(Almost) Envy-Free, Proportional and Efficient Allocations of an Indivisible Mixed Manna *

Extended Abstract

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ABSTRACT

We study the problem of finding fair and efficient allocations of a set of indivisible items to a set of agents, where each item may be a good (positively valued) for some agents and a bad (negatively valued) for others, i.e., a mixed manna. As fairness notions, we consider arguably the strongest possible relaxations of envy-freeness and proportionality, namely envy-free up to any item (EFX and EFX_0), and proportional up to the maximin good or any bad (PropMX and PropMX_0). Our efficiency notion is Pareto-optimality (PO).

We study two types of instances: (*i*) *Separable*, where the item set can be partitioned into goods and bads, and (*ii*) *Restricted mixed goods (RMG)*, where for each item j, every agent has either a non-positive value for j, or values j at the same $v_j > 0$. We obtain polynomial-time algorithms for the following:

- Separable instances: PropMX_0 allocation.
- RMG instances: Let *pure bads* be the set of items that everyone values negatively.
  - PropMX allocation for general pure bads.
  - EFX+PropMX allocation for identically-ordered pure bads.
  - EFX+PropMX+PO allocation for identical pure bads.

Finally, if the RMG instances are further restricted to *binary mixed goods* where all the v_j's are the same, we strengthen the results to guarantee EFX_0 and PropMX_0 respectively.

KEYWORDS

Fair division; Mixed Manna; Proportionality; Envy-freeness; Pareto-optimality

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1 INTRODUCTION

The problem of fair division is concerned with allocating items to agents in a *fair* and *efficient* manner. Formally introduced by Steinhaus [13], fair division is an active area of research studied across fields like computer science and economics. Most work has focused on the fair division of *goods*, *i.e.,* items that give non-negative *value* (or utility) to the agents when consumed. However, several practical scenarios involve *bads* (or chores), which impose a *cost* (or disutility) to the agent to whom they are allocated. Generalizing both settings, we study fair division of a set M of indivisible items, where each $j \in M$ can be a good for some agents and a bad for others – a *mixed manna.* Examples of mixed manna include splitting assets and liabilities when dissolving a partnership, dividing tasks among various team members, and deciding teaching assignments between faculty. The agents' valuation functions are additive, i.e. for an agent i and set of goods S, $v_i(S) = \sum_{j \in S} v_{ij}$ where $v_{ij} \in \mathbb{R}$ is the value of agent i for item j. If $v_{ij} \geq 0$, j is a good for i, otherwise j is a bad for i.

Two quintessential fairness notions are of that of envy-freeness (EF) [7, 14] and proportionality (Prop) [13]. EF requires that every agent (weakly) prefers her own allocation than anyone else's, while Prop requires that every agent gets at least a $\frac{1}{n}$-fraction of her total value for all items. Since neither exists always, we consider relaxations of these notions, namely *EF up to any item (EFX)* [6] and *proportionality up to any item (PropX)* [1] respectively. We say that an allocation is EFX if each agent i does not envy another agent k after either removal of a positively-valued good from k's bundle or removal of a bad from i's bundle, and it is PropX if every agent can receive her proportional share after the addition to her bundle of any one good not in her bundle or the removal of any one bad assigned to her.

PropX allocations always exist for bads [10] but need not exist for goods [1, 12]. However, the notion of *proportionality up to the maximin good (PropM)* [2] always exists for goods when agents' valuations are additive. Combining the strongest possible guarantees, we define the notion of *proportionality up to the maximin good or any bad (PropMX)* for the mixed manna setting. An allocation is said to be PropMX if every agent can receive her proportional share after the addition to her bundle of the *maximin* positively-valued good for that agent, or after the removal of any one bad assigned to her.

In addition to being fair, it is important for allocations to be *efficient*. Our notion of efficiency is that of *Pareto optimality (PO)*, and we call an allocation PO if no other allocation makes an agent better off without making someone else worse off.

Motivated by the fact that mixed manna is significantly harder to handle than the goods (bads) manna [4, 9], we aim to achieve all of EFX, PropMX, and PO in a single allocation, computable in polynomial time for special yet important classes of valuations.

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2 CONTRIBUTIONS

We study the problem of computing EFX+PO and PropMX+PO allocations for special case of mixed manna instances. To define these, let us first partition the items into three sets: the set M^+ of *mixed goods*, which are valued positively by at least one agent; the set M^0 of *dummy bads*, which are not valued positively by any agent but may be valued at zero by some; and set M^- of *pure bads* which are valued negatively by all agents.

We consider the following:

- *Separable* instances: The set of items can be partitioned into goods (non-negatively valued) and bads (negatively valued), i.e. we can partition M into $M^{\geq 0}$ and M^- such that $v_{ij} \geq 0$ for all $j \in M^{\geq 0}$ and $v_{ij} < 0$ for all $j \in M^{\geq 0}$, for every agent $i \in N$.
- *Restricted Mixed goods (RMG)* instances: For every item $j \in M^+$ there exists a value $v_j > 0$ such that if an agent values j positively, then she values it at v_j. Furthermore, if $v_j = v_{j'}$ for all $j, j' \in M^+$, then the instance is called a *binary mixed goods (BMG)* instance.

Note that the separable and RMG instances are incomparable.

Focusing on M^-, an instance is called *identical ordering (IDO)*, if all agents have the same ordinal preference for all items, i.e., there exists an ordering of the items in M^- such that for all agents $i \in N$, $v_{i1} \leq v_{i2} \leq \cdots \leq v_{im}$. A special case of an IDO instance is the *identical* setting, in which for every $j \in M$, $v_{ij} = v_{i'j}$ for all $i, i' \in N$.

An allocation is called PropMXif for all $i \in N$ either:

(i) $v_i(\mathbf{x}_i) + d_i(\mathbf{x}) \geq \text{Prop}_i$, where $d_i(\mathbf{x}) = \max_{i' \neq i} \min_{\substack{j \in \mathbf{x}_{i'} \\ v_{ij} > 0}} v_{ij}$, or

(ii) $\forall c \in \mathbf{x}_i$ such that $v_{ic} < 0$, $v_i(\mathbf{x}_i - c) \geq \text{Prop}_i$.

We also consider a slightly stronger fairness notion than EFX, which we call EFX_0. The difference between the definitions of EFX and EFX_0 is that EFX allows for the envy of an agent i towards agent h to disappear after removing any positively-valued item from the bundle of h, whereas in EFX_0 this envy must disappear after removing any non-negative valued item. It is easy to see that any EFX_0 allocation is EFX, but not vice-versa. Furthermore, we consider the notion of PropMX_0which is related to PropMX in a similar way.

We observe that EFX (resp. EFX_0) implies PropMX (resp. PropMX_0), and therefore whenever we get an allocation that is EFX, we also get that the allocation is PropMX, but the converse is not true.

Our first result focuses on instances with *pure goods* and *restricted valuations*. In such instances, for every $j \in M$, there exists a $v_j > 0$ s.t. $v_{ij} \in \{0, v_j\}$ for every agent $i \in N$. As a warm-up, we present a polynomial-time algorithm which returns an EFX+PO allocation for such instances, based on the ency cycle elimination procedure, which assigns goods to carefully chosen vertices of the envy-graph.

Theorem 2.1. *Given a fair division instance of pure goods with restricted valuations, an allocation that is EFX, PO and maximizes the utilitarian social welfare can be computed in polynomial-time.*

We then generalize our result by extending it to the mixed manna setting. We present a polynomial-time algorithm which returns an EFX+PO allocation for RMG instances with identical bads. The algorithm proceeds in three phases, first assigning all items in M^+ using the algorithm of Theorem 2.1, then M^0 and finally M^-, again utilizing the envy-graph for the final phase. We show that PO is preserved because no cycles ever appear in the envy graph and thus there is no need for a bundle reallocation to eliminate envy cycles. In the special case where the RMG instance is actually BMG, we show the stronger $\text{EFX}_0$and PropMX_0guarantees instead.

Theorem 2.2. *Given a fair division instance with restricted mixed goods and identical bads, an allocation that is EFX, PropMX, PO and maximizes the social welfare be computed in polynomial-time. In the special case of a binary mixed goods instance, a modification to this algorithm returns, in polynomial time, an allocation that is EFX_0, PropMX_0, PO and maximizes the social welfare.*

Next, we extend our results by considering more general cases of the set of pure bads M^-. We present an algorithm that uses a slightly different approach to allocating M^- than the algorithm of Theorem 2.2 and maintains EFX even when the pure bads are identically ordered (IDO).

Theorem 2.3. *Given a fair division instance of restricted mixed goods and IDO bads, an EFX allocation can be computed in polynomial-time. In the special case of a binary mixed goods instance, a modification to this algorithm returns, in polynomial time, an EFX_0 allocation.*

To extend our results to the case of general pure bads, we need a reduction from instances of general pure bads to IDO instances, which appears in Li et al. [10], and is commonly used in designing algorithms for MMS fair allocations [3, 5, 8]. First, we show that the reduction continues to hold in the mixed manna setting.

Lemma 2.4. *If there exists a polynomial time algorithm that given any mixed manna instance with IDO pure bads computes a PropMX (resp. PropMX_0) allocation, then there exists a polynomial time algorithm that given any mixed manna instance (with general pure bads) computes a PropMX (resp. PropMX_0) allocation.*

Using Lemma 2.4, we obtain the following result.

Theorem 2.5. *Given a fair division instance of restricted mixed goods, a PropMX allocation can be computed in polynomial-time. Furthermore, for the case of binary mixed goods, a PropMX_0 allocation can be computed in polynomial-time.*

Finally, we turn our attention to separable instances, a setting which is orthogonal to the previous settings considered. In separable instances, all agents agree on the set of goods and bads.

Theorem 2.6. *Given a separable fair division instance (N, M, V), a PropMX_0 allocation can be computed in polynomial-time.*

In addition to our previous results, we show via a counterexample that one cannot hope to obtain a PropMX_0+PO allocation, even for instances with only goods.

3 CONCLUSION

We studied the fair and efficient allocation of an indivisible mixed manna. We measured efficiency via Pareto-optimality, and fairness via EFX and PropMX, where PropMX combines PropM for goods and PropX for bads. We obtained polynomial time algorithms which find allocations that satisfy a mix of these guarantees for separable, restricted mixed goods, and binary mixed goods instances.

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