the Shapley value:

Theorem 5.8. *Given game $\Gamma=(N,v)$ with veto decomposition $v=\sum_S a_S v_S$, graph $G=(N,E)$, $i\in N$ and $S\subseteq N$ denote by $NC(i,S)$ the set of nodes in $S\setminus\widehat{N(i)}$ and by $Cov(i,S)=S\cap\widehat{N(i)}$. Then*

$$HSh[v](i)=\sum_{i\in S}\frac{a_S}{|NC(i,S)|+1}+\sum_{\substack{i\in N(S)\\ i\notin S}}\frac{a_S|Cov(i,S)|}{(|NC(i,S)|+1)(|S|+1)}$$

$$Help(i)=\sum_{i\in S}\frac{a_S(|Cov(i,S)|-1)}{|S|(|NC(i,S)|+1)}+\sum_{\substack{i\in N(S)\\ i\notin S}}\frac{a_S|Cov(i,S)|]}{(|NC(i,S)|+1)(|S|+1)}$$

Proof. By the linear decomposition of games as combination of veto games, we only need to compute the Helping Shapley value of i for the S-veto game v_S. Clearly, if $i\notin S\cup N(S)$, then $HSh[v_S](i)=0$. Indeed, in this case i can help no coalition T contain S, since $\widehat{N(i)}\cap S=\emptyset$). So we concentrate on the case $i\in S\cup N(S)$. There are two subcases:

- $i\in S$. Then i is pivotal iff all elements of $NC(i,S)$ appear before i in π. The probability of this happening is $\frac{1}{|NC(i,S)|+1}$.
- $i\in N(S)\setminus S$. Then i is pivotal iff all elements of $NC(i,S)$ appear before i in π and some element of $S\cap N(i)$ appears after i in π. That is

Node v	HSh(v)
7,9,12	0.3153
13	0.2940
16	0.2791
5	0.2042
4,6,8,10,11	0.1829
17,18	0.0940
1,3,14,15,19	0.0791
2	0.0598

Node v	$Help(v)$
16	0.2791
13	0.2341
7,9,12	0.2
4,6,8,10,11	0.1230
5	0.0888
3,14,15	0.0791
1,19	0.0449
17,18	0.0341
2	0

Node v	$Sh(v)$
5,7,9,12	0.1153
2,4,6,8,10,11,13,17,18	0.0598
1,3,14,15,16,19	0

Figure 3: (Unnormalized) values of (a). the Helping Shapley value HSh (b) Helping centrality. (c). Shapley value.

(a). **all elements of** $NC(i,S)$ **apppear before** i **in** π (an event which happens with probability $1/(|NC(i,S)|+1)$), but it is not the case that

(b). i **appears after all elements of** S **in** π. The probability of (b) happening is $1/(|S|+1)$.

To derive the second formula we simply use formula (1). □

5.1 Helping Centralities in Terrorist Networks

We next highlight our helping centrality measure to the terrorist network in Figure 1. Other experiments are also possible, but their presentation is deferred (for lack of space) to the journal version of the paper. We could estimate helping centrality using sampling techniques similar to those for the Shapley value, but in this case of the 9/11 network exact computations are actually feasible. We have implemented a simple Python script for computing measures Sh, HSh and $Help$. The results are presented in Figure 3.

To reduce computational overhead, we may use the following definition: Call two nodes *helping equivalent* iff (a). They have the same set of skills (b). The families of multisets of skills of their neighbors are identical (as multisets). For Figure 1 helping equivalence splits V into the following equivalence classes: $\{1,19\}$, $\{2\}$, $\{3,15\}$, $\{18\}$, $\{17\}$, $\{4,11\}$, $\{5\}$, $\{16\}$, $\{14\}$, $\{13\}$, $\{9,12\}$, $\{6,8,10\}$, $\{7\}$. It is easy to see that equivalent nodes have identical Helping Shapley values and Helping Centralities.

The Shapley value is the least discriminating of the three measures, still it tells us an interesting fact: on their own, pilots are the most important, followed by those that know martial arts, followed by those with no skill (and Shapley value 0).

Interestingly, Helping Shapley Value (which tallies the contribution of both node's skill and those of its neighbors) and Helping Centrality (which only measures the second factor) give different predictions: in the first case the most important nodes are three pilot nodes 7,9,12, whereas the node with most useful connections seems to be 16, which has no skills of its own ! At the other extreme, node 2 is identified by both measures as the least helpful node. Its HSh value is nonzero (as 2 has some skills that may help) but its Helping Centrality is 0, as its neighbors have no skills.

That 13 is "more helping" than 7,9,12 could seem counterintuitive at first, but this fact can be easily explained: nodes 7,9,12 have the highest Shapley value (0.1153) of all nodes, hence their skill (pilots) makes them the most likely to be able to contribute to forming an attacking team *on their own*, without having to enlist their neighbors. In contrast, node 13 has much less valuable own skills (Shapley value 0.0598) but can contribute valuable neighbors.

6 COMPLEXITY OF HELPING CENTRALITIES.

Computing helping centralities is, as expected, intractable:

Theorem 6.1. *The following problem is #P-complete:*

- *[INPUT:] Graph G, CSG Γ, and player $x \in N$.*
- *[COMPUTE:] The Helping Shapley value $HSh[v](x)$.*

Proof. Since computing the Shapley value of *CSG* is #*P*-complete [4], the result follows by chosing G = the empty graph on V. ◻

On the other hand helping centralities are fixed-parameter tractable:

Theorem 6.2. *The following problems*

- *[INPUT:] A CSG $\Gamma = (N, v)$ and a player $i \in N$.*
- *[TO COMPUTE:] $HSh[v](i)$ and $Help[v](i)$.*

parameterized by k, the cardinality of the largest skill set required by any task, are fixed parameter tractable.

The proof of this result is quite similar to that of Theorem 4.1, and is deferred to the full (arXiv) version.

We also give in the sequel a class of games on which not only HSh is efficiently computable, but it has a simple closed-form formula:

Definition 6.3. A *pure skill game* is a CSG where, for every $t \in T$, $|T_t| = 1$ (every task presumes a single skill with multiplicity 1).

Theorem 6.4. *For pure skill games v and player $x \in V$*

$$Help(x) = \sum_{t \in T_{N(x)} \setminus T_x} \frac{w_t}{|P(t)| + 1}$$

Proof. We reduce again the proof by linearity to the setting when the game is a single-task game $t = \{s\}$ of profit 1. *Because the game is a pure skill game*, the fact that x be pivotal for *HSh* is equivalent to requiring that none of the elements before x in π are in $P(s)$, while x does not have s, but has a neighbor that has, i.e. $t \in T_{N(x)} \setminus T_x$ and x is the first element from $\{x\} \cup P(t)$ in permutation π. In a random permutation this happens with probability $1/(|P(t)| + 1)$. ◻

7 RELATED WORK

Our work combines several important lines of research: the extensive literature on (game-theoretic) centrality measures (see [7, 35] for reviews from different perspectives) and that on compact representation frameworks for cooperative games [6, 9, 11, 13, 16].

Ideas related to the use of compact representations in defining notions of network centrality have been considered (implicitly or explicitly) in previous literature, e.g. [30, 37]. This last paper is, perhaps, the closest in spirit to our approach. They undertake a comprehensive study of classes of network centralities and identify axiomatic foundations for various representational frameworks. Compared to this work our focus is, however, different: we are not interested in comparing multiple frameworks, but strive to include capabilities/tasks explicitly into the representational framework, and identify one framework which does just that.

Finally, a related problem is *team formation* in the knowledge discovery literature (e.g.[20, 21]). These papers often assume a skills/tasks framework similar to ours. Their concerns are, however, different: team formation is viewed not as a game-theoretic problem but as an optimization problem, and efficient (approximation) algorithms are provided.

8 CONCLUSIONS, OPEN PROBLEMS.

Our work provides two important conceptual contributions:

(1). Giving *an explicit framework for representing capabilities to perform tasks* in measures of network centrality, and
(2). Proposing *the new notion(s) of helping centrality.*

Several open issues arise: first of all, we have only used one of the several formalisms for CSG games. A more extensive investigation of the representational power of (other) families is in order. So are the computational and experimental aspects of helping centralities. The problem of defining representational formalisms that are able to naturally model all "reasonable" centrality measures is interesting. Also, note that Helping Centrality is missing from the list of #*P*-complete results of Theorem 6.1. We leave its complexity open.

Finally, the Shapley value has a nice axiomatic characterization [29]. The axiomatic approach to coalitional measures has developed into an important direction in coalitional game theory, and has recently been adapted to centrality measures as well. A natural question is whether our Helping Shapley value has a similar axiomatic characterization. To attempt such a characterization define:

Definition 8.1. Given graph $G = (V, E)$, a function $f : \Gamma[V] \to \mathbb{R}^V$ satisfies the axioms of

- **linearity** if for every player $x \in V$ and any two games v_1, v_2 on V, $f[v_1 + v_2](x) = f[v_1](x) + f[v_2](x)$ and, for every $\alpha \in \mathbb{R}$, $f[\alpha v_1](x) = \alpha \cdot f[v_1](x)$.
- **null helping** if for every game v on V and $i \in V$ s.t. for every $S \subseteq V$, $v(S \cup \{i\} \cup N(i)) = v(S)$ then $f[v](i) = 0$.
- **veto game symmetry** if for any veto game v_S and players x, y , $f[v_S](x), f[v_S](y) > 0$ then $f[v_S](x) = f[v_S](y)$.

One can easily prove the following:

Theorem 8.2. *HSh satisfies linearity and null helping.*

Unfortunately while the Shapley value satisfies veto game symmetry, this is **not** true for the Helping Shapley value:

Example 8.3. Consider the star network S_n in figure 2 (b). Then in the unanimity game v_N on S_n (corresponding to $S = \{1, 2, \ldots, n\}$) we have $HSh[v_S](1) = 1 > 0$, $HSh[v_S](i) = \frac{1}{n-1} > 0$ for all $i = 2, \ldots, n$. Indeed, $i \geq 2$ is pivotal for π iff all other nodes in $S \setminus \{1, i\}$ appear before i in π. This happens with probability $1/(n - 1)$.

This mismatch has implications for the axiomatic characterization of HSh: the uniqueness of the ordinary Shapley value follows from veto game symmetry, which is established via an *equal treatment* axiom. The lack of veto symmetry means that we cannot adapt this classical proof to HSh. It is an interesting open question to give an axiomatization of HSh.

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