Position-Based Matching with Multi-Modal Preferences

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ABSTRACT

Many models have been proposed for computing a one-to-one matching between two equal-sized sets/sides of agents, each assigned with one preference list of the agents in the opposite side. The most prominent one might be the Stable Matching model. Recently, the Stable Matching model has been extended to the multi-modal setting [6, 13, 29], where each agent has more than one preference lists, each representing a criterion based on which the agents of the opposite side are evaluated. We use a layer to denote the set of preference lists of agents, which are based on the same criterion. Thus, the single modal matching problem has only one layer. This setting finds applications in many real-world scenarios. However, it turns out that stable matchings might not exist with multi-modal preferences and the determination is NP-hard and W-hard with respect to several natural parameters. Here, we introduce three position-based matching models, which minimize the "dissatisfaction score". We define four dissatisfaction scores, which measure matchings from different perspectives. The first model minimizes the total respective dissatisfaction score over all layers, while the second minimizes the maximum of the respective score over all layers. The third model seeks for a matching M which is Layer Pareto-optimal, meaning that there does not exist a matching M', which is at least as good as M with respect to the respective dissatisfaction score in all layers, but is strictly better in at least one layer. We present diverse complexity results for these three models, among others, polynomial-time solvability for the first model. We also investigate the generalization which given an upper bound on the dissatisfaction score, computes a matching involving subsets of agents and a subset of layers. Hereby, we mainly focus on the parameterized complexity with respect to parameters such as the size of agent subsets, or the size of the layer subset and achieve fixed-parameter tractability as well as intractability results.

KEYWORDS

Social Choice; Stable Matching; Computational Complexity; Parameterized Complexity.

ACM Reference Format:
Yinghui Wen, Aizhong Zhou, and Jiong Guo. 2022. Position-Based Matching with Multi-Modal Preferences. In *Proc. of the 21st International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2022), Online, May 9–13, 2022*, IFAAMAS, 9 pages.

1 INTRODUCTION

Matching problems have received a considerable amount of attention from both economics and computer science communities and have been studied for several decades, due to their rich applications in the real world, for instance, assignment of students to colleges [1, 2], kidney patients to donors [28, 35, 36], or refugees to host countries [7, 8].

One of the most studied models is the two-sided preference-based matching. Given two disjoint sets U and W, each member from U has a preference order of the members in W based on some criterion, and vice versa. A matching is a one-to-one or one-to-many assignment of $u \in U$ to $w \in W$. We call the members of U men and the members of W women. There are many different criteria for determining whether a matching is good enough. The most famous one is the so-called stable matching, which was introduced by Gale and Shapley [20]. If a matching M has no blocking pairs, then M is stable, where a blocking pair of a matching M is a pair of agents who prefer each other to the partners matched by M. Besides stable matching, criteria like popular matching [5, 15, 25] and Pareto-optimal matching [3, 4, 12] also have been studied. The former tries to find a matching preferred by most members. The latter seeks for a matching M such that no other matching is strictly better than M.

The traditional preference-based model only allows each agent to provide a single preference list, which is insufficient in many real applications. Imagine a scenario where the applicants and companies choose each other. Applicants may judge a company from different criteria, such as salary, location, office environment, reputation, etc. Applicants may give a preference list for each criterion rather than a global one, and the task is to find a matching satisfying all preference lists. Another example is the marriage problem. Every agent can easily give a series of preference lists based on different criteria, such as age, country, job, hobbies, etc. A common way to deal with these scenarios is to let the agents aggregate their preferences by themselves and report their global preferences, which has been studied for decades [9–11, 17, 24, 27, 32–34, 37]. Another way is to find a matching with the multi-modal setting, which is the method used in this paper. Aziz et al. [6] investigated stable matching with joint probability, and discussed applications and computational complexity of their model. Chen et al. [13] studied stable matching with multi-modal preferences. They introduced three new models and focused on computational complexity aspects of the models with respect to these new scenarios. Miyazaki and Okamoto [29] studied the same scenarios as Chen et al. under stable marriage with incomplete preferences, and found that even when the length of preference lists is bounded by 4 for both men and women sides, this problem is NP-hard. Jain and Talmon [23] studied committee selection with multi-modal preferences.

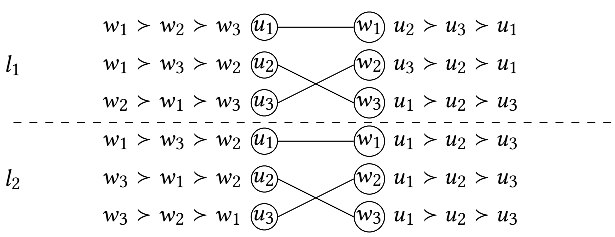
In this paper, we studied two-sided one-to-one matching with multi-modal preferences, where we are given as input a set of preference profiles (we call each of these profiles a layer). In a certain layer, agents from both sides list their preferences with

Proc. of the 21st International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2022), P. Faliszewski, V. Mascardi, C. Pelachaud, M.E. Taylor (eds.), May 9–13, 2022, Online.

respect to the same criterion. The question is whether there exists a good matching which takes all preference profiles into account. We seek for position-based matchings, which satisfies a model $\mathcal{M}$ under a positional scoring rule $\mathcal{R}$. We define three models and four scoring rules. Every model can be combined with every rule, meaning that there are twelve combinations.

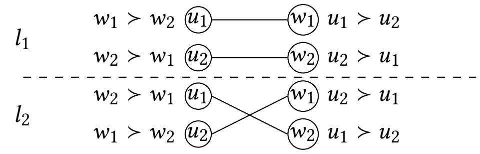
Given $2n$ agents in $U \cup W$ with n for each side, $|U| = |W| = n$, we call the agents in U men and agents in W women. An agent b is the partner of an agent a from the opposite side in a matching M if M matches them together, and we use $M(a)$ to denote the partner of a. We use $P^l_a(b)$ to denote the position of b in a's preference list in layer l, where a and b are agents from different sides. The four scoring rules are defined based on different dimensions, called **Regret score**, **Pair score**, **Balanced score**, and **Egalitarian score**, respectively. We write them as Reg, Pair, Balc, and Egal. Given an agent a, a matching M, and a layer l, the dissatisfaction score of a in l with respect to M, denoted as $\text{Reg}(a, M, l)$, is set equal to $P^l_a(M(a))$. $\text{Pair}(a, M, l)$ stands for the dissatisfaction of a pair of agents a and $M(a)$ matched by M in l, equal to $P^l_a(M(a)) + P^l_{M(a)}(a)$. The third rule $\text{Balc}(a, M, l)$ is designed to measure the total dissatisfaction score of the gender of an agent a, which equals to $\sum P^l_b(M(b))$ with $b \in U$ if a is a man or $b \in W$ if a is a woman. The last scoring rule $\text{Egal}(a, M, l)$ focuses on the overall dissatisfaction score in layer l, which equals to $\sum P^l_b(M(b))$ with $b \in U \cup W$. It is easy to observe that two agents from the same pair have the same Pair-scores, and agents from the same side have the same Balc-scores. Thus in the following, we use "$\mathcal{R}$-score of a pair/side" to denote the $\mathcal{R}$-score of any agent in the pair/side. For instance, given a layer l and a matching M, the Reg-score of the pair $\{u, w\}$ equals to $\text{Reg}(u, M, l)$ or $\text{Reg}(w, M, l)$. We use "$\mathcal{R}$-score of a layer" to denote the maximum $\mathcal{R}$-score among all agents in this layer, written as $\mathcal{R}(M, l)$ with respect to a matching M and a layer l. For instance, given two sets U and W, a matching M and a layer l, the Regret-score of layer l with respect to M is $\text{Reg}(M, l) = \max_{x \in U \cup W} \text{Reg}(x, M, l)$. See Example 1.1 for a concrete example of the models and rules.

Example 1.1. Consider two sets of agents, $U = \{u_1, u_2, u_3\}$ and $W = \{w_1, w_2, w_3\}$, two layers l_1 and l_2, and a matching $M = \{\{u_1, w_1\}, \{u_2, w_3\}, \{u_3, w_2\}\}$.

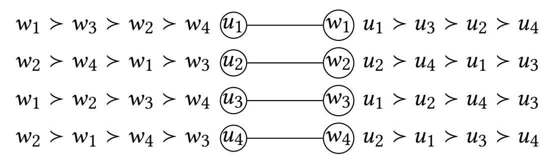


The $\mathcal{R}(u_1, M, l_1)$-score is 1, 4, 4, 10 for $\mathcal{R}$ = Reg, Pair, Balc, Egal, respectively. The corresponding $\mathcal{R}(M, l_1)$-score is 3, 4, 6, 10, respectively. The corresponding $\mathcal{R}(M, l_2)$-score is 3, 5, 6, 10, respectively.

Reg, Balc, and Egal have already been studied under stable marriage setting [14, 19, 21, 22]. Pair is designed to fill the gap between Reg and Balc. We aim to find matchings minimizing the respective dissatisfaction scores based on the following reasons. First, it is natural that every agent tends to be matched with a partner, who is ranked as high as possible in the agent's preference list. For comparison, positional scoring rules form an important set of election rules for voting problems, and have been extensively examined for decades. Second, in contrast to the single preference setting, stable matchings may not exist with multiple preference profiles. For instance, the following instance has two pairs of agents u_1, u_2, w_1, w_2 with preference lists in two layers l_1 and l_2:



There is only one stable matching $M_1 = \{\{u_1, w_1\}, \{u_2, w_2\}\}$ for l_1, which is not stable for l_2. Similarly, there is only one stable matching $M_2 = \{\{u_1, w_2\}, \{u_2, w_1\}\}$ for l_2, which is not stable for l_1. Therefore, there is no stable matching in both l_1 and l_2. In contrast, position-based matchings always exist[1]. Third, stable matchings often tend to suboptimal solutions for most agents. Consider two sets of agents, $U = \{u_1, u_2, u_3, u_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$ with preference lists are set as follows:



There are two stable matchings; one is shown above and another is $M_1 = \{\{u_1, w_1\}, \{u_2, w_2\}, \{u_3, w_4\}, \{u_4, w_3\}\}$. In both matchings, half of the agents can only be matched to partners, who are ranked in the bottom of the preference lists, while, there exists a non-stable matching $M_2 = \{\{u_1, w_3\}, \{u_2, w_4\}, \{u_3, w_1\}, \{u_4, w_2\}\}$ with half of the agents being matched with the first person and the others being matched with the second person in their preference lists. Therefore, the above position-based rules may provide matchings, which pay more value on collective optimality. In summary, we propose the position-based matchings as a legitimate alternative for computing matchings with multi-modal preference profiles.

The three models are denoted as LSum-$\mathcal{R}$, LMax-$\mathcal{R}$, and LPareto-$\mathcal{R}$, where $\mathcal{R} \in \{\text{Reg}, \text{Pair}, \text{Balc}, \text{Egal}\}$. Given an integer d, LSum-$\mathcal{R}$ tries to find a matching M, such that for each agent $a \in U \cup M$, the sum of $\mathcal{R}(a, M, l)$ over all layers l is at most d. LMax-$\mathcal{R}$ seeks for a matching M where the maximum of $\mathcal{R}(M, l)$ over all layers is at most d. LPareto-$\mathcal{R}$ tries to figure out whether there exists a Layer Pareto-optimal matching, where a matching M is Layer Pareto-optimal if there does not exist a matching M', such that $M \neq M'$, $\mathcal{R}(M', l) \leq \mathcal{R}(M, l)$ for all layers, and $\mathcal{R}(M', l) < \mathcal{R}(M, l)$ for at least one layer. See Example 1.2 for a concrete example.

Example 1.2. Consider the same situation as Example 1.1. For $d = 3, 6, 10$, M is an LMax-$\mathcal{R}$ matching with $\mathcal{R}$ = {Reg, Balc, Egal},

[1]Note that each computation problem can be formulated as an optimization version or a decision version. Here, we mean the optimization version, that is, there is always an optimal matching in each instance. However, the score of the matching could be very large. Later, while we investigate the complexity of the problems, we have to formulate the problems as decision problems, since computational complexity only refers to decision problems. With an upper-bound posed on the score of matchings, there might be no matching satisfying this bound in some instances.

respectively. For $d = 4$, M is not an LMax-Pair matching because in l_2, $\text{Pair}(M, l_2)$ is 5.

For $d = 6$, M is an LSum-Reg matching because for every agent $a \in U \cup W$ it holds that $\text{Reg}(a, M, l_1)+ \text{Reg}\ (a, M, l_2) \leq 6$, but M is not an LSum-Pair matching since $\text{Pair}(u_2, M, l_1) + \text{Pair}(u_2, M, l_2) = 4 + 3 = 7 > 6$.

M is not an LPareto-Balc matching, because there is a matching $M' = \{\{u_1, w_3\}, \{u_2, w_1\}, \{u_3, w_2\}\}$, such that in each layer, the Balc score of M' is at most 6, which is at most the score of M, and is better than M in l_1, $\text{Balc}(M', l_1) = 5 < \text{Balc}(M, l_1) = 6$. After checking all matchings, we can conclude that M is an LPareto-$\mathcal{R}$ matching with $\mathcal{R} \in$ {Reg, Pair, Egal}.

Note that the definition of Layer Pareto-optimal is similar as the one given by Jain and Talmon [23], who defined the Layer Pareto-optimal for multiwinner elections with multi-modal preferences. LPareto-$\mathcal{R}$ focuses on the Pareto-optimality of layers rather than agents, which differs from the so-called Pareto-optimal matching [3, 4, 12]. There might be another way to define Pareto-optimality under the multi-modal setting, that is, no matching is better for a particular agent/pair/gender in some layers, while being just at least as good for all agents/pairs/genders in all layers. This definition follows from the idea of traditional Pareto-optimal matchings. We also investigate the models under this definition, denoted as TLPO-$\mathcal{R}$ and find out that the computational complexity of TLPO-$\mathcal{R}$ is the same as LPareto-$\mathcal{R}$. We can adapt the algorithms for LPareto-Reg and LPareto-Pair to TLPO-Reg and TLPO-Pair, and prove the hardness results for TLPO-Balc and TLPO-Egal in a similar way as for LPareto-Balc and LPareto-Egal. In this paper, we omit the definition and the results for TLPO-$\mathcal{R}$, since we aim to keep the definition of Layer Pareto-optimal consistent with Jain and Talmon's. A natural extension of our models is to introduce weights for the layers to represent their significances. For instance, each layer has a weight, and the target is to find a matching taking both the scoring rules and the layer weights into account. In this paper, we only consider the basic setting, all layers with unit weight.

We study the computational complexity of the three models. On one hand, we find that when the rules of Reg and Pair are used, the three models admit polynomial-time algorithms. With a reduction to the Minimum weighted perfect matching problem, LSum-$\mathcal{R}$ can be solved by the Hungarian method [26], with the only exception of the Balc rule. On the other hand, we prove that it is NP-hard to determine whether a matching is LMax-Balc or LMax-Egal. LPareto-Balc and LPareto-Egal are co-NP-hard, since for a given a matching M, there exists no effective algorithm to determine whether M is an LPareto-Balc or LPareto-Egal matching.

We also study two generalizations of the multi-modal matching problems. One is to check whether there are subsets of the agents, which can form a matching satisfying the dissatisfaction score bound. More precisely, given disjoint agent sets U and W, a set of layers L with $|L| = \beta$, and an integer k, the task is to decide whether there are $U' \subseteq U$ and $W' \subseteq W$ with $|U'| = |W'| = k \leq n$, a profile collection $L_{U' \cup W'}$ where $U \setminus U'$ and $W \setminus W'$ are removed from L, such that there exists a matching M satisfying the score bound posed by the models. We also consider the maximum subset of the preference profiles, that is, whether there is a subset $L' \subseteq L$, where $|L'| = t \leq \beta$, such that there is a matching M satisfying the models with profiles in L'. We call these variants (k, t)-$\mathcal{M}$, where $\mathcal{M} \in$ {LSum-$\mathcal{R}$, LMax-$\mathcal{R}$ }[2].

Example 1.3. Consider the same situation as Example 1.1. If $d = 4$, M is not an LMax-Pair matching because in l_2, $\text{Pair}(M, l_2)$ is 5, which is greater than d, but is a $(3, 1)$-LMax-Pair matching, since all three pairs of agents have the Pair-scores less than d in l_1.

We demonstrate that the generalizations are NP-hard with respect to LSum-$\mathcal{R}$ and LMax-$\mathcal{R}$. Then, we turn to examine parameterized complexity. Unfortunately, we obtain W[1]-hardness, W[2]-hardness and para-NP-hardness results for most parameters except n, where n is the number of men/women. Refer to Table 1 for an overview of our results. Due to lack of space, some proofs are deferred to the full version.

2 PRELIMINARIES

Let $U = \{u_1, \cdots, u_n\}$ and $W = \{w_1, \cdots, w_n\}$ be two n-elements disjoint sets of agents. We call the members in U men, and the members in W women. Let $L = \{l_1, \cdots, l_\beta\}$ be a set of preference profiles, where β is a positive integer. We call each of these preference profiles a layer. A layer l_i is a collection of the preference lists of all agents. For a layer l_i with $1 \leq i \leq \beta$ and an agent $u \in U$, the preference list of u in l_i is a linear order of the members in W, denoted as $\succ_u^{l_i}$. The symbol $\succ_w^{l_i}$ for $w \in W$ is defined analogously. A matching $M \subseteq \{\{u, w\} | u \in U \wedge w \in W\}$ is a set of pairwise disjoint pairs. We say M is a perfect matching if $|M| = n$. If $\{u, w\} \in M$, we say that w is the partner of u matched by M, denoted as $M(u)$, and vice versa.

2.1 Multi-Modal Matchings

Now, we formally define the three models. Let U be the set of men, W the set of women, L the set of layers, and $\mathcal{R} \in$ {Reg, Pair, Balc, Egal}.

Definition 2.1. **(LSum-$\mathcal{R}$ matching)** Given an integer d, a matching M is an LSum-$\mathcal{R}$ matching if, for each agent $a \in (U \cup W)$, it holds that $\sum_{l \in L} \mathcal{R}(a, M, l) \leq d$.

LMax-$\mathcal{R}$-matching is a matching that the maximum $\mathcal{R}$-score in each layer is at most d.

Definition 2.2. **(LMax-$\mathcal{R}$ matching)** Given an integer d, a matching M is an LMax-$\mathcal{R}$ matching if, for each layer $l \in L$, it holds that $\mathcal{R}(M, l) \leq d$.

An LPareto-$\mathcal{R}$ matching is a matching which is Layer Pareto-optimal. We use a similar definition as Jain and Talmon, who defined the Layer Pareto-optimal in multiwinner election with multimodal preferences [23].

Definition 2.3. **(LPareto-$\mathcal{R}$ matching)** A matching M is an LPareto-$\mathcal{R}$ matching if there is no matching $M' \neq M$, such that: (1) In each layer $l \in L$, $\mathcal{R}(M', l) \leq \mathcal{R}(M, l)$, and (2) in at least one layer l, $\mathcal{R}(M', l) < \mathcal{R}(M, l)$. We call M' a **dominating matching** for M.

Now we formally define the problem of finding an LSum-$\mathcal{R}$ matching. The other two problems are defined analogously.

[2]We do not consider (k, t)-LPareto-$\mathcal{R}$ since every instance of LPareto-$\mathcal{R}$ admits a Layer Pareto-optimal matching, as shown in Section 4.

Table 1: Overview of our results. The FPT results with respect to n are trivial, so we omit it. The FPT result marked by (*) is trivial, since both parameters are greater than n. "Para-NP-hard" stands for NP-hardness even with the corresponding parameter being a constant. $\mathcal{M}$ stands for LSum-$\mathcal{R}$ or LMax-$\mathcal{R}$ and $\mathcal{R} \in \{$Reg, Pair, Balc, Egal$\}$. Here, $\bar{k} = n - k$, $\bar{t} = \beta - t$, and $\bar{d} = d' - d$, where d' is the minimum $\mathcal{R}$-score, which can be achieved for the LSum(LMax)-$\mathcal{R}$ instance (U, W, L, d').

<table>
<tr><th colspan="2"></th><th>Regret</th><th>Pair</th><th>Balanced</th><th>Egalitarian</th></tr>
<tr><td colspan="2">LSum-$\mathcal{R}$</td><td colspan="2">P (Thm. 4.2)</td><td>?</td><td>P (Thm. 4.2)</td></tr>
<tr><td colspan="2">LMax-$\mathcal{R}$</td><td colspan="2">P (Thm. 4.3)</td><td>NP-hard (Thm. 4.5)</td><td>NP-hard (Thm. 4.4)</td></tr>
<tr><td colspan="2">LPareto-$\mathcal{R}$</td><td colspan="2">P (Thm. 4.6)</td><td colspan="2">co-NP-hard (Thm. 4.7)</td></tr>
<tr><td rowspan="4">(n,t)-$\mathcal{M}$-$\mathcal{R}$</td><td>t</td><td colspan="4">W[1]-hard (Thm. 4.11)</td></tr>
<tr><td>$\bar{t}$</td><td colspan="4">W[2]-hard (Thm. 4.12)</td></tr>
<tr><td>d</td><td colspan="2" rowspan="2">para-NP-hard (Thm. 4.11)</td><td colspan="2" rowspan="2">FPT(*)</td></tr>
<tr><td>$\bar{d}$</td></tr>
<tr><td rowspan="4">(k,β)-$\mathcal{M}$-$\mathcal{R}$</td><td>k</td><td colspan="4">W[1]-hard (Thm. 4.8)</td></tr>
<tr><td>d</td><td colspan="2">para-NP-hard (Thm. 4.8)</td><td colspan="2">W[1]-hard (Thm. 4.8)</td></tr>
<tr><td>$\bar{k}$</td><td colspan="2">W[2]-hard (Thm. 4.9)</td><td colspan="2" rowspan="2">W[1]-hard (Thm. 4.10)</td></tr>
<tr><td>$\bar{d}$</td><td colspan="2">para-NP-hard (Thm. 4.9)</td></tr>
</table>

> ***LSum-$\mathcal{R}$ Matching Problem (LSum-$\mathcal{R}$)***
> **Input**: Two sets of agents U and W of n agents each, a set of preference profiles L, and a positive integer d.
> **Output**: A perfect LSum-$\mathcal{R}$ matching, if exists; otherwise, "No".

We also investigate a generalization of LSum-$\mathcal{R}$ and LMax-$\mathcal{R}$. Given two subsets $U' \subseteq U$ and $W' \subseteq W$, we define the preference profiles collection $L_{U' \cup W'}$ as the preference lists resulting by removing all $x \in \{\{U \setminus U'\} \cup \{W \setminus W'\}\}$ from the lists in L.

Similarly, we only define the problem of finding a (k, t)-LSum-$\mathcal{R}$ matching. (k, t)-LMax-$\mathcal{R}$ is defined analogously.

> ***(k, t)-LSum-$\mathcal{R}$ Matching Problem ((k, t)-LSum-$\mathcal{R}$)***
> **Input**: Two sets of agents U and W of n agents each, a set of preference profiles L with $|L| = \beta$, and three integers d, k, t with $k \leq n$ and $t \leq \beta$.
> **Output**: Two subsets $U' \subseteq U$ and $W' \subseteq W$ with $|U'| = |W'| = k$, and a subset $L' \subseteq L_{U' \cup W'}$ with $|L'| = t$, such that LSum-$\mathcal{R}$ on (U', W', L', d) does not return "No"; otherwise, "No".

2.2 Parameterized Complexity

Parameterized complexity provides a refined exploration of the connection between problem complexity and various problem-specific parameters. A parameterized problem is *fixed-parameter tractable (FPT)* with respect to a parameter k, if there is an $O(f(k) \cdot |I|^{O(1)})$-time algorithm solving the problem, where I denotes the whole input instance and f can be any computable function. Parameterized problems can be classified into many classes with W[1] and W[2] being the basic fixed-parameter intractability classes. For more details on parameterized complexity, we refer to [16, 18, 31]. We study the parameterized complexity of (k, t)-LSum(LMax)-$\mathcal{R}$, and consider the following parameters: $n = |U| = |W|$, k, $\bar{k} = n - k$, t, $\bar{t} = \beta - t$, d, and $\bar{d} = d' - d$, where d' is the minimum $\mathcal{R}$-score, which can be achieved for the LSum(LMax)-$\mathcal{R}$ instance (U, W, L, d').

3 STRUCTURAL PROPERTIES

We first prove some useful structural properties, which are useful for the following complexity study. This section is divided into two parts. We first show a property of instances with only one layer, and then we show that every LSum(LMax)-$\mathcal{R}$ instance with $\mathcal{R}$ being Reg or Pair has an equivalent LSum(LMax)-$\mathcal{R}$ instance with only one layer.

3.1 Special Case $\beta = 1$

The following observation follows from the definitions of LSum-$\mathcal{R}$ and LMax-$\mathcal{R}$.

Observation 3.1. *If there is only one layer, then LSum-$\mathcal{R}$ is equivalent to LMax-$\mathcal{R}$.*

Then we explore the relation between LSum-Egal and LSum-Balc when there is only one layer.

Lemma 3.1. *Given an LSum-Egal instance (U, W, L, d) with $|L| = 1$, we can construct in polynomial time an equivalent LSum-Balc instance $(U \cup P, W \cup Q, L', d')$ with $|L'| = 1$.*

3.2 From Multiple Layers to Single Layer

Next, we prove that with $\mathcal{R}$ being Reg, Pair, Balc or Egal, every LSum-$\mathcal{R}$ instance can be reduced to a one-layer instance.

Lemma 3.2. *Every LSum-$\mathcal{R}$ instance (U, W, L, d) can be reduced in polynomial time to a new equivalent LSum-$\mathcal{R}$ instance with only one layer and $\mathcal{R} \in \{Reg, Pair, Balc, Egal\}$.*

Finally, we prove that with $\mathcal{R}$ being Reg or Pair, every LMax-$\mathcal{R}$ instance can be reduced to a one-layer instance.

Lemma 3.3. *Every LMax-Reg(Pair) instance (U, W, L, d) can be reduced in polynomial time to a new equivalent LMax-Reg(Pair) instance with only one layer.*

4 COMPLEXITY RESULTS

We first present classical complexity results of LSum-$\mathcal{R}$, LMax-$\mathcal{R}$ and LPareto-$\mathcal{R}$. Next, we consider the parameterized complexity of the more general (k,t)-$\mathcal{M}$-$\mathcal{R}$ with $\mathcal{M}$= {LSum, LMax}.

4.1 LSum-$\mathcal{R}$

We first prove that when there is only one layer, LSum-$\mathcal{R}$ with $\mathcal{R}$ being Reg, Pair or Egal can be solved in polynomial time by reducing them to the Minimum weighted perfect matching problem (MWPM). Given a bipartite graph $G = (V_F \cup V_R, E)$ with V_F and V_R being two disjoint vertex sets, each edge $e \in E$ having an integer weight $h(e) \geq 0$, and a positive integer d, MWPM tries to find a perfect matching M with $\sum_{e \in M} h(e) \leq d$. A perfect matching in a graph is a set of disjoint edges saturating all vertices. MWPM can be solved in polynomial time with Hungarian method (also known as the Kuhn–Munkres algorithm) [26, 30].

Theorem 4.1. *LSum-$\mathcal{R}$ with $\mathcal{R}$ being Reg, Pair or Egal is polynomial-time solvable when there is only one layer.*

Proof. Given an LSum-Egal instance (U, W, L, d), we can reduce it to an equivalent MWPM instance $(V_F \cup V_R, E)$. First, for each man $u \in U$, we construct a vertex $v_f \in V_F$, and for each woman a vertex $v_r \in V_R$. We add all possible edges between V_F and V_R. The only difference concerning the constructions of the three LSum-$\mathcal{R}$ instances lies in the weights of the edges. Given a pair $\{u, w\}$ and their corresponding vertices v_f, v_r, we set the weight $h(e)$ of $e = \{v_f, v_r\}$ as follows:

- [For Egal] $h(e) = P^l_u(w) + P^l_w(u)$.
- [For Pair] $h(e) = 0$, if $P^l_u(w) + P^l_w(u) \leq d$; otherwise, $h(e) = \infty$.
- [For Reg] $h(e) = 0$, if $P^l_u(w) \leq d$ and $P^l_w(u) \leq d$; otherwise, $h(e) = \infty$.

Then the construction is complete. The equivalence between the instances of LSum-$\mathcal{R}$ and MWPM is obvious. The construction can be done within $O(2n + n^2) = O(n^2)$ time. Since the Hungarian method needs polynomial time, we can conclude that, when there is only one layer, that is, $\beta = 1$, LSum-$\mathcal{R}$ with $\mathcal{R}$ being Reg, Pair, or Egal is solvable in polynomial-time. □

By Lemmas 3.2 and Theorem 4.1, we can get the following theorem.

Theorem 4.2. *LSum-Reg, LSum-Pair, and LSum-Egal are in P.*

4.2 LMax-$\mathcal{R}$

In analog to Theorem 4.2, Lemma 3.3 and Theorem 4.1 imply the following result.

Theorem 4.3. *LMax-Reg and LMax-Pair are in P.*

Next we show LMax-Egal is NP-hard by reducing the 3sat problem to LMax-Egal. Given a variable set V and a clause set C with each clause containing exactly three literals, 3sat asks whether there exists a satisfying truth assignment that sets at least one literal in each clause to be true.

Theorem 4.4. *LMax-Egal is NP-hard.*

Proof. Given a 3sat instance $(V = \{v_1, \cdots, v_n\}, C = \{c_1, \cdots, c_m\})$, we create for each variable $v_i \in V$, two pairs of agents, namely, $u_i, \bar{u_i} \in U$ and $w_i, \bar{w_i} \in W$. Then, create two sets P and Q of auxillary agents, $P = \{p_1, \cdots, p_{6n+2}\}$ and $Q = \{q_1, \cdots, q_{6n+2}\}$. Then, the LMax-Egal instance has $4n + 12n + 4 = 16n + 4$ agents, where $P \cup U$ forms the man side and $Q \cup W$ the woman side.

Next, we create m layers, one for each clause $c \in C$, where the preference lists of each $x \in P \cup Q$ are the same in all layers. The preference list of each $p_i \in P$ has the following form: $q_i \succ \overrightarrow{Q \setminus \{q_i\}} \succ \overrightarrow{W}$, with $\overrightarrow{S}$ denoting an arbitrary but fixed ordering of a set S. The preference lists of $q_i \in Q$ are set accordingly. For two agents $u_i, \bar{u_i} \in U$ which are created for the same variable v_i, we create $2m$ preference lists of the same form, two lists for each layer l, $\succ^l_{u_i}$ and $\succ^l_{\bar{u_i}}$: $w_i \succ q_1 \succ \bar{w_i} \succ \overrightarrow{Q \setminus \{q_1\}} \succ \overrightarrow{W \setminus \{w_i, \bar{w_i}\}}$, where w_i and $\bar{w_i}$ are also created for $v_i \in V$. The preference lists of w_i and $\bar{w_i}$ then have the form: $u_i \succ \bar{u_i} \succ \overrightarrow{P} \succ \overrightarrow{U \setminus \{u_i, \bar{u_i}\}}$ and each layer has also exactly one such list for each of w_i and $\bar{w_i}$. Next, we make the following modifications to the lists $\succ^l_{u_i}$ and $\succ^l_{\bar{u_i}}$ according to the occurrence of variables in clauses. In each layer l_j, which is according to a clause c_j, we exchange the positions of $\bar{w_i}$ and q_1 in $\succ^{l_j}_{\bar{u_i}}$ for each variable v_i occurring in c_j positively; if v_i occurs negatively in c_j then we exchange the positions of q_1 and $\bar{w_i}$ in $\succ^{l_j}_{u_i}$; if v_i does not occur in c_j, then no change is done to the lists. □

By using a similar technique as in the proof of Lemma 3.1, we can reduce LMax-Egal to LMax-Balc and get the following result.

Theorem 4.5. *LMax-Balc is NP-hard.*

4.3 LPareto-$\mathcal{R}$

In this part we show the computational complexity of LPareto-$\mathcal{R}$. First, we show a basic observation that an LPareto-$\mathcal{R}$ matching exists for every instance.

Observation 4.1. *Given an instance of LPareto-$\mathcal{R}$, an LPareto-$\mathcal{R}$ matching always exists for $\mathcal{R}$ being Reg/Pair/Balc/Egal.*

Note that, by Observation 4.1, the decision version of LPareto-$\mathcal{R}$ is easy to solve for $\mathcal{R} \in$ {Reg, Pair, Balc, Egal }; it returns "Yes" for all instances. However, the constructive version admits different complexity behaviors for Reg, Pair, Balc, and Egal. The constructive version requires to output a Layer Pareto-optimal matching, as defined in this paper. Now, we show LPareto-Reg and LPareto-Pair admit polynomial-time solving strategies.

Theorem 4.6. *LPareto-Reg and LPareto-Pair are in P.*

Proof. The basic idea is that, given an arbitrary M, we search for a matching dominating M. If there is no such matching, then M is returned as an LPareto-Reg(Pair) matching; otherwise, we repeat this process for the dominating matching. If the search for a dominating matching for a given matching is polynomial-time doable, this problem can be solved in polynomial time. The algorithm of finding a dominating matching for M is shown in Algorithm 1. Recall that the $\mathcal{R}$-score of a layer, denoted as $\mathcal{R}(M, l)$ with respect to a matching M and a layer l, equals to the maximum $\mathcal{R}$-score of all agents in this layer, that is, $\mathcal{R}(M, l) = \max_{a \in U \cup W}\{\mathcal{R}(a, M, l)\}$. Given a triple $(n, L, \{d_1, \cdots, d_\beta\})$ with n and d_i being integers, we construct a bipartite graph G=$(U \cup W, E)$ with n pairs of vertices, i.e., $|U| = |W| = n$, and there is an edge between $u_i \in U$ and $w_j \in W$, if both $P^{l_q}_{u_i}(w_j) \leq d_q$ and $P^{l_q}_{w_j}(u_i) \leq d_q$ for all $l_q \in L$ under Reg, or $P^{l_q}_{u_i}(w_j) + P^{l_q}_{w_j}(u_i) \leq d_q$ for all $l_q \in L$ under Pair.

Algorithm 1 Finding a dominating matching for M

Input: Set of preference profiles L, a perfect matching M
Output: M' which dominates M

```
Let d_i = R(M, l_i) with 1 ≤ i ≤ β and R being Reg or Pair
for j = 1 to β do
    For 1 ≤ i ≤ β, let d'_i = d_i
    d'_j = d'_j − 1
    Construct a bipartite graph G with (n, L, {d'_1, ··· , d'_β})
    Find a perfect matching M_P of G
    if M_P ≠ ∅ then
        return M_P
    end if
end for
```

We can use the Hungarian Method to compute a maximum matching of a bipartite graph in polynomial time. Then Algorithm 1 runs in polynomial time. Thus, we can solve LPareto-Reg(Pair) by first finding an arbitrary perfect matching M and then applying Algorithm 1 to improve it. Since each application of Algorithm 1 decreases the Reg(Pair)-score of at least one layer by at least one, and the maximum Reg(Pair)-score of one layer is $n(2n)$. Therefore, the whole progress is in polynomial time, and LPareto-Reg(Pair) is in P. □

Now we investigate the computational complexity of LPareto-Egal and LPareto-Balc. Unfortunately, these problems seem to be at least co-NP-hard, since the LPareto-Egal-Determine problem is co-NP-hard, which given an instance of LPareto-Egal and a matching M_0, decides whether M_0 is a solution of LPareto-Egal, that is, whether there is no other matching M dominating M_0. We define the LPareto-Balc-Determine problem in the similar way.

Theorem 4.7. *LPareto-Egal-Determine and LPareto-Balc-Determine are co-NP-hard.*

Proof. To prove this theorem, we need to prove its complementary problem is NP-hard, that is, given an instance (U, W, L, M_0), deciding whether there is a matching M dominating M_0. We call it LPareto-Egal/Balc-Dominating. We establish the NP-hardness by reducing 3-partition to this problem. Given a set of $3m$ integers $\{a_1, \cdots, a_{3m}\}$ with the total sum of the integers being mB and each a_i satisfying $B/4 < a_i < B/2$, 3-partition decides whether this set of integers can be partitioned into m subsets such that the sum of the numbers in each subset is equal to B and each subset contains exactly three integers. 3-partition is strongly NP-hard, that is, it remains NP-hard even if B can be bounded by a polynomial of m.

We first prove this theorem for the Egalitarian score. Given a 3-partition instance $(A = \{a_1, \cdots, a_{3m}\}, B)$, we create for each integer $a_i \in A$, $m + 1$ pairs of agents, namely, $u^j_i \in U$ and $w^j_i \in W$ with $0 \leq j \leq m$. Then, create two sets P and Q of auxillary agents $P = \{p_1, \cdots, p_{3mD-B}\}$ and $Q = \{q_1, \cdots, q_{3mD-B}\}$ with $D = (3mB + m + 2)(m + 1)$. Then the LPareto-Egal-Dominating instance has $3m(m + 1) + 3mD - B$ agents per side, where $P \cup U$ forms the man side and $Q \cup W$ the woman side.

Next, we create $m + 1$ layers, $l_1, \cdots, l_{m+1}$, among which the first m layers are created for the m subsets. The preference lists of each $x \in P \cup Q$ are firstly set the same in all layers. For each $p_i \in P$, the preference list has the following form: $q_i \succ \overrightarrow{Q \setminus \{q_i\}} \succ \overrightarrow{W}$, with $\overrightarrow{S}$ denoting an arbitrary but fixed ordering of a set S. The preference lists of $q_i \in Q$ are set accordingly. Then, we switch in $\succ^{l_{m+1}}_{q_i}$, p_i with the agent at the last position for $1 \leq i \leq 3mD - B$. For the agents $u^j_i \in U$ with $0 \leq j \leq m$ which are created for the same integer a_i, we create $(m + 1)(m + 1)$ preference lists of the same form and add $m + 1$ lists to each layer. The preference list of u^j_i has the following form, where $\{w^0_i, \cdots, w^m_i\}$ are also created for $a_i \in A$:

$$\succ^{l_s}_{u^j_i}: w^0_i \succ \cdots \succ w^m_i \succ \overrightarrow{Q} \succ \overrightarrow{W \setminus \{w^0_i, \cdots, w^m_i\}}, \qquad \forall 0 \leq s \leq m$$

The preference lists of $w^j_i \in W$ with $0 \leq j \leq m$ then have the following form and each layer has also exactly $m + 1$ such lists:

$$\succ^{l_s}_{w^j_i}: p_1 \succ \cdots \succ p_{3mB} \succ u^0_i \succ \cdots \succ u^m_i \succ p_{3mB+1} \succ \cdots \succ p_{3mD-B} \succ \overrightarrow{U \setminus \{u^0_i, \cdots, u^m_i\}}, \qquad \forall 0 \leq s \leq m$$

Next, we make some modifications in the above preference lists. In each layer l_j with $1 \leq j \leq m$, which is according to a subset, we do the following modifications for $\succ^{l_j}_w$ with $w \in W$.

- In $\succ^{l_j}_{w^0_j}$, exchange the positions of u^0_j and p_1, where p_1 is at the first position in $\succ^{l_j}_{w^0_j}$.
- In $\succ^{l_j}_{w^0_i}$ with $1 \leq i \leq 3m$ and $i \neq j$, exchange the positions of u^0_i and $p_{3mB+B-m}$, where $p_{3mB+B-m}$ is the auxillary agent at the $(3mB + 1 + B)$-th position in $\succ^{l_j}_{w^0_i}$.
- In $\succ^{l_j}_{w^j_i}$ with $1 \leq i \leq 3m$, exchange the positions of u^0_i and $p_{3mB+1-a_i}$, where $p_{3mB+1-a_i}$ is the auxillary agent at the $(3mB + 1 - a_i)$-th position in $\succ^{l_j}_{w^j_i}$.

Next, we make modifications for the last layer l_{m+1}. For each $1 \leq i \leq 3m$ and $0 \leq j \leq m$, we switch u^j_i with the agent at the last position in $\succ^{l_{m+1}}_{w^j_i}$, that is, u^j_i is the worst agent that w^j_i can be matched to in layer l_{m+1}. Finally, we set $M_0 = \{\{u^j_i, w^j_i\} | u^j_i \in U, w^j_i \in W\} \cup \{\{p_i, q_i\} | p_i \in P, q_i \in Q\}$. □

4.4 (k,t)-LSum-$\mathcal{R}$ and (k,t)-LMax-$\mathcal{R}$

For a given bound d, there can be instances of LSum-$\mathcal{R}$ and LMax-$\mathcal{R}$, which admit no satisfying matching.[3] In this case, it is desirable to seek for a "maximum" matching, that is, a matching satisfying the score bound with subsets of agents and/or a subset of layers. It turns out that even in the case of taking subsets of agents and keeping the layers unchanged or of taking a subset of layers and keeping the sets of agents unchanged, LSum-$\mathcal{R}$ and LMax-$\mathcal{R}$ become NP-hard for all scoring rules. Thus, we investigate their parameterized complexity and achieve both fixed-parameter tractable and intractable results. The FPT result for (k,β)-$\mathcal{M}$-$\mathcal{R}$ with respect to n is trivial, so we omit it.

Theorem 4.8. *Even with $\beta = 1$, (k,β)-$\mathcal{M}$-$\mathcal{R}$ with $\mathcal{M}$= {LSum, LMax} is W[1]-hard with respect to k under all scoring rules, and para-NP-hard with respect to d under Reg and Pair, and W[1]-hard with respect to d under Balc and Egal.*

Proof. We establish this theorem by a reduction from Clique. Given a graph G=(V, E), Clique asks whether there exists in G a complete subgraph with k' vertices. Clique is W[1]-hard with respect to k' [18]. Given an instance (G, k') of Clique with $G = (V, E)$ and $k' > 1$, we construct a (k,1)-$\mathcal{M}$-$\mathcal{R}$ instance $(U, W, \{l\}, d)$ as follows. We create, for each vertex $v_i \in V$, one agent u_i in U and one agent w_i in W. In the only layer l, we construct the following preference lists for u_i and w_i.

$$\succ^l_{u_i}:\overrightarrow{W(V \setminus N(v_i))} \succ w_i \succ \overrightarrow{W(N(v_i))}$$
$$\succ^l_{w_i}:u_i \succ \overrightarrow{U \setminus \{u_i\}}$$

Here, $N(v_i)$ denotes the neighbors of v_i in G, and for a subset $V' \subseteq V$, $W(V')$ and $U(V')$ denote the sets of W-agents and U-agents, respectively, which are created according to the vertices in V'. $\overrightarrow{U \setminus \{u_i\}}$ denotes the ordering where the agents in $U \setminus \{u_i\}$ are sorted according to the increasing order of their indices. Set $d = 1, 2, k'$ and $2k'$ under Reg, Pair, Balc, and Egal, respectively, and $k = k'$. □

Theorem 4.9. *Even with $\beta = 1$, (k,β)-$\mathcal{M}$-$\mathcal{R}$ with $\mathcal{M}$= {LSum, LMax} is W[2]-hard with respect to $\bar{k}$, and para-NP-hard with respect to $\bar{d}$ under Reg and Pair.*

Proof. We establish this theorem by a reduction from Dominating set. Given a graph $G = (V, E)$, Dominating set asks whether there is a size-k' subset of V, denoted as D, such that every $v \in V$ is in D or a neighbor of at least one member of D. Dominating set is W[2]-hard with respect to parameter k' [18]. Denote the degree of a vertex v as $deg(v)$, and we may assume that $\forall v \in V$, $deg(v) = r \geq 1$. Let $n = |V|$.

Given a Dominating set instance (G,k') with $G = (V, E)$, we construct a (k,1)-$\mathcal{M}$-$\mathcal{R}$ instance $(U \cup P, W \cup Q, \{l\}, d)$ as follows. Let $d = r + 1$ for Reg, or $d = 2(r + 1)$ for Pair. For each $v_i \in V$, we create $r + 1$ man agents u_i^j in U and $r + 1$ woman agents w_i^j in W with $0 \leq j \leq r$. Then create two sets of auxillary agents $P = \{p_1, \cdots, p_{n\times(k'+d)}\}$ and $Q = \{q_1, \cdots, q_{n\times(k'+d)}\}$. This means that there are $k' + d$ agents in P and $k' + d$ agents in Q for each $1 \leq i \leq n$. Then, let $P_i = \{p_{(i-1)(k'+d)+1}, \cdots, p_{i(k'+d)}\}$ and $Q_i = \{q_{(i-1)(k'+d)+1}, \cdots, q_{i(k'+d)}\}$.

Next, we create the preference lists of the agents. Add for each $p_i \in P$, the preference list $\succ^l_{p_i}$: $q_i \succ \overrightarrow{Q \setminus \{q_i\}} \succ \overrightarrow{W}$, and for each $q_i \in Q$, the preference list $\succ^l_{q_i}$: $p_i \succ \overrightarrow{P \setminus \{p_i\}} \succ \overrightarrow{U}$ to the preference profile l, where $\overrightarrow{S}$ denotes an arbitrary but fixed ordering of a set S. For each $1 \leq i \leq n$, we add the following preference lists to l, where $n^i(j)$ is the index of the vertex which is the j-th neighbor of v_i for $1 \leq j \leq r$:

$$\succ^l_{u_i^0}:w_i^0 \succ \overrightarrow{Q_i} \succ \overrightarrow{Q \setminus Q_i} \succ \overrightarrow{W \setminus \{w_i^0\}}$$
$$\succ^l_{u_i^j}:w_i^0 \succ w_i^1 \succ \cdots \succ w_i^{r-1} \succ w^0_{n^i(j)} \succ w_i^r \succ \overrightarrow{Q} \succ \overrightarrow{W \setminus \{w_i^0, \cdots, w_i^r\} \cup \{w^0_{n^i(j)}\}}$$
$$\succ^l_{w_i^0}:u_i^0 \succ \overrightarrow{P_i} \succ \overrightarrow{P \setminus P_i} \succ \overrightarrow{U \setminus \{u_i^0\}}$$
$$\succ^l_{w_i^j}:u_i^0 \succ u_i^1 \succ \cdots \succ u_i^{r-1} \succ u^0_{n^i(j)} \succ u_i^r \succ \overrightarrow{P} \succ \overrightarrow{U \setminus \{u_i^0, \cdots, u_i^r\} \cup \{u^0_{n^i(j)}\}}$$

There are totally $n \times (k' + d) + n \times (r + 1)$ pairs of agents, and $2n \times (k' + d) + 2n \times (r + 1)$ preference lists in the layer l. Finally, we set $k = |U| + |P| - k'$, then $\bar{k} = k'$ and $\bar{d} = (r + 2) - (r + 1) = 1$ for Reg or $\bar{d} = 2(r + 2) - 2(r + 1) = 2$ for Pair, with $r + 2$ (or $2(r + 2)$) being the minimum Reg(or Pair)-score before deleting the agents. Clearly, the construction is doable in polynomial time. □

Theorem 4.10. *Even with $\beta = 1$, (k,β)-$\mathcal{M}$-$\mathcal{R}$ with $\mathcal{M}$= {LSum, LMax} is W[1]-hard with respect to $\bar{k}$ and $\bar{d}$ under Egal and Balc.*

Proof. Here, we only prove this theorem for $\mathcal{R}$ being Egal. $\mathcal{R}$ being Balc can be proved in a similar way. We give a reduction from Clique. Given a Clique instance $(G = (V, E), k')$ with $|V| = n$ and $|E| = m$, we construct an (k,1)-$\mathcal{M}$-$\mathcal{R}$ instance as follows. We create one pair of agents for each $v_i \in V$, that is, u^{v_i} and w^{v_i}. For each $e_i \in E$, we create two pairs of agents, $u_1^{e_i}, w_1^{e_i}, u_2^{e_i}, w_2^{e_i}$. Create two sets of auxillary agents P, Q with $|P| = |Q| = (d^* + 10k')$, where $d^* = 10(n - k') + 6\frac{k'(1+k')}{2} + 7(m - \frac{k'(1+k')}{2})$. There are totally $2n + 4m + 2(d^* + 10k')$ agents.

Now we set the preference lists of the agents. Add for each $p_i \in P$, the preference list $\succ^l_{p_i}$: $q_i \succ \overrightarrow{Q \setminus \{q_i\}} \succ \overrightarrow{W}$, and for each $q_i \in Q$, the preference list $\succ^l_{q_i}$: $p_i \succ \overrightarrow{P \setminus \{p_i\}} \succ \overrightarrow{U}$ to the preference profile l, where $\overrightarrow{S}$ denotes an arbitrary but fixed ordering of a set S. For each $v_i \in V$, we add the following preference lists to l, where $Q_i = \{q_{8(i-1)+1}, \cdots, q_{8i}\}$:

$$\succ^l_{u^{v_i}}:\overrightarrow{Q_i} \succ w^{v_i} \succ \overrightarrow{Q \setminus Q_i} \succ \overrightarrow{W \setminus \{w^{v_i}\}}$$
$$\succ^l_{w^{v_i}}:u^{v_i} \succ \overrightarrow{P} \succ \overrightarrow{U \setminus \{u^{v_i}\}}$$

For each edge $e_i = \{v_s, v_t\}$, we add the following preference lists to l.

$$\succ^l_{u_1^{e_i}}: w_1^{e_i} \succ w^{v_s} \succ w^{v_t} \succ w_2^{e_i} \succ \overrightarrow{Q} \succ \overrightarrow{W \setminus \{w_1^{e_i}, w_2^{e_i}, w^{v_s}, w^{v_t}\}}$$

[3]Note that each instance of LPareto-$\mathcal{R}$ has a Layer Pareto-optimal matching with respect to the respective scoring rules.

$$\succ^l_{u_2^{e_i}}: w_1^{e_i} \succ q_{8n+i} \succ w_2^{e_i} \succ \overrightarrow{Q \setminus \{q_{8n+i}\}} \succ \overrightarrow{W \setminus \{w_1^{e_i}, w_2^{e_i}\}}$$
$$\succ^l_{w_1^{e_i}}: u_1^{e_i} \succ u_2^{e_i} \succ \overrightarrow{P} \succ \overrightarrow{U \setminus \{u_1^{e_i}, u_2^{e_i}\}}$$
$$\succ^l_{w_2^{e_i}}: u_1^{e_i} \succ u_2^{e_i} \succ \overrightarrow{P} \succ \overrightarrow{U \setminus \{u_1^{e_i}, u_2^{e_i}\}}$$

Finally, let $k = |U| + |P| - k'$, then $\bar{k} = k'$, and let $d = 3d^* + 10k'$, then $\bar{d} = 10k' + \frac{k(1+k')}{2}$ with $d + \bar{d}$ being the minimum Egal-score of this instance before removing agents. □

In the following we turn to investigate the parameterized complexity of (n,t)-$\mathcal{M}$-$\mathcal{R}$, that is, selecting t out of β layers to form a new instance of LSum-$\mathcal{R}$ or LMax-$\mathcal{R}$ and search for a matching satisfying the $\mathcal{R}$-score. Under such a setting, (n,t)-LSum-$\mathcal{R}$ with $\mathcal{R}$ being Reg/Pair/Egal and (n,t)-LMax-$\mathcal{R}$ with $\mathcal{R}$ being Reg/Pair are FPT with respect to β. That is, by enumerating all subsets $L' \subseteq L$, we can reduce an (n,t)-$\mathcal{M}$-$\mathcal{R}$ instance to an equivalent LSum-$\mathcal{R}$ or LMax-$\mathcal{R}$ instance (U, W, L', d) and apply Theorem 4.2 or 4.3. The time of enumerating all subsets of L is within $O(2^\beta)$.

Theorem 4.11. *(n,t)-$\mathcal{M}$-$\mathcal{R}$ with $\mathcal{M}$ ∈ {LSum, LMax} is W[1]-hard with respect to t under all four scoring rules, and para-NP-hard with respect to d or $\bar{d}$ under Reg and Pair.*

Proof. We give a reduction from Set packing. Given an universe V and a family C of subsets of V, and an integer k', Set packing seeks for a family $C' \subseteq C$ of k' pairwise disjoint sets. Set packing is W[1]-hard with respect to parameter k' [18].

Given a Set packing instance (V,C,k') with $|V| = n'$ and $|C| = m$, we construct an (n,t)-$\mathcal{M}$-$\mathcal{R}$ instance $(U \cup P, W \cup Q, L, d)$ as follows. For each $v_i \in V$, we create $2m$ pairs of agents, $u_i^j, \bar{u}_i^j \in U$ and $w_i^j, \bar{w}_i^j \in W$ with $1 \le j \le m$. We create two sets of auxillary agents P and Q with $|P| = |Q| = d^*$, with d^* being set as follows:

- [For LMax-$\mathcal{R}$] let $d^* = 2, 4, 3mn', 6mn'$ under Reg, Pair, Balc, and Egalitarian, respectively.
- [For LSum-$\mathcal{R}$] let $d^* = 2n, 4n, 3mn'^2, 6mn'^2$ under Reg, Pair, Balc, and Egalitarian, respectively.

Next, we create m layers, one for each subset $c_j \in C$. The preference lists of each $x \in P \cup Q$ are the same in all layers. For each $p_i \in P$, the preference list has the following form: $q_i \succ \overrightarrow{Q \setminus \{q_i\}} \succ \overrightarrow{W}$, with $\overrightarrow{S}$ denoting an arbitrary but fixed ordering of a set S. The preference lists of $q_i \in Q$ are set accordingly. For agents $u_i^j, \bar{u}_i^j \in U$ and $w_i^j, \bar{w}_i^j \in W$ with $1 \le i \le n'$ and $1 \le j \le m$, which are created for the same element v_i, we create $4m$ preference lists of the same form and add 4 lists to each layer. The preference lists of u_i^j and $\bar{u}_i^j$ have the following form: $w_i^j \succ \bar{w}_i^j \succ \overrightarrow{Q \setminus \{q_{d^*}\}} \succ q_{d^*} \succ \overrightarrow{W \setminus \{w_i^j, \bar{w}_i^j\}}$, where w_i^j and $\bar{w}_i^j$ are also created for $v_i \in V$. The preference lists of w_i^j and $\bar{w}_i^j$ are set analogously. Next, we make modifications according to the occurrence of elements in subsets. In each layer l_j, which is according to a subset c_j, we do the following modifications if v_i occurs in c_j.

- For $j = i$, exchange $\bar{u}_i^j$ with p_{d^*} in $\succ^{l_j}_{w_i^j}$.
- For $j \neq i$, exchange u_i^j with p_{d^*} in $\succ^{l_j}_{w_i^j}$.

Finally, set $d = d^*$ under Reg, Pair, and $d = 2d^*$ under Balc, Egalitarian. Set $t = k'$ under all four scoring rules. □

Theorem 4.12. *(n,t)-$\mathcal{M}$-$\mathcal{R}$ with $\mathcal{M}$ ∈ {LSum, LMax} is W[2]-hard with respect to $\bar{t}$ under all four scoring rules.*

5 CONCLUDING REMARKS

We introduce three models for position-based matching with multimodal preferences under four scoring rules. A collection of polynomial-time tractable and intractable results have been achieved: Under rules of Reg and Pair, all three models admit polynomial-time algorithms. Under rules of Balc and Egal, LSum-Egal is known to be polynomial-time solvable, while there is no polynomial-time algorithm for LMax-$\mathcal{R}$ and LPareto-$\mathcal{R}$ unless P=NP.

The classical complexity of one problem remains open, that is, LSum-Balc. We want to mention that this problem is not equivalent to Two-weighted maximum weighted matching (TMWM), which is the Maximum weighted matching problem with exactly two weights assigned to each edge. The target of TMWM is to find a matching, such that the sum of the first weights of all matching edges and the sum of the second weights of all matching edges are both at most d. The only difference between LSum-Balc and TNWM is that the weights of TMWM are allowed to be exponential in n, the number of vertices. Thus, we can prove TMWM is NP-hard by reducing the Partition problem to it, while the same method does not apply to LSum-Balc.

It might be interesting to examine the parameterized complexity of LMax-$\mathcal{R}$ and LPareto-$\mathcal{R}$ under Balc and Egal with respect to β, where β is the number of layers. Since we only focus on parameterized complexity, it might be interesting to examine the approximation complexity of our models.

We only investigate the position-based models. Actually, more models can be adapted to the corresponding version with multimodal preferences. Besides stable matching which has been studied in [13], other models such as popular matching [5, 15, 25] and Pareto-optimal matching [3, 4, 12] might be suitable candidates.

ACKNOWLEDGMENTS

We thank the AAMAS-22 reviewers for their constructive comments. All authors are supported by the National Natural Science Foundation of China (Grants No.61772314, 61761136017 and 62072275.).

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