Pareto Optimal and Popular House Allocation with Lower and Upper Quotas

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ABSTRACT

In the house allocation problem with lower and upper quotas, we are given a set of applicants and a set of projects. Each applicant has a strictly ordered preference list over the projects she finds acceptable, while the projects are equipped with a lower and an upper quota. A feasible matching assigns the applicants to the projects in such a way that a project is either matched to no applicant or to a number of applicants between its lower and upper quota.

In this model we study two classic optimality concepts: Pareto optimality and popularity. We show that finding a popular matching is hard even if the maximum lower quota is 2 and that finding a perfect Pareto optimal matching, verifying Pareto optimality, and verifying popularity are all NP-complete even if the maximum lower quota is 3. We complement the last three negative results by showing that the problems become polynomial-time solvable when the maximum lower quota is 2, thereby answering two open questions of Cechlárová and Fleiner [16]. Finally, we also study the parameterized complexity of all four mentioned problems.

KEYWORDS

house allocation; Pareto optimal matchings; popular matchings

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1 INTRODUCTION

Many university courses involve team-based project work. In such courses, a set of projects is offered, and each student submits a list of projects she finds acceptable. Ideally, the student also ranks these projects in order of her preference. Naturally, the number of students ideally assigned to a specific project strongly depends on the project itself. The lecturer responsible for the project thus might restrict the number of students to an interval. Projects that did not awake sufficient interest in the students are then dropped, while the other projects start with a number of assigned students that falls into the prescribed interval. Such quota constraints also arise in various other contexts involving the centralized formation of groups, including organizing team-based leisure activities, opening facilities to serve a community, and coordinating rides within car-sharing systems. In these and similar applications, the goal is to fulfill some optimality condition under the assumption that the number of participants for each open activity is within the prescribed limits of the activity.

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1.1 Problem Formulation and Solution Concepts

The mathematical formulation of this problem is known as the house allocation problem with lower and upper quotas. In a house allocation instance, we are given a two-sided market, where one side A represents applicants, while the other side P represents projects. Each applicant has a strictly ordered preference list of the projects she finds acceptable. Furthermore, each project $p \in P$ has a lower quota ℓ_p and an upper quota u_p.

In a feasible matching, a project is either *open* or *closed*. The central feasibility requirement is that the number of applicants assigned to an open project must lie between its lower and upper quota, whilst a closed project has no assigned applicant. Each applicant is assigned to at most one project. We define the optimality of a matching with respect to the satisfaction level of the agents. In this paper, we study two well-known notions from the broad topic of matchings under preferences: Pareto optimality and popularity.

A matching M is *Pareto optimal* if there is no matching M', in which no applicant is matched to a project she considers worse, while at least one applicant is matched to a project she considers better than her project in M. A matching M is *popular* if there is no matching M' that would win a head-to-head election against M, where each applicant casts a vote based on her preferences on her assigned project in M and in M'.

1.2 Related Work

Arulselvan et al. [2] derived several complexity results for the maximum weight many-to-one matching problem with project closures and lower and upper quotas, i.e., a project is either open and both the lower quota and upper quota are fulfilled or the project is closed. However, their model excludes agent preferences. In Table 1 we display a structured overview of existing work in the field of matchings under preferences with lower and upper quotas.

Stable matchings. In the classic hospitals residents problem [26, 30], the underlying model is a bipartite many-to-one matching problem involving preferences on both sides, and the goal is to find a stable matching, which is a matching where no hospital-resident pair could improve their situation by being assigned to each other. This model has been combined with lower and upper quotas in several papers. Hamada et al. [31] considered a version where hospitals cannot be closed and presented a polynomial-time algorithm to find a stable solution, while Mnich and Schlotter [35]

	Stability	Pareto optimality	Popularity
1-sided, no project closures	open	[28]	open
1-sided, project closures	[11]	[16, 20, 32, 36], our paper	our paper
2-sided, no project closures	[31, 35]	[39]	[33, 37]
2-sided, project closures	[8, 11]	open	open

Table 1: Overview of the existing literature in the most related settings. The four models differ in how many of the two sides are equipped with preferences, and in the possibility of project closures. Note that stability is defined for one-sided preferences such that hospitals do not differentiate between applicants, but aim to fill their quota.

<table>
<tr><th></th><th>$l_{\max} \leq 2$</th><th>$l_{\max} \leq 3$</th><th>P_{open}</th></tr>
<tr><td>POP-HA$_L^U$</td><td>NP-c. Thm. 3</td><td>coNP-h. Thm. 1</td><td rowspan="2">NP-h. Cor. 3</td></tr>
<tr><td>PERPO-HA$_L^U$</td><td rowspan="3">P Cor. 1</td><td rowspan="2">NP-c. Thm. 1</td></tr>
<tr><td>POPV-HA$_L^U$</td><td rowspan="2">P Thm. 7</td></tr>
<tr><td>POV-HA$_L^U$</td><td>NP-c. [16]</td></tr>
</table>

Table 2: Overview of our results in classic complexity. The four problems studied are finding a popular matching, finding a perfect Pareto optimal matching, verifying popularity, and verifying Pareto optimality. The columns $l_{\max} \leq 2$ and $l_{\max} \leq 3$ indicate the cases where the maximum lower quota of any project is 2 or 3, respectively. The column P_{open} indicates the complexity of deciding whether there is matching of our desired type that opens exactly the projects in P_{open}.

studied the fixed parameter tractability of finding an approximately stable solution in no-instances identified by Hamada et al. [31]. The model of Biró et al. [8] permitted hospital closures, and was shown to lead to NP-hardness. Very recently, the setting of Biró et al. [8] was further investigated by Boehmer and Heeger [11], who conducted a parameterized study in the original hospitals residents setting, and also in the house allocation setting, where hospitals only have a preference for filling their quota, but do not mind which applicant is assigned to them. They also answered an open question of [8] by showing that a stable matching in the hospitals residents problem with lower quota at most 2 can be found in polynomial time. For a further overview on matchings with quotas and constraints we refer the reader to a recent survey by Aziz et al. [3].

Pareto optimal matchings. Pareto optimality is one of the most studied concepts in coalition formation and hedonic games [4, 7, 13, 23], and it has also been defined in the context of various matching markets [6, 9, 14, 15]. As shown by Abraham et al. [1], in the one-to-one house allocation model, a maximum size Pareto optimal matching can be found in polynomial time. Pareto optimality of matchings with lower and upper quotas on projects was studied in four papers. Motivated by a school choice application with regional constraints, Goto et al. [28] analyzed the case of so-called hierarchical lower quotas that must be obeyed. Monte and Tumennasan [36] considered the case of project closures with complete lists, while the model of Kamiyama [32] allowed incomplete lists as well. In all three works, it was shown that a Pareto optimal matching can always be found using a variant of the famous serial dictatorship algorithm. Cechlárová and Fleiner [16] extended this algorithm to the case when an applicant can be assigned to more than one project. They also showed for the many-to-one case with lower and upper quotas that it is NP-hard to compute a maximum size Pareto optimal matching if the maximum lower quota is at least 4, furthermore that it is NP-complete to verify if a matching is Pareto optimal if the maximum lower quota is at least 3. This led the authors to ask whether both of these problems stay intractable if no lower quota exceeds 2. Finally, Darmann et al. [20] study the simplified group activity selection problem as a variant case of the house allocation with lower quotas model, in which agents have non-strict preference lists and a void activity they can be assigned to. In their model, Darmann et al. [20] study the computational complexity of computing (among others) core stable, envy-free, or Pareto optimal matchings. In this paper, we show that both problems are indeed polynomial-time solvable if the maximum lower quota is 2, while finding a maximum size Pareto optimal matching is NP-complete if the maximum lower quota is 3. Regarding two-sided instances, Sanchez-Anguix et al. [39] conducted experiments to derive an approximate Pareto optimal solution with workload balance as an additional requirement.

Popular matchings. Popularity as an optimality principle has been on the rise recently [18, 25, 29] in the matchings under preferences literature. On instances with two-sided preferences, Brandl and Kavitha [12] and Gopal et al. [27] studied popularity for many-to-many and many-to-one matching problems with upper quotas only. For the model introduced in the latter paper, the complexity of deciding whether a popular matching exists is still open. Krishnapriya et al. [33] and Nasre and Nimbhorkar [37] investigated popular matchings in the hospital residents problem with lower and upper quotas, but without the option to close hospitals. They proved that whenever a feasible matching exists, a popular matching has to exist as well. In the house allocation setting, even with weights and upper quotas, the problem of computing a popular matching is tractable as shown by Sng and Manlove [40].

1.3 Our Contribution and Techniques

We provide an analysis of both Pareto optimal and popular matchings in the setting of the house allocation problem with lower and upper quotas, and project closure, and derive tractability results for both classic and parameterized complexity. Due to space restrictions, we only sketch the main idea of most proofs in the body of the paper, and provide the full proof in the appendix.

Table 2 displays a comprehensive overview of our results for classic complexity. We answer both open questions of Cechlárová and Fleiner [16] and show that a Pareto optimal matching can be verified and a perfect Pareto optimal matching can be found in polynomial time if the maximum lower quota is 2. Further, these results also apply to the work of Darmann et al. [20] thus also showing that a maximum size Pareto optimal matching (or group activity selection in their notation) can be found if the maximum lower quota is 2, improving on their result for maximum lower quota 1 and answering one of their open questions. Our positive parameterized results also apply to their problem of computing or verifying Pareto optimal assignments. Further, our work initiates

<table>
<tr><th></th><th>n</th><th>m</th><th>m_{quota}</th><th>m_{open}</th><th>m_{closed}</th></tr>
<tr><td>POP-HA$_L^U$</td><td>W[1]-h. Thm. 4</td><td>FPT Thm. 5</td><td>?</td><td rowspan="2">coNP-h. Thm. 10</td><td rowspan="4">W[1]-h. Thm. 11</td></tr>
<tr><td>PERPO-HA$_L^U$</td><td rowspan="3">FPT Cor. 2</td><td rowspan="3" colspan="2">FPT Thm. 8</td></tr>
<tr><td>POPV-HA$_L^U$</td><td rowspan="2">W[1]-h. Thm. 9</td></tr>
<tr><td>POV-HA$_L^U$</td></tr>
</table>

Table 3: Overview of our parameterized results. The columns are the parameters we use in the respective cases. The first parameter n is the number of applicants, m is the number of projects, and $m_{\mathbf{quota}}$ is the number of projects with a lower quota greater than 1. The parameter $m_{\mathbf{open}}$ asks for a matching that opens exactly $m_{\mathbf{open}}$ projects, while $m_{\mathbf{closed}}$ asks for a matching closing exactly $m_{\mathbf{closed}}$ projects.

the study of the popular house allocation problem with lower quotas by showing that even if the maximum lower quota is 2 it is NP-hard to find a popular matching. However, we also present a polynomial time algorithm to verify if a given matching is popular if no lower quota exceeds 2, while the same problem is shown to be NP-hard for maximum lower quota 3. We then reduce all three problems to the maximum weight matching problem of Arulselvan et al. [2], for which we firstly observe a simple reduction to the general factor problem introduced by Dudycz and Paluch [21], and secondly design a faster algorithm for our special cases by combining results from [21] and gadget techniques established by Cornuéjols [17].

We then identify tractable sub-cases via the power of parameterized complexity, as demonstrated by Table 3. Here we again use the connection to maximum weight matchings and show how to use a treewidth-based algorithm of Arulselvan et al. [2] to get fixed parameter tractability when parameterized by the number of applicants. Further we give a flow-based algorithm to prove fixed parameter tractability when parameterized by m_{quota}, the number of projects with a lower quota greater than 1. Since these two algorithms are for the maximum weight matching problem, they also apply to a recently introduced model in the area of multi-robot task allocation by Aziz et al. [5]. Finally, by a reduction to the parametric integer programming problem [22], we also show that the problem of finding a popular matching is fixed parameter tractable when parameterized by the number of projects.

2 PRELIMINARIES

In this section we formally introduce our notation and the problems we consider.

We are given a set A of n applicants, a set P of m projects, and a bipartite graph $G = (A\dot{\cup}P, E)$, with A and P being the two sides of the bipartition. The *degree* $\deg_v$ of a vertex $v \in A \cup P$ equals the number of vertices v is adjacent to in G and the *neighborhood* N_v is the set of these adjacent vertices. We define Δ_A to be the maximum degree of any vertex in A and Δ_P to be the maximum degree of any vertex in P. In our model, each project $p \in P$ is equipped with a *lower quota* $\ell_p \in \mathbb{N}$ and an *upper quota* $u_p \in \mathbb{N}$. We refer to the maximum lower quota among all projects as $\ell_{\max}$ and the maximum upper quota as $u_{\max}$.

A *matching* $M \subseteq E$ is a set of edges so that each applicant $a \in A$ is incident to at most one edge in M, while each project $p \in P$ is either incident to no edge in M or its degree in the graph $(A\dot{\cup}P, M)$ is at least ℓ_p and at most u_p. If applicant a is assigned to project p in M, then we write $M(a) = p$ and $a \in M(p)$. A matching M assigns each applicant a a project in N_a or a itself. The notation $M(a) = a$ serves convenience and it expresses that the applicant a is unmatched. Conversely, the quota requirement for the projects can be expressed as $\ell_p \leq |M(p)| \leq u_p$ or $|M(p)| = 0$ for each project $p \in P$. We call a project $p \in P$ with $|M(p)| = 0$ *closed* and a project p with $\ell_p \leq |M(p)| \leq u_p$ *open.* A matching M is *perfect* if $M(a) \neq a$ for all $a \in A$, i.e., all applicants are matched to a project in M.

Each applicant $a \in A$ has a strict order $\succ_a$ over $N_a \cup \{a\}$, which we call the *preference list* of a. For each $a \in A$ and $p \in N_a$ we assume that $p \succ_a a$, which translates into applicant a listing only the projects that are acceptable to her, in other words, they are more preferable to her than staying unmatched. If the applicant is clear from the context, we simply write $\succ$ for her preference list.

We now introduce our two optimality concepts based on applicants' preferences. Given a matching M, we say that matching M' *dominates* M if there is no $a \in A$ with $M(a) \succ_a M'(a)$ and there is an $a \in A$ with $M'(a) \succ_a M(a)$. We call matching M *Pareto optimal* if there is no matching that dominates M. We call matching M' *more popular* than matching M if $|\{a \in A \mid M'(a) \succ_a M(a)\}| > |\{a \in A \mid M(a) \succ_a M'(a)\}|$. Matching M is *popular* if there is no other matching that is more popular than M.

We are now ready to define the four problems we tackle in this paper. The first one of these is the popular matching problem.

Popular house allocation with lower and upper quotas (POP-HA$_L^U$)

Input: Graph $G = (A\dot{\cup}P, E)$, preferences $(\succ_a)_{a \in A}$, and quotas $\ell, u\colon P \to \mathbb{N}$.

Question: Does G have a popular matching?

As an example of this problem, we introduce a small gadget instance we also use in a later proof, representing an instance of the famous Condorcet cycle.

Observation 1. *Consider the instance $\mathcal{I}$ of POP-HA$_L^U$ with three applicants a_1, a_2, a_3 and three projects p_1, p_2, p_3, each with a lower and upper quota of 3, such that the preference list of a_1 is $p_1 \succ_{a_1} p_2 \succ_{a_1} p_3$, the preference list of a_2 is $p_2 \succ_{a_2} p_3 \succ_{a_2} p_1$ and the preference list of a_3 is $p_3 \succ_{a_3} p_1 \succ_{a_3} p_2$, i.e., the preference lists are just cyclically shifted between the applicants. This instance does not admit a popular matching.*

Proof. Any non-perfect matching has to be empty in this instance due to the lower quota of 3 on each project. Thus, matching all three applicants to any of the projects would lead to a matching that is preferred by all three applicants. Therefore, we only need to consider the three perfect matchings

- M_1 with $M_1(p_1) = \{a_1, a_2, a_3\}$,
- M_2 with $M_2(p_2) = \{a_1, a_2, a_3\}$,
- M_3 with $M_3(p_3) = \{a_1, a_2, a_3\}$.

Then the following statements hold.

- The applicants a_2 and a_3 prefer M_3 to M_1, while a_1 prefers M_1 to M_3, making M_3 more popular than M_1.
- The applicants a_1 and a_2 prefer M_2 to M_3, while a_3 prefers M_3 to M_2, making M_2 more popular than M_3.
- The applicants a_1 and a_3 prefer M_1 to M_2, while a_2 prefers M_2 to M_1, making M_1 more popular than M_2.

Thus, this instance admits no popular matching. □

Besides this we also study the complexity of verifying whether a given matching is popular.

Popularity verification in house allocation with lower and upper quotas (popv-ha$_L^U$)

Input: Graph $G = (A \dot\cup P, E)$, preferences $(\succ_a)_{a \in A}$, quotas $\ell, u \colon P \to \mathbb{N}$, and matching M.

Question: Does G have a matching M' that is more popular than M?

Thirdly we study the problem of finding a Pareto optimal matching covering all applicants. We remind the reader that (non-perfect) Pareto optimal matchings can be found in polynomial time using a variant of the serial dictatorship method [16, 32, 36].

Perfect Pareto optimal house allocation with lower and upper quotas (perpo-ha$_L^U$)

Input: Graph $G = (A \dot\cup P, E)$, preferences $(\succ_a)_{a \in A}$, and quotas $\ell, u \colon P \to \mathbb{N}$.

Question: Does G have a Pareto optimal matching that matches all applicants in A?

Finally we also study the verification version of Pareto optimality.

Pareto optimality verification in house allocation with lower and upper quotas (pov-ha$_L^U$)

Input: Graph $G = (A \dot\cup P, E)$, preferences $(\succ_a)_{a \in A}$, quotas $\ell, u \colon P \to \mathbb{N}$, and matching M.

Question: Does G have a matching M' that dominates M?

3 CONNECTION TO WEIGHTED MATCHINGS

Before diving into the main theorems of our paper, we present two auxiliary lemmas, allowing us to reduce our problems popv-ha$_L^U$, perpo-ha$_L^U$, and pov-ha$_L^U$ to the following weighted many-to-one matching problem, defined by Arulselvan et al. [2].

Weighted bipartite matching with lower and upper quotas (w-ha$_L^U$)

Input: Graph $G = (A \dot\cup P, E)$ with quotas $\ell, u \colon P \to \mathbb{N}$, weight function $w \colon E \to \mathbb{R}$, and a bound $W \in \mathbb{R}$.

Question: Is there a matching M with $\sum_{e \in M} w(e) \geq W$?

First, we give a reduction from perpo-ha$_L^U$ to w-ha$_L^U$.

Lemma 1. *For each perpo-ha$_L^U$ instance $\mathcal{I}$, there is a w-ha$_L^U$ instance $\mathcal{I}'$ on the same graph, such that each maximum weight matching in $\mathcal{I}'$ corresponds to a Pareto optimal matching in $\mathcal{I}$, with a maximum number of matched applicants. The instance $\mathcal{I}'$ can be computed in polynomial time from $\mathcal{I}$.*

Proof. Given any $a \in A$ with preference list $p_1 \succ_a \cdots \succ_a p_k$, we define the weight of the edge between a and any p_i for $i = 1, \ldots, k$ to be $(k - i) + mn$. Let M be a maximum weight matching in this new instance. First M has to be a maximum matching, since any larger matching would lead to a vertex being matched that was previously unmatched and thus increasing the weight of the matching by at least $mn - (n-1)m > 0$. Furthermore the matching has to be Pareto optimal, since any matching dominating it would obviously lead to a matching of larger weight. □

Lemma 1 shows that in order to check if a perfect Pareto optimal matching exists in $\mathcal{I}$, it is sufficient to find a maximum weight matching in $\mathcal{I}'$ and check if it is perfect. However, Lemma 1 does not hold in the reverse direction: not all perfect Pareto optimal matchings of $\mathcal{I}$ translate into a maximum weight matching in $\mathcal{I}'$.

For popv-ha$_L^U$ and pov-ha$_L^U$, we show similar statements as first observed by Biró et al. [10] for one-to-one popular matchings. Our results state that a matching is popular / Pareto optimal if and only if it is a maximum weight matching in a certain weighted graph.

Lemma 2. *For each popv-ha$_L^U$ /pov-ha$_L^U$ instance $\mathcal{I}$ with matching M there is a w-ha$_L^U$ instance $\mathcal{I}'$ on the same graph, such that a matching is more popular than M / dominates M in $\mathcal{I}$ if and only if it has a larger weight than M in $\mathcal{I}'$. The instance $\mathcal{I}'$ can be computed in polynomial time from $\mathcal{I}$.*

Proof sketch. As a sketch, we show the construction for the popv-ha$_L^U$ case. The exact calculations and the pov-ha$_L^U$ case are in the appendix. We start by defining a modified vote function. For applicant a and projects $p_1, p_2 \in N_a$ let

$$\text{vote}_a(p_1, p_2) = \begin{cases} 2, & \text{if } p_1 \succ_a p_2, \\ 1, & \text{if } p_1 = p_2, \\ 0, & \text{if } p_2 \succ_a p_1. \end{cases}$$

Note that based on the definition of popularity, matching M' is more popular than matching M if $|\{a \in A \mid M'(a) \succ_a M(a)\}| > |\{a \in A \mid M(a) \succ_a M'(a)\}|$, which is equivalent to $\sum_{a \in A} \text{vote}_a(M'(a), M(a)) > n$. Further, let $U(M) := \{a \in A \mid M(a) = a\}$ be the set of applicants left unmatched by M.

For our weighted matching instance $\mathcal{I}'$ we now take the same graph as in the popv-ha$_L^U$ instance $\mathcal{I}$, and introduce the weight function $w \colon E \to \{0, 1, 2\}$ such that for any $a \in A \setminus U(M)$ and project $p \in N_a$ we set $w(\{a, p\}) = \text{vote}_a(p, M(a))$, further for any $a \in U(M)$ and project $p \in N_a$ we set $w(\{a, p\}) = 1$. We claim that a matching M' is more popular than M in $\mathcal{I}$ if and only if $w(M') > n - |U(M)|$ in $\mathcal{I}'$. Since the weight of M in $\mathcal{I}'$ is exactly $n - |U(M)|$, this is sufficient to show. The intuition behind this is how agents contribute to $w(M)$ and $w(M')$.

- each $a \in A$ with $M'(a) \succ_a M(a)$ adds 1 to $w(M') - w(M)$
- each $a \in A$ with $M(a) \succ_a M'(a)$ subtracts 1 from $w(M') - w(M)$
- each $a \in A$ with $M'(a) = M(a)$ contributes the same to $w(M)$ and $w(M)$

Thus $w(M') > w(M)$ is achieved if and only if M' is more popular than M. □

4 CONSTANT LOWER QUOTAS

In this section, we show that all four of our problems become intractable when the maximum lower quota is 3 and the maximum degree is constant. Furthermore, pop-ha$_L^U$ is NP-complete even if the maximum lower quota is 2. We contrast this result by showing that the other three problems become polynomial-time solvable if the maximum lower quota is 2.

4.1 Lower Quota 3

We begin by showing that all our problems are hard for maximum lower quota 3 by modifying a reduction from exact cover by 3-sets by Cechlárová and Fleiner [16, Theorem 6] for their NP-hardness proof of POV-HA$_L^U$.

EXACT COVER BY 3-SETS (X3C)

Input: Set $X = \{x_1, \ldots, x_{3m}\}$ and set-system $\mathcal{T} = \{T_1, \ldots, T_n\} \subseteq 2^X$ such that $|T_i| = 3$ for all $i \in [n]$.

Question: Is there a subset $T' \subseteq \mathcal{T}$ such that T' is a partition of X?

THEOREM 1. *The problems* POPV-HA$_L^U$ *and* PERPO-HA$_L^U$ *are* NP-*complete, while* POP-HA$_L^U$ *is* coNP-*hard when* $l_{max} = 3 = u_{max}$.

4.2 Lower Quota 2

Next, we complement the results of Section 4.1 and show that all problems except for POP-HA$_L^U$ become polynomial-time solvable, while POP-HA$_L^U$ remains NP-complete for lower quota 2. For the former result, we use the general factor problem and a recent result from the world of factor theory.

GENERAL FACTOR PROBLEM

Input: Graph $G = (V, E)$ with edge weights $w: E \to \mathbb{R}$ and demands $B_v \subseteq \{0, \ldots, |V|\}$ for each $v \in V$.

Task: Find a subgraph H of maximum weight such that $\deg_H(v) \in B_v$, for each $v \in V$, where $\deg_H(v)$ is the degree of the vertex v in H.

In a recent paper by Dudycz and Paluch [21], a pseudo-polynomial algorithm was given for the restricted case where the maximum gap, i.e., the maximum number of missing adjacent values in any degree list B_v, is at most 1.

PROPOSITION 4.1 (DUDYCZ AND PALUCH [21]). *If there is no* $v \in V$ *and* $p \geq 2$ *such that* $k \in B_v, k+1, \ldots, k+p \notin B_v$ *and* $k+p+1 \in B_v$, *then the* GENERAL FACTOR PROBLEM *can be solved in* $O(W|E|\,|V|^6)$ *time, where* W *is the maximum edge weight.*

This theorem leads to the following corollary.

Corollary 1. *Given a* W-HA$_L^U$ *instance with* $l_{max} = 2$, *a maximum weight matching can be computed in polynomial time if the highest edge weight is polynomial in the size of the graph.*

PROOF. This immediately follows by setting $B_a = \{0, 1\}$ for each applicant $a \in A$ and $B_p = \{0, \ell_p, \ldots, u_p\}$ [1] for each project $p \in P$. Since $\ell_p \leq 2$, the gaps are of size at most 1. □

We further show how to combine the methods of Cornuéjols [17] and Dudycz and Paluch [21] to design a faster algorithm for our special case, which is also easier to implement by giving a Turing reduction to the maximum weight matching problem in graphs without quotas. This reduction is achieved by constructing gadgets similar to the one designed by Cornuéjols [17] and by exploiting a key lemma of Dudycz and Paluch [21] on the existence of augmenting paths and the structure of larger weight matchings.

THEOREM 2. POPV-HA$_L^U$ *and* POV-HA$_L^U$ *can be solved in* $O(|V|^3|E|)$ *time, while* PERPO-HA$_L^U$ *can be solved in* $O(|V|^3|E|^2)$ *time if* $l_{max} = 2$.

[1] Note that in the paper of Cechlárová and Fleiner [16], the applicants technically had capacities as well, which can be easily implemented by setting $B_a = \{0, \ldots, c(a)\}$ for each applicant a with capacity $c(a)$.

Following this, we show that it is NP-complete to determine whether a popular matching exists if $l_{\max} = 2 = u_{\max}$. For this we use a simple graph transformation translating popular matchings in non-bipartite instances to popular matchings with lower quota 2.

THEOREM 3. POP-HA$_L^U$ *is* NP-*complete even if* $l_{max} = 2 = u_{max}$ *and* $\Delta_P = 2$.

5 PARAMETERIZED COMPLEXITY

In this section, we study the parameterized complexity of our four problems with regard to five different parameters. These parameters are identical to the ones used by Boehmer and Heeger [11]: n, the number of applicants, m, the number of projects, m_{quota}, the number of projects with a lower quota greater than 1, m_{closed}, the number of closed projects in the matching, and finally, m_{open}, the number of open projects in the matching. In real life instances—such as in the bachelor project allocation at Hasso Plattner Institute [38], where 70 students choose from 15 projects—one would expect the number of projects being allocated to be small in comparison to the number of students.Depending on the instance, either m_{open} or m_{closed} would seem like a very suitable candidate for a parameterized algorithm. On the other hand, while n is often large, our W[1]-hardness result for this parameter for POP-HA$_L^U$ eliminates the possibility of any fixed parameter algorithm for any smaller, maybe more realistic, sub-parameter of n. For a brief introduction to parameterized complexity, we refer to Cygan et al. [19] and Section **??** in the appendix.

5.1 Parameterization by n

First we observe the following theorem from Arulselvan et al. [2, Theorem 4], which was later improved by Marx et al. [34].

PROPOSITION 5.1 (ARULSELVAN ET AL. [2]). *In an instance* $\mathcal{I}$ *of* W-HA$_L^U$ *such that the underlying graph has a treewidth* **tw**, *a maximum weight matching can be found in* FPT *time in* $\mathbf{tw} + u_{max}$.

From this proposition, the inequalities $\mathbf{tw} \leq \min(n, m)$ and $u_{\max} \leq n$, and our results in Section 3 follows the fixed parameter tractability with regard to the parameter n.

Corollary 2. W-HA$_L^U$, POPV-HA$_L^U$, POV-HA$_L^U$, *and* PERPO-HA$_L^U$ *are in* FPT *when parameterized by* n.

While this result relies on the machinery of tree decompositions and treewidth, we further complement this result by showing that the maximum weight matching admits a kernelization if the weights are encoded in unary. For this we exploit that if a subset of applicants could be matched to more than n different projects achieving the same weight, then we can delete at least one of these projects and not change the maximum weight we can reach.

Observation 2. *An instance of* W-HA$_L^U$ *with maximum weight* W *admits a kernelization with* $O(2^n n^2 W)$ *applicants and projects.*

Even though it follows from the above observation that the other three considered problems become fixed parameter tractable when parameterized by n, POP-HA$_L^U$ remains W[1]-hard. We show this by modifying the proof of Boehmer and Heeger [11, Theorem 2], who reduce from the classic MULTICOLORED INDEPENDENT SET problem to prove that finding a stable matching is W[1]-hard when parameterized by n.

THEOREM 4. *POP-HA$_L^U$ parameterized by n is* W[1]*-hard.*

5.2 Parameterization by m

Next we turn to POP-HA$_L^U$ and show that it is fixed parameter tractable when parameterized by the number of projects m. For this, we reduce POP-HA$_L^U$ to the PARAMETRIC INTEGER PROGRAM problem.

PARAMETRIC INTEGER PROGRAM

Input: Matrices $B \in \mathbb{Q}^{n\times m}, C \in \mathbb{R}^{m\times k}$, and a vector $d \in \mathbb{R}^k$.

Question: For any $b \in \mathbb{Z}^m$, such that $Cb \le d$, does there exist an $x \in \mathbb{Z}^n$ such that $Bx \le b$?

The feasibility of a PARAMETRIC INTEGER PROGRAM can be decided in $O(f(n,m)poly(||B,C,d||_\infty,k))$ time, as shown by Eisenbrand and Shmonin [22]. Using this we can now show that POP-HA$_L^U$ is indeed in FPT when parameterized by m. We first construct a matrix C and vector d such that all feasible matchings are represented by solutions to $Cb \le d$. Then we construct the matrix B such that any solution to $Bx \le b$ represents a feasible matching that is more popular than the matching represented by b, thus ensuring that the parametric integer program is feasible if and only if no popular matching exists. To bound the dimensions of the matrices we use that we can assign each agent a type based on their preferences, of which there can be at most $O((m+1)!)$.

THEOREM 5. *POP-HA$_L^U$ is in* FPT *when parameterized by m.*

PROOF SKETCH. Our goal is to encode the matching instance in such a way that the resulting PARAMETRIC INTEGER PROGRAM is feasible if and only if no popular matching exists. For this we choose the matrix B and vector d so that all possible matchings can be represented by a vector b satisfying $Cb \le d$. Further the matrix B should be chosen in such a way that the vector x represents a matching that is more popular than the matching represented by b.

Notation. Since we parameterize by m, each applicant $a \in A$ can be uniquely identified by her preference structure over P. There are at most $O((m+1)!)$ different preference structures, hence we can partition A into $t \in O((m+1)!)$ types. Let $A_1, \ldots, A_t$ be this partition such that any two applicants in the same set have identical preference lists. We refer to the projects that appear in the preference lists of applicants in A_i as N_i. Furthermore we slightly alter the notation of the previous sections to follow [10] and define the vote of a vertex as

$$\text{vote}_a(p_1,p_2) = \begin{cases} 1, & \text{if } p_1 \succ_a p_2, \\ 0, & \text{if } p_1 = p_2, \\ -1, & \text{if } p_2 \succ_a p_1. \end{cases}$$

By the definition of popularity, matching M' is more popular than M if and only if $\sum_{a\in A} \text{vote}_a(M'(a), M(a)) \ge 1$ holds. For easier notation for any type $i \in [t]$ we define $\text{vote}_i(p_1,p_2) = \text{vote}_a(p_1,p_2)$ where $a \in A_i$ is an applicant of type i.

Construction of C. As noted earlier, our goal is to construct the linear program represented by B and d in such a way that every feasible matching is a solution to this linear program. To reach this, for each $i \in [t]$ and $p \in N_i$ we create a variable x_i^p that should indicate how many applicants of type A_i are matched to project p. Moreover we add one variable x_i^i indicating the number of unmatched applicants of type i. Furthermore for each project $p \in P$ we create a variable o_p that should indicate whether project p is open or closed. This now leads us to the following linear program

$$\sum_{p\in N_i\cup\{i\}} x_i^p = |A_i|, \qquad \text{for each } i \in [t] \quad (1)$$

$$\sum_{i\in[t]:\,p\in N_i} x_i^p - o_p u_p \le 0, \qquad \text{for each } p \in P \quad (2)$$

$$\sum_{i\in[t]:\,p\in N_i} x_i^p - o_p \ell_p \ge 0, \qquad \text{for each } p \in P \quad (3)$$

$$x_i^p \ge 0, \qquad \text{for each } i \in [t] \text{ and } p \in N_i \quad (4)$$

$$o_p \in [0,1], \qquad \text{for each } p \in P \quad (5)$$

Here Constraint (1) enforces that all applicants are either matched or unmatched. With Constraints (2) and (3) we ensure that the number of applicants matched to an open project is between its lower and upper quota and the number of applicants matched to a closed project is 0. Note that each feasible matching is a solution to this ILP. Currently, any solution to this ILP is of the form $(x_1^{p_1}, \ldots, x_t^{p_m}, o_{p_1}, \ldots, o_{p_m})$. This, however, is not enough to fully model the ILP we need for B. As a first step, for each variable of the form x_i^p we add a second copy. Furthermore we add $2t$ variables $b_1, \ldots, b_{2t}$ with $b_{2i} = |A_i| = b_{2i+1}$, then we add $2m$ variables that are forced to be 0, and one variable that is forced to be -1. After this, each solution b is of the form $(|A_1|, |A_1|, \ldots, |A_t|, |A_t|, x_1^{p_1}, x_1^{p_1}, \ldots, x_t^{p_m}, x_t^{p_m},$ $o_{p_1}, \ldots, o_{p_m}, \underbrace{0, \ldots, 0}_{2m \text{ times}}, -1)$.

Construction of B. We design B to ensure that there is a matching M' that is more popular than the matching M induced by b. For any type $i \in [t]$ and projects $p, p' \in N_i \cup \{i\}$, we create a variable $x_i^{p\to p'}$ that should indicate the number of applicants of type i who were matched to project p in M and are matched to project p' in M'. Furthermore, for each project $p \in P$, we again create a variable o'_p indicating whether project p is open or closed in M'. This now allows us to construct the final ILP.

$$\sum_{p,p'\in N_i\cup\{i\}} x_i^{p\to p'} = |A_i|, \quad \text{for each } i \in [t] \qquad (6)$$

$$\sum_{p'\in P\cup\{i\}} x_i^{p\to p'} = x_i^p, \quad \text{for each } i \in [t] \text{ and } p \in N_i \qquad (7)$$

$$\sum_{i\in[t],p\in P} x_i^{p\to p'} - o'_{p'} u_{p'} \le 0, \quad \text{for each } p' \in P \qquad (8)$$

$$\sum_{i\in[t],p\in P} x_i^{p\to p'} - o'_{p'} \ell_{p'} \ge 0, \quad \text{for each } p' \in P \qquad (9)$$

$$\sum_{i\in[t]p,p'\in N_i\cup\{i\}} -\text{vote}_i(p',p) x_i^{p\to p'} \le -1, \qquad (10)$$

$$x_i^{p\to p'} \ge 0, \quad \text{for each } i \in [t] \text{ and } p, p' \in N_i \qquad (11)$$

$$o'_p \in [0,1], \quad \text{for each } p \in P \qquad (12)$$

Constraint (6) ensures that each applicant has a new partner in M', and Constraint (7) guarantees that each applicant matched to some p in M now has a new partner. Constraints (8) and (9) enforce the lower and upper quota constraints, and finally Constraint (10) ensures that M' is more popular than M. ☐

5.3 Parameterization by m_{quota}

While for POP-HA$_L^U$ we were able to show fixed parameter tractability with regard to m, we now improve this parameter for the other three problems by considering m_{quota}, i.e., the number of projects with a lower quota greater than 1. In order to solve this problem we turn to a special subcase of our matching problems defined by Boehmer and Heeger [11], namely the task of deciding whether, given a certain set of projects P_{open}, there is a matching with our desired property that opens exactly the projects in P_{open}. We show that finding a maximum weight matching that opens exactly the projects in P_{open} is polynomial-time solvable. First we give an algorithm for simply finding a matching that dominates a given matching and afterwards we modify this algorithm to find a maximum weight matching. For this algorithm we utilize the FEASIBLE FLOW WITH DEMANDS problem, which can easily be reduced to the well-known maximum flow problem, see for instance Erickson [24].

FEASIBLE FLOW WITH DEMANDS

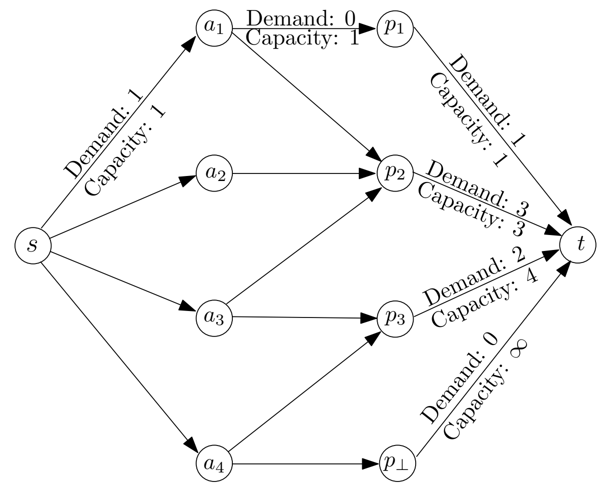
Input: Directed graph $G = (V, E)$, capacities $C\colon E \to \mathbb{N}$, and demands $D\colon E \to \mathbb{N}$.

Question: Is there a flow $f\colon E \to \mathbb{N}$ such that $D(e) \leq f(e) \leq C(e)$ for all $e \in E$?

THEOREM 6. *Given an instance $\mathcal{I}$ of POV-HA$_L^U$ with input matching M and a set $P_{open} \subseteq P$, we can decide in polynomial time whether a matching M' exists that dominates M and opens exactly the projects in the set P_{open}.*

PROOF SKETCH. We construct an instance of FEASIBLE FLOW WITH DEMANDS to solve the problem, see Figure 1 for an example of this construction. The proof of correctness is in the appendix. First, we assume that some $a \in A$ is given who will be the designated applicant to receive a better partner in M'. Now we create a directed bipartite graph G_a with vertices $A \cup P_{\text{open}} \cup \{p_\perp, s, t\}$, where $p_\perp$ will represent the unmatched applicants. We connect s to all vertices in A with demand and capacity of 1 on the edge. Next we connect a to all projects in P_{open} that a prefers to $M(a)$ and all applicants in $A \setminus \{a\}$ get connected to all projects in P_{open} which they prefer to $M(a)$ as well as to $M(a)$ if $M(a)$ is in P_{open}, further if they are unmatched in M, we also connect them to $p_\perp$. All of these edges have no demand and a capacity of 1. Finally we connect every project $p \in P_{\text{open}}$ to t, each with a demand of ℓ_p and with a capacity of u_p and we connect $p_\perp$ to t with no demand and infinite capacity. We solve the FEASIBLE FLOW WITH DEMANDS problem for all $a \in A$ in G_a and if some $a \in A$ exists for which the demands are satisfiable, we return the respective induced matching as M'. That is, if a flow of 1 goes from a to p, we match a to p in M' and if a flow of 1 goes from a to $p_\perp$, we leave a unmatched in M'. Otherwise we return that our matching is Pareto optimal. ☐

Next we show how to generalize this idea to prove that finding a maximum weight matching that opens exactly the projects in P_{open} can be done in polynomial time. For this we need the slightly more complicated but still polynomial-time solvable MAX-COST CIRCULATION problem, see Tardos [41].



$\ell_{p_1} = \ 1 = \ u_{p_1}$
$\ell_{p_2} = \ 3 = \ u_{p_2}$
$\ell_{p_3} = \ 2, \ \ u_{p_3} = 4$
$\ell_{p_4} = \ 2 = \ u_{p_4}$

$a_1 : p_2 \succ p_1 \succ p_3$
$a_2 : p_2 \succ p_4$
$a_3 : p_3 \succ p_2 \succ p_4$
$a_4 : p_3$

Figure 1: Consider an instance of POV-HA$_L^U$ with four applicants $a_1, \ldots, a_4$ and four projects $p_1, \ldots, p_4$ with quotas and preference lists as shown below the graph. If we consider the matching $M(a_1) = p_1, M(a_2) = p_4, M(a_3) = p_4$, and $M(a_4) = a_4$ with $P_{\text{open}} = \{p_1, p_2, p_3\}$, the above instance is the FEASIBLE FLOW WITH DEMANDS instance created in Theorem 6.

MAX-COST CIRCULATION

Input: Directed graph $G = (V, E)$, costs $p\colon E \to \mathbb{R}$, capacities $C\colon E \to \mathbb{N}$, and demands $D\colon E \to \mathbb{N}$.

Task: Find a flow $f\colon E \to \mathbb{N}$ of maximum cost such that for any $v \in V$ it holds that $\sum_{(v,w)\in E} f((v,w)) - \sum_{(w,v)\in E} f((w,v)) = 0$, i.e., all flow gets conserved and such that for any edge $e \in E$ it holds that $D(e) \leq f(e) \leq C(e)$.

THEOREM 7. *Given an instance $\mathcal{I}$ of W-HA$_L^U$ and a set $P_{open} \subseteq P$, we can find in polynomial time a maximum weight matching that opens exactly the projects in P_{open}.*

Using this theorem, we can calculate maximum weight matchings in FPT time in m_{quota}. To do so, we use the fact that projects with a lower quota of at most 1 are always open and then iterate through all projects with a lower quota of at least 2.

THEOREM 8. *Given an instance $\mathcal{I}$ of W-HA$_L^U$, a maximum weight matching can be computed in $O(2^{m_{quota}} poly(|\mathcal{I}|))$, where $|\mathcal{I}| = nm+$ the input size of the weights.*

Using the lemmas in Section 3 this now implies that POPV-HA$_L^U$, PERPO-HA$_L^U$, and POV-HA$_L^U$ are all fixed parameter tractable when parameterized by m_{quota}.

5.4 Parameterization by m_{open}

The next parameter we investigate is m_{open}, the number of projects that are open in the output matching. We show hardness for all our problems, by reducing from either X3C or the EXACT UNIQUE HITTING SET problem.

THEOREM 9. *Given an instance $\mathcal{I}$ of POV-HA$_L^U$/POPV-HA$_L^U$ with a matching M and a parameter m_{open}, it is both* W[1]*-hard to decide whether there is a matching that dominates / is more popular than M and opens exactly m_{open} projects. This remains true even if M opens exactly* 1 *project.*

This approach also generalizes to POP-HA$_L^U$ and PERPO-HA$_L^U$ with the difference that here it is even hard to decide whether there is a desired matching with only one open project.

THEOREM 10. *Given an instance $\mathcal{I}$ of POP-HA$_L^U$/PERPO-HA$_L^U$, it is* coNP*-hard to decide whether there is a popular / perfect Pareto optimal matching that opens exactly* 1 *project.*

As a corollary of our previous proof, we obtain that given a set $P_{\text{open}} \subseteq P$, it is coNP-hard to determine whether a perfect Pareto optimal matching or a popular matching opening exactly the projects in P_{open} exists.

Corollary 3. *Deciding whether there exists a popular matching / perfect Pareto optimal matching that opens exactly the projects in a given subset $P_{open} \subseteq P$ is* coNP*-hard, even if $|P_{open}| = 1$.* [2]

5.5 Parameterization by m_{closed}

As our final parameter we turn to m_{closed}, the number of closed projects in our desired matching and show that all four problems become W[1]-hard when parameterized by m_{closed}. For these results, we reduce from the classic problems MULTICOLORED INDEPENDENT SET and MULTICOLORED CLIQUE. We present the result for POV-HA$_L^U$ here, while the other three results are in the appendix.

THEOREM 11. *Given an instance $\mathcal{I}$ of PERPO-HA$_L^U$, POV-HA$_L^U$, POP-HA$_L^U$, or POPV-HA$_L^U$ and parameter m_{closed}, it is* W[1]*-hard to decide whether a matching of our desired type exists that closes exactly m_{closed} projects.*

PROOF. To show the hardness of POV-HA$_L^U$ we reduce from the multicolored clique problem. For this we are given a graph $G = (V, E)$ with a partition into color classes $V_1, \dots, V_k$ and our goal is to find a clique adhering to this partition. For simpler notation we assume that the graph induced by each color class is an independent set. This allows us to use clique and multicolored clique interchangeably and we can ensure that edges are always between two different colors in our construction.

Construction. First as our projects we include

- for each $v \in V$, a *vertex project* p_v with lower and upper quota $k - 1$;
- for each edge $e \in E$, an *edge project* p_e with lower and upper quota 2;
- for each color $c \in [k]$, a *color project* p_c with lower and upper quota $k - 1$.

Our applicants will be the following.

- For each color $c \in [k]$ and vertex $v \in V_c$, we add $k - 1$ *vertex applicants* $a_v^1, \dots, a_v^{k-1}$, each with preference list $p_c \succ p_v$.
- For each edge $e = \{u, v\} \in E$, we include two *edge applicants*, a_e^u with preference list $p_u \succ p_e$ and a_e^v with list $p_v \succ p_e$.

Our matching M now matches all vertex applicants to their corresponding vertex project, i.e., it matches all applicants in $\{a_v^1, \dots, a_v^k\}$ to p_v, and it matches all edge applicants to their edge project. Finally we set $m_{\text{closed}} := \binom{k}{2}$

$\Rightarrow$. Let us assume we have a clique $C = \{v_1, \dots, v_k\}$ with $v_c \in V_c$ for $c \in [k]$. Then we take the matching M' such that

- For any color $c \in [k]$ the vertex applicants $a_{v_c}^1, \dots, a_{v_c}^{k-1}$ are matched to p_c, thus improving over M for all of them.
- For any vertex $v \in V \setminus C$, i.e., v is not in the clique, we match the vertex applicants $a_v^1, \dots, a_v^{k-1}$ are matched to p_v.
- For any edge $e = (v, u)$ such that v and u are in the clique, we match a_e^v to p_v and a_e^u to p_u, thus improving their matching.
- For any edge $e = (v, u)$ such that at least one of v and u is not in the clique, we match a_e^v and a_e^u to p_e.

The matching M' does not make any applicant worse, adheres to the lower and upper quotas, due to each vertex in the clique having exactly $k - 1$ neighbors in the clique and closes exactly the $\binom{k}{2}$ projects corresponding to the edges in the clique.

$\Leftarrow$. Assume that we have a matching dominating M, which closes exactly $\binom{k}{2}$ projects. It is easy to see that the only way to close $\binom{k}{2}$ projects while simultaneously matching all applicants is to close $\binom{k}{2}$ edge projects, matching the edge applicants to the corresponding vertex projects and the vertex applicants to the color projects. This however implies that all vertices corresponding to promoted vertex projects must have an edge to all the other vertices, thus forming a multicolored clique.

The corresponding reductions for the three other problems can be found in the appendix. □

6 OPEN QUESTIONS

There are two major open questions and future research directions that could be derived from our paper. Firstly, the question whether POP-HA$_L^U$ is in FPT when parameterized by m_{quota} is still open. Secondly, even after the papers of Arulselvan et al. [2], Dudycz and Paluch [21], and now our paper it is still open whether W-HA$_L^U$ with maximum lower quota 2 or the general factor problem with gap at most 1 can be solved in polynomial time, i.e., by eliminating the linear factor on W, the largest weight.

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