Fair and Truthful Mechanism with Limited Subsidy

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ABSTRACT

The notion of *envy-freeness* is a natural and intuitive fairness requirement in resource allocation. With indivisible goods, such fair allocations are unfortunately not guaranteed to exist. Classical works have avoided this issue by introducing an additional divisible resource, i.e., money, to subsidize envious agents. In this paper, we aim to design a truthful allocation mechanism of indivisible goods to achieve both fairness and efficiency criteria with a limited amount of subsidy. Following the work of Halpern and Shah, our central question is as follows: to what extent do we need to rely on the power of money to accomplish these objectives? We show that, when agents have matroidal valuations, there is a truthful allocation mechanism that achieves envy-freeness and utilitarian optimality by subsidizing each agent with at most 1, the maximum marginal contribution of each item for each agent. The design of the mechanism rests crucially on the underlying matroidal M-convexity of the Lorenz dominating allocations. For superadditive valuations, we show that there is a truthful mechanism that achieves envy-freeness and utilitarian optimality, with each agent receiving a subsidy of at most m; furthermore, we show that the amount m is necessary even when agents have additive valuations.

CCS CONCEPTS

• **Theory of computation → Algorithmic mechanism design**.

KEYWORDS

Fair Division; Mechanism Design with Money; Envy-freeness

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1 INTRODUCTION

Consider a group of employees with preferences over their shifts; some may prefer to work in the morning, whereas others may prefer to work in the afternoon. All employees are willing to work, but they may differ in the extent to which they like each time slot. How can shifts be scheduled such that the resulting allocation is fair among employees? This question falls under the realm of the fair division problem, whereby indivisible resources are distributed among heterogeneous participants.

The notion of fairness that has been extensively studied in the literature is *envy-freeness* [22]. It requires that no agent wants to swap their bundle with that of another agent. When the resource to be allocated is divisible, the classical result ensures the existence of an envy-free allocation [41]; when the resource is indivisible, envy-freeness is not a reasonable goal. A relevant example is the case of one item and two agents: no matter how we allocate the single item, the agent who gets nothing envies the other. Hence, the only "fair" solution is to give nothing to both agents.

One way to circumvent this issue is monetary compensation. In the preceding example, the employer may attempt to balance the inequality, e.g., by compensating employees who are assigned to the night shifts. Another example is a governmental body that subsidizes health workers in rural and remote areas.

In mechanism design with money, envy-freeness can indeed be achieved by the well-known Vickrey–Clarke–Groves (VCG) auction mechanism [17, 26, 42] in cases when each agent's valuation is superadditive [36]. In principle, this mechanism is guaranteed to be envy-free, truthful, and utilitarian optimal if one allocates enough money to participants assuming each agent's valuation is superadditive; we will formalize this argument in Section 4. In several applications, however, the resulting outcome of VCG may be unsatisfactory in the following two respects. First, the social planner may have a limited amount of money that can be used to subsidize the participants; for example, employees are usually paid additional compensation up to some limit. Second, when some agent has a non-superadditive valuation, VCG fails to satisfy envy-freeness.

Our contributions In this paper, we study the allocation mechanisms of indivisible items with limited subsidies. Formally, we work in the setting of Halpern and Shah [28]. There, a set of indivisible items together with subsidies are to be distributed among agents who have quasi-linear preferences. The objective is to bound the amount of subsidies necessary to accomplish envy-freeness, assuming that the maximum value of the whole set of items is at most the number m of items for each agent.[1] Although Halpern and Shah [28] and subsequent works [4, 14, 15] are mostly concerned with fairness criteria, we take the mechanism-design perspective: in practice, agents may behave strategically rather than truthfully when reporting their preferences. The goal of this paper is to analyze the amount of subsidies required to accomplish the three basic desiderata of a mechanism: truthfulness, envy-freeness, and utilitarian optimality.

If agents are broadly expressive, i.e., the family of agents' valuations satisfies the so-called *convexity* condition (see [29, Definition 1]), Groves mechanisms are known to be the unique family of mechanisms that satisfy truthfulness and utilitarian optimality [29]. Hence, our hopes are centered on such mechanisms for a rich class of valuations. Although VCG fails to satisfy envy-freeness for monotone submodular valuations [19], Pápai [36] showed that it is envy-free when agents have superadditive valuations, i.e., when agents' preferences do not exhibit substitutability. These results have immediate implications for our setting. We show that, for superadditive valuations, there is a truthful mechanism that achieves envy-freeness and utilitarian optimality, with each agent receiving a subsidy of at most m; furthermore, we show that the amount m is necessary even when agents have additive valuations; see Section 4. In Section 5 of the full version [25], we further observe that, even if an arbitrarily large amount of money is available for use, no mechanism can achieve truthfulness, envy-freeness, and utilitarian optimality simultaneously.

In practice, items may not complement each other, but rather they can be *substitutes.* For example, employees want to work in some time slots, but working all day long is not preferable because of overwork. To capture such a phenomenon, it is natural to consider the class of submodular valuations. Although the impossibility result of combinatorial auction immediately applies to these valuations, our question is whether there is any *well-structured* subclass of submodular valuations that guarantees a desired mechanism.

A subclass of monotone submodular valuations that arises in a number of applications is that of matroidal valuations, i.e., submodular functions with dichotomous marginals. This class of valuations provides a versatile framework that can describe various fair division problems with substitute preferences [11, 35]. A notable example of matroidal valuations is when agents' valuations are governed by uniform matroids, i.e., valuations are binary additive up to some capacity. Taking an example of a Food Bank problem [2], recipients of the service may either like each item or not and may not increase their utilities after receiving a certain amount of foods. Similarly, when allocating courses to students, students are typically able to enjoy a limited number of courses. A more general scenario can be further expressed by the matroidal valuations associated with laminar matroids [20]. When employees are allocated to tasks of various types, they may either approve or disapprove of each task depending on their abilities and can perform certain combinations of tasks under hierarchical constraints; e.g., each employee can be assigned at most two tasks in the morning, three tasks in the afternoon, and four tasks in a day.

Matroidal valuations can also capture other situations when the binary marginal gain follows a more complicated discipline. For instance, consider when a social planner desires to allocate public housing to people in a way that is fair across different social/ethnic groups; One way to achieve group fairness is to model groups as agents and set each group's valuation to be the optimal value of assignments of items to group members [10]. This situation corresponds to the case when agents have binary assignment valuations, a subclass of matroidal valuations. See [11] for further applications of matroidal valuations.

The class of matroidal valuations turns out to be fruitful in the standard setting of fair division without subsidy [6, 8, 9, 11, 27]; particularly, they do admit an allocation rule that is truthful, approximately fair, and efficient. Babaioff et al. [6] very recently designed such a mechanism, called the prioritized egalitarian (PE) mechanism. With ties broken according to a prefixed ordering over the agents, the mechanism returns a *clean Lorenz dominating allocation*, i.e., an allocation whose valuation vector (weakly) Lorenz dominates those under the other allocations and whose bundles include no redundant items that can be removed without decreasing the agents' valuations.

Now, returning to our setting, is it possible to design a desired mechanism with a limited amount of subsidy? We observe that a mere extension of the previously known mechanism [6, 27] does not achieve these properties; informally, by distributing the commonly desirable good (namely, money), some agents, who do not desire any item, may be incentivized to pretend to envy others in order to get subsidized (see Example 3.2). Nevertheless, in Section 3, we are able to design a polynomial-time implementable mechanism, the so-called *subsidized egalitarian* (SE) mechanism, satisfying truthfulness, envy-freeness, and utilitarian optimality with total amount of subsidy at most $n - 1$. Note that this total amount cannot be improved as the worst-case guarantee; e.g., consider one item and n agents; if all agents desire the single item, subsidy 1 must be given to every agent but the one who gets the item to achieve envy-freeness.

Our mechanism resembles the classic VCG in the sense that it *punishes* agents who may potentially decrease others' valuations, while the classic VCG mechanism itself is not directly applicable when we require a limited amount of subsidy (see Example **??** of Section 4). At a high-level, the mechanism hypothetically distributes 1 dollar to each agent and implements the auction over the set of clean Lorenz dominating allocations (**cLD**). By contrast with the PE mechanism, the actual allocation can be taken *arbitrarily*[2] from these allocations; then, each agent who benefits from the allocation pays 1 dollar back to the mechanism designer. A further, perhaps surprising, remark is that, in the output of the SE mechanism, the final utility of each agent *does not* change according to the choice

[1] Halpern and Shah [28] dealt with additive valuations and assumed that the maximum marginal value of each single item is at most 1; our works deals with valuations that are not necessarily additive, and assumes a more general condition $v_i(M) \leq m$ for each agent i.

[2] Note that an appropriate tie-breaking is necessary in the context of fair division without subsidy; see Example 4 of Babaioff et al. [5], which shows that truthfulness is not ensured if a mechanism chooses an arbitrary clean Lorenz dominating.

of an allocation. More strikingly, we show in Theorem 3.7 that the final utility guaranteed by the mechanism is invariant under permutations of agent names.

In Section 3.4, we further discuss our setting without the free-disposal assumption, i.e., each item has to be allocated to some agent. Examples include assigning papers to reviewers and allocating shifts to medical workers. Unfortunately, we observe that, even when agents have binary additive valuations, no truthful and envy-free mechanism allocates all items and returns a Lorenz dominating allocation with each agent being subsidized by at most 1. However, dropping the truthfulness requirement, we show that there is a polynomial-time algorithm that accomplishes envy-freeness and utilitarian optimality while each agent is subsidized at most 1 and all items are allocated to some agent for matroidal valuations. Of independent interest, we also prove in the full version [25, Appendix A.1] that the resulting allocation of the algorithm satisfies an approximate fairness notion, called envy-freeness up to any good (EFX). Due to the space restrictions, we defer the omitted proofs to the full version [25].

Related work The idea of compensating an indivisible resource allocation with money has been prevalent in classical economics literature [3, 30–32, 38–40]. Most classical literature, however, has not considered a situation in which the number of items to be allocated exceeds the number of agents, in contrast to the rich body of recent literature on the multi-demand fair division problem. Halpern and Shah [28] recently extended the model to the multi-demand setting wherein multiple items can be allocated to one agent. Brustle et al. [14] proved that for additive valuations in which the value of each item is at most 1, giving at most 1 to each agent is sufficient to eliminate envies; they also showed that, for monotone valuations, an envy-free allocation with subsidy $2(n-1)$ for each agent exists, assuming that the maximum marginal contribution of each item is 1 for each agent. Note that our work is the first to show that for valuations that are not necessarily additive, envy-freeness and completeness can be accomplished by giving each agent at most 1 subsidy.

Due to its practical importance, a setting where agents have few marginal utility values for the goods has attracted a great deal of attention in various contexts of fair division problems [1, 6–9, 11–13, 24, 27]. One fundamental class of such valuations is the class of binary additive valuations where each agent either approves an item or not [12, 27]. The class of matroidal valuations properly includes that of binary additive valuations and has been studied in several recent works of fair division [6, 8, 9, 11].

2 MODEL

We model fair division with a subsidy as follows. For $k \in \mathbb{N}$, we denote $[k] = \{1, \ldots, k\}$. Let $N = [n]$ be the set of given n agents and let $M = [m]$ be the set of given m indivisible goods. Each agent i has a *valuation function* $v_i : 2^M \to \mathbb{R}_+$ with $v_i(\emptyset) = 0$, where $\mathbb{R}_+$ is the set of non-negative reals. For notational simplicity, we write $v_i(e)$ instead of $v_i(\{e\})$ for all $e \in M$. In this paper, we assume that valuation functions are *monotone*: $v_i(X) \leq v_i(Y)$ for any $X \subseteq Y \subseteq M$. Further, we assume a value-giving oracle for each v_i, i.e., each $v_i(X)$ for $X \subseteq [m]$ can be computed in polynomial time. We focus upon the following classes of valuation functions:

General: We assume that the maximum valuation is bounded, i.e., $v_i(M) \leq m$ holds for all $i \in N$;

Superadditive: A subclass of general valuations, where $v_i(X) + v_i(Y) \leq v_i(X \cup Y)$ holds for any $i \in N, X, Y \subset M$ s.t. $X \cap Y = \emptyset$;

Additive: A subclass of the superadditive valuations, where $v_i(X) = \sum_{e \in X} v_i(e)$ holds for any $X \subseteq M$, $i \in N$;

Binary additive: A subclass of the additive valuations, where $v_i(e) \in \{0, 1\}$ for any $e \in M$ and $i \in N$; we say that agent i *wants* item e if $v_i(e) = 1$;

Matroidal: A superclass of the binary additive valuations, where (i) the marginal contribution $v_i(X \cup \{e\}) - v_i(X)$ is either 0 or 1 for all $X \subsetneq M$ and $e \in M \setminus X$, and (ii) v_i is submodular, i.e., $v_i(X) + v_i(Y) \geq v_i(X \cup Y) + v_i(X \cap Y)$ holds for all $X, Y \subseteq M$.

We remark that a matroidal valuation function is a rank function of a matroid, i.e., a function $r : 2^E \to \mathbb{Z}_+$ such that, for all $X, Y \subseteq E$, (i) $X \subseteq Y \Rightarrow r(X) \leq r(Y) \leq |Y|$, and (ii) $r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$. In [6, 11], this class of valuation functions is referred to as submodular valuations with dichotomous marginals or matroid rank valuations. For a matroidal valuation v_i, each set $X \subseteq M$ such that $v_i(X) = |X|$ is called an *independent set*.

Allocations An allocation of goods is an ordered subpartition of M into n bundles. We denote an allocation by $A = (A_1, \ldots, A_n)$ such that $A_i \subseteq M$ for all $i \in N$ and $A_i \cap A_j = \emptyset$ for any $i \neq j$. In allocation A, agent i receives a bundle A_i of goods. We will deal with two types of allocation: (1) a *complete* allocation (that is, every good must be allocated to some agent), and (2) an incomplete allocation (that is, we can leave some goods unallocated). For an (incomplete) allocation A, we use A_0 to denote the set of unallocated items $M \setminus \bigcup_{i \in N} A_i$.

We introduce notions of efficiency that we use in this paper. The *utilitarian social welfare* of an allocation A is $\sum_{i \in N} v_i(A_i)$, and A is a *utilitarian optimal* allocation if it maximizes the utilitarian social welfare among all allocations. A refinement of utilitarian optimality is Lorenz dominance: given allocations A and B, we say that *A Lorenz dominates B* if, for every $k \in [n]$, the sum of the smallest k values in $(v_1(A_1), \ldots, v_n(A_n))$ is at least as large as that of $(v_1(B_1), \ldots, v_n(B_n))$, i.e., if $v_{i_1}(A_{i_1}) \leq \cdots \leq v_{i_n}(A_{i_n})$ and $v_{j_1}(B_{j_1}) \leq \cdots \leq v_{j_n}(B_{j_n})$ (where $\{i_1, \ldots, i_n\} = \{j_1, \ldots, j_n\} = [n]$), then $\sum_{\ell=1}^{k} v_{i_\ell}(A_{i_\ell}) \geq \sum_{\ell=1}^{k} v_{j_\ell}(B_{j_\ell})$ holds for each k. A *Lorenz dominating allocation* is an allocation that Lorenz dominates every other allocation. The following proposition holds from the definition of Lorenz dominance with $k = n$.

Proposition 2.1. *Every Lorenz dominating allocation is utilitarian optimal.*

Lorenz dominance is also an egalitarian fairness notion in the sense that the least happy agent becomes happier to the greatest extent possible. Another allocation that often achieves the sweet spot of efficiency and fairness is a maximum Nash welfare (MNW) [16]. We say that A is a *maximum Nash welfare (MNW)* allocation if it maximizes the number of agents receiving positive utility and, subject to that, maximizes the product of the positive utilities, i.e., $\prod_{i \in N : v_i(A_i) > 0} v_i(A_i)$. It is known that for matroidal valuations, the set of Lorenz dominating allocations coincides with the set of MNW allocations (more generally, that minimizing a symmetric strictly convex function) [6, 11, 23]. Note that a Lorenz dominating allocation always exists for matroidal valuation functions, whereas it may not exist in general.

To find efficient allocations, it is often necessary to avoid redundancy in allocations. An allocation A is called *clean* if $v_i(A_i \setminus \{e\}) < v_i(A_i)$ for any $i \in N$ and $e \in A_i$. Note that any allocation can be transformed into a clean one without changing valuations by removing items of zero marginal gain from respective agents. For matroidal valuations, A is clean if and only if $v_i(A_i) = |A_i|$ for each $i \in [n]$ (see also [11]). Thus, for matroidal valuations, an allocation A is clean Lorenz dominating if and only if for every clean allocation B, the total size of the smallest k bundles in A is at least as large as that of B for each $k \in [n]$; we will use this in Section 3.

Fairness with a subsidy Our goal is to achieve an envy-free allocation of indivisible goods using a limited amount of *subsidy*, which is an additional divisible good. We denote by $p = (p_1, \ldots, p_n) \in \mathbb{R}^n_+$ a subsidy vector, whose ith entry p_i is the amount of subsidy received by agent i. For allocation A and a subsidy vector p, we call (A, p) an allocation with a subsidy; we assume that each agent has a standard quasi-linear utility, i.e., the utility of agent i, who obtains a bundle X and subsidy p_i, is equal to: $v_i(X) + p_i$. The envy-freeness for an allocation with a subsidy is defined as follows:

Definition 2.2. An allocation with a subsidy (A, p) is *envy-free* if $v_i(A_i) + p_i \geq v_i(A_j) + p_j$ for all agents $i, j \in N$.

An allocation A is called *envy-freeable* if there exists a subsidy vector p such that (A, p) is envy-free. Halpern and Shah [28] prove that an allocation is envy-freeable if and only if there is no permutation of bundles that results in a higher social welfare. Note that this condition only guarantees the optimality with fixed bundle sets, and hence envy-freeability is weaker than utilitarian optimality. The characterization can be stated in terms of the nonexistence of positive-weight cycles in envy graphs (see Theorem 3.15).

Mechanisms In each subsequent section, we assume that a valuation function of each agent is taken from some specified function class V. For example, in Section 3, we let V be the set of all matroidal functions on M. A *valuation profile*, or just a *profile*, is a tuple $(v_1, \ldots, v_n) \in V^N$ of the valuation functions of the all agents in N. For resource allocation with a subsidy, a *mechanism* is a mapping from valuation profiles to *outcomes*, i.e., allocations with a subsidy. A mechanism first asks each agent to report a valuation function and then outputs an allocation with subsidy on the basis of the reported valuations. We notice that the reported valuations may be different from the true ones.

Some agents may have incentives to report a false valuation function to obtain a larger utility. To prevent such manipulation, truthfulness is a standard requirement for mechanisms. A mechanism is *truthful* if reporting the true valuation function maximizes the agent's utility, given the fixed reports of the other agents. A more precise definition is as follows: for every agent i, every profile $(v_1, \ldots, v_n) \in V^N$, and every $v'_i \in V$, if we denote by (A, p) and (A', p') the outputs of the mechanism for the profiles $(v_1, \ldots, v_i, \ldots, v_n)$ and $(v_1, \ldots v'_i, \ldots, v_n)$, respectively, then $v_i(A_i) + p_i \geq v_i(A'_i) + p'_i$.

We say that a mechanism satisfies property P if it outputs an allocation satisfying P. For example, a mechanism satisfies envy-freeness if it outputs an envy-free outcome, and similarly for other properties such as MNW, completeness, and utilitarian optimality.

3 MATROIDAL VALUATIONS

In previously explained applications such as shift scheduling, goods usually have substitute properties; therefore, we are interested in the setting with submodular valuation functions. For such a setting, can we design a mechanism that simultaneously achieves truthfulness, efficiency, and fairness with small amount of subsidies?

Generally, the impossibility result of the combinatorial auction applies to monotone submodular valuations [19]; we are, however, able to answer this question affirmatively for the class of matroid rank valuations, i.e., submodular functions with dichotomous marginals. By setting the domain V of the valuation functions as matroidal functions, we can show that giving at most 1 subsidy to each agent suffices to accomplish these goals. Note that such a mechanism has not been shown to exist even for binary additive valuations. Our main theorem in this section is stated as follows:

Theorem 3.1. *For matroidal valuations, there is a polynomial-time implementable mechanism that is truthful, utilitarian optimal, and envy-free with each agent receiving subsidy* 0 *or* 1*, and the total subsidy being at most* $n - 1$.

Before presenting our mechanism, let us illustrate the difficulty that arises by subsidizing agents: The following simple example shows that we have to give some subsidy to agents who want nothing even when the agents have binary additive valuations.

Example 3.2. Consider two agents, Alice and Bob, and one item $M = \{e\}$ with each agent either wanting the item or not (i.e., valuation for the item is either 0 or 1). Suppose that there is a mechanism that is truthful, envy-free, and utilitarian optimal. Consider two profiles P_1 and P_2. In P_1, both agents want the single item. In this case, the outcome must be such that one agent receives the item and the other receives nothing. Without loss of generality, we assume that Alice receives the item. By envy-freeness, Bob must obtain at least 1 subsidy. In P_2, Alice reports that she wants the item but Bob does not; then, the item must be allocated to Alice who wants the item by utilitarian optimality. Now, it appears that no subsidy is needed in P_2 because agents do not envy each other. However, it turns out that we *do* have to subsidize the agent who wants nothing; otherwise, Bob benefits by misreporting that he wants the item. Hence, we must allocate and subsidize as follows:

P_1:

	Alice	Bob
$v_i(e)$:	$\boxed{1}$	1
subsidy:	0	1

P_2:

	Alice	Bob
$v_i(e)$:	$\boxed{1}$	0
subsidy:	0	1

Note that, for a Lorenz dominating allocation, we can easily compute the amount of subsidy required to make it envy-free by the results of Halpern and Shah [28] (Theorem 3.15 in Section 3.4); however, as we observed in Example 3.2, the mechanism should account for an exponential number of profiles if it aims to compute the minimum amount of additional subsidies to achieve truthfulness. Rather, the mechanism "generously" distributes subsidies.

Our mechanism, which we refer to as *subsidized egalitarian* (SE), proceeds as follows. First, it arbitrarily chooses a clean Lorenz dominating allocation that coincides with a clean MNW and is thus guaranteed to exist under matroidal valuations [6, 11]; then, it subsidizes agents with the following condition: the valuation of allocated bundle is (i) the same as the worst (clean) Lorenz dominating allocation and (ii) not the largest among the agents. The mechanism thus ensures that the utility of agent i is equal to the valuation of the worst clean Lorenz dominating allocation plus 1 if she is not the one who receives the largest bundle.

Recall that, for matroidal valuations, allocation A is clean if and only if $v_i(A_i) = |A_i|$ for any $i \in N$. For a profile $P = (v_1, \dots, v_n)$, let $\mathbf{cLD}[P]$ be the set of clean Lorenz dominating allocations. To ease notation, we often omit the argument P if no confusion will arise. Formally, our mechanism is summarized as follows.

Subsidized Egalitarian
1. Allocate items according to an arbitrarily chosen $A \in \mathbf{cLD}$.
2. Give 1 subsidy to each $i \in N$ if (i) $|A_i| = \min_{B \in \mathbf{cLD}} |B_i|$ and (ii) $|A_i| < \max_{j \in N} |A_j|$.

The mechanism returns a utilitarian optimal allocation according to the property of Lorenz dominating allocations. Clearly, the subsidy for each agent is 0 or 1. The total subsidy is at most $n - 1$ since at least one agent (who receives $\max_{j \in N} |A_j|$ items) gets no subsidy. Remarkably, we observe that the difference between the valuations of the best and the worst Lorenz dominating allocations is at most one for every agent (Proposition 3.6) and that the utility of each agent does not change with the choice of an allocation in Step 1 (Proposition **??**). Hence, the utility of each agent is at least the valuation of the best clean Lorenz dominating allocation.

Here, we note that the SE mechanism imposes the condition (ii) $|A_i| < \max_{j \in N} |A_j|$ in Step 2 to avoid giving all agents subsidy 1. In fact, a variant of the SE mechanism in which condition (ii) is removed fulfills all properties required by Theorem 3.1 except that the total subsidy is at most n, instead of $n - 1$. It is also worth noting that, without subsidy, simply picking an arbitrary allocation in $\mathbf{cLD}$ does not guarantee truthfulness [5, Example 4]; namely, an appropriate tie-breaking rule is necessary to achieve such property.

Proof outline We will prove that the SE mechanism satisfies the desired properties in Theorem 3.1 through the following steps. First, we will provide the structural properties of $\mathbf{cLD}$ and prove that the SE mechanism is polynomial time implementable in Lemma 3.8. We will further show that the mechanism is envy-free and truthful. Throughout, we assume that all agents have matroidal valuations.

As a preparation for the proof of Theorem 3.1, we introduce some notations. For an allocation A, let $\mathrm{sv}(A)$ be a size vector $(|A_1|, \dots, |A_n|)$ and let $\mathrm{sv}^{\uparrow}(A)$ be a vector obtained from $\mathrm{sv}(A)$ by rearranging its components in increasing order. Recall that a clean allocation A is Lorenz dominating if and only if for each clean allocation B, it holds that $\sum_{i=1}^{k} \mathrm{sv}^{\uparrow}(A)_i \geq \sum_{i=1}^{k} \mathrm{sv}^{\uparrow}(B)_i$ for each $k \in [n]$. Note that $\mathrm{sv}^{\uparrow}(A)$ is unique across all $A \in \mathbf{cLD}[P]$ according to the definition of $\mathbf{cLD}[P]$.

For any finite set E and any $i \in E$, a characteristic vector χ_i is an E-dimensional vector whose ith entry is 1 and whose other entries are all 0. For two vectors $x, y \in \mathbb{Z}^E$, we define $\mathrm{supp}^+(x - y) := \{i \in E \mid x(i) > y(i)\}$ and $\mathrm{supp}^-(x - y) := \{i \in E \mid x(i) < y(i)\}$. For a valuation function v_i and $X \subseteq M$, a set function $v_i|_X$ defined as $v_i|_X(Y) = v_i(X \cap Y)$ for all $Y \subseteq M$ is called a restriction of v_i to X.

Recall that, for a matroidal valuation function v_i, a subset $X \subseteq M$ is called *independent* if $v_i(X) = |X|$. The family of independent sets of any matroidal function is known to satisfy the following *augmentation property*: if both X and Y are independent and $|X| < |Y|$, then there exists an item $e \in Y \setminus X$ such that $X \cup \{e\}$ is also independent. A maximal independent set is called a *base*; by the augmentation property, all bases have the same cardinality.

3.1 Structure of non-redundant Lorenz dominating allocations

We first present the following lemma, shown in the proof of [6, Lemma 17], concerning an operation that moves an allocation closer to another allocation in terms of size vectors. Note that this operation can be interpreted as an augmenting path in the exchange graph of a matroid intersection (see, e.g., [37] for details). In the lemma, we treat that the unassigned items are virtually assigned to agent 0. Recall that, for an allocation A, we use A_0 to denote the set of unallocated items. Also, we assume that v_0 is the rank function of the free matroid, i.e., $v_0(X) = |X|$ for any $X \subseteq M$.

Lemma 3.3. *Let A and B be two clean allocations, and let i be an agent. If $|A_i| > |B_i|$, there exists a sequence of clean allocations $C^0, C^1, \dots, C^r$ with the following properties:*

(i) $C^0 = B$, $k^0 = i$,
(ii) e^t *is an item such that* $e^t \in A_{k^{t-1}} \setminus C^{t-1}_{k^{t-1}}$ *and* $C^{t-1}_{k^{t-1}} \cup \{e^t\}$ *is independent for* $v_{k^{t-1}}$ $(t = 1, \dots, r)$,
(iii) $k^t \in N \cup \{0\}$ *is the index such that* $e^t \in C^{t-1}_{k^t}$ $(t = 1, \dots, r)$,
(iv) C^t *is the allocation that is obtained from* C^{t-1} *by transferring* e^t *from* k^t *to* k^{t-1} $(t = 1, \dots, r)$,
(v) $|A_{k^r}| < |B_{k^r}|$.

Note that C^t is a clean allocation obtained by transferring an item t times from the allocation B. In the transferring process, k^{t-1} loses one item (e^{t-1}) in the $(t-1)$st transfer and receives one item (e^t) in the tth transfer.

Note that $\mathrm{sv}(C^t) = \mathrm{sv}(C^0) + \chi_{k^0} - \chi_{k^t}$ if $k^t \in N$; additionally, if allocation B is utilitarian optimal, then k^r must be in N. A key structure of $\mathbf{cLD}$ is the *M-convex* structure of size vectors. A nonempty set $S \subseteq \mathbb{Z}^E$ is said to be *M-convex* if it satisfies the following (simultaneous) *exchange property*:

(B-EXC) For any $x, y \in S$ and $i \in \mathrm{supp}^+(x - y)$, there exists some $j \in \mathrm{supp}^-(x - y)$ such that $x - \chi_i + \chi_j \in S$ and $y + \chi_i - \chi_j \in S$.

It is known that M-convex sets are also characterized in terms of the following (seemingly weaker but actually equivalent) exchange property [33]:

(B-EXC$_+$) For any $x, y \in S$ and $i \in \mathrm{supp}^+(x - y)$, there exists some $j \in \mathrm{supp}^-(x - y)$ such that $y + \chi_i - \chi_j \in S$.

An M-convex set S is *matroidal M-convex* if $|x_e - y_e| \leq 1$ for any $x, y \in S$ and any $e \in E$. In other words, an M-convex set is matroidal if it is obtained from some matroid on E by translating the characteristic vectors of the bases by the same integral vector. Lemma 3.3 implies that the set of size vectors of the clean allocations and the clean utilitarian optimal allocations satisfy (B-EXC$_+$).

Lemma 3.4. *The following sets are M-convex:*

$$S_1 = \{(|A_0|, |A_1|, \dots, |A_n|) \mid A \text{ is clean allocation}\} \quad \text{and}$$
$$S_2 = \{(|A_1|, \dots, |A_n|) \mid A \text{ is clean utilitarian optimal allocation}\}.$$

Note that, for each of the above M-convex sets S_i for $i = 1, 2$, the following problems are solvable in polynomial time via matroid intersection [18]:

(Initialization) computing an element of S_i, and
(Membership) deciding whether a given size vector is in S_i.

Also, for a given vector x in S_1 or S_2, there is a polynomial time algorithm that finds an allocation whose size vector is equal to x; indeed, we can find such an allocation by computing a clean utilitarian optimal allocation for the profile $P' = (v'_1, \ldots, v'_n)$ such that $v'_i(X) = \min\{v_i(X), x_i\}$ for each $i \in N$ and $X \subseteq N$.

Frank and Murota [23, Theorem 5.7] proved that the set of *increasingly maximal elements*[3] of an M-convex set is a matroidal M-convex set. Further, they showed that, in the matroidal M-convex set, an increasingly maximal element that minimizes a linear function can be found in polynomial time if (Initialization) and (Membership) for the M-convex set can be solved in polynomial time. By combining this with the facts that S_2 is a M-convex set and an increasingly maximal element corresponds to a clean Lorenz dominating allocation, we obtain the following lemma.[4]

Lemma 3.5. *The set of size vectors corresponding to clean Lorenz dominating allocations $S^* := \{\mathrm{sv}(A) \mid A \in \mathbf{cLD}\}$ is a matroidal M-convex set. Additionally, for a given weight $w \in \mathbb{R}^N$ a minimum-weight clean Lorenz dominating allocation* $\arg\min_{s\in S^*} \sum_{i\in N} w_i s_i$ *can be found in polynomial time.*

Since S^* is a matroidal M-convex set, the difference between values of the best and the worst clean Lorenz dominating allocations for each agent is at most one.

Proposition 3.6. $\max_{B,C\in\mathbf{cLD}}(|B_i|-|C_i|) \in \{0,1\}$ *for any* $i \in N$.

The resulting utilities under the SE mechanism are thus invariant under permutations of agent names. Formally, we say that a mechanism is *weakly anonymous* if for any permutation $\sigma\colon [n] \to [n]$, $v_i(A_i) + p_i = v_{\sigma(i)}(A'_{\sigma(i)}) + p'_{\sigma(i)}$ where (A, p) and (A', p') are the outcomes of the mechanism when applied to $P = (v_i)_{i\in[n]}$ and $P_\sigma = (v_{\sigma(i)})_{i\in[n]}$ respectively. Note that this property is weak in the sense that the valuations of the allocated bundles may change.

Theorem 3.7. *The SE mechanism is weakly anonymous.*

Note that $\min_{B\in\mathbf{cLD}} |B_i|$ can be computed in polynomial time for each i using Lemma 3.5, e.g., by setting the weight w as $w_i = 0$ and $w_j = 1$ for all $j \in N \setminus \{i\}$. Hence, the outcome of the SE mechanism can be computed in polynomial time.

Lemma 3.8. *The SE mechanism is polynomial-time implementable.*

Furthermore, the M-convex structure of **cLD** leads to some properties that are useful to prove the truthfulness of the SE mechanism. Due to space limitation, we defer the details to Section 3.1 of the full version [25].

3.2 Envy-freeness of the SE mechanism

Here, we prove that the SE mechanism is envy-free. We remark that the matroidal M-convex structure of **cLD** is not sufficient to prove it because the structure gives no information about the value of a bundle received by anyone other than oneself. Instead, we obtain envy-freeness of the SE mechanism by exploiting Lemma 3.3.

Lemma 3.9. *The SE mechanism is envy-free.*

Proof. Let (A, p) be the pair of clean allocation and subsidy vector returned by the SE mechanism. To obtain a contradiction, suppose that i envies j, i.e., $v_i(A_i)+p_i < v_i(A_j)+p_j$. We separately consider the following three cases: $v_i(A_i) > v_i(A_j)$, $v_i(A_i) < v_i(A_j)$, and $v_i(A_i) = v_i(A_j)$.

Case 1. Suppose that $v_i(A_i) > v_i(A_j)$. This case is impossible since $v_i(A_i) + p_i < v_i(A_j) + p_j$ and $p_i, p_j \in \{0, 1\}$.

Case 2. Suppose that $v_i(A_i) < v_i(A_j)$. As A is a clean allocation, $|A_i|$ must be strictly smaller than $|A_j|$. By the matroid augmentation property, there exists an item $e \in A_j$ such that $v_i(A_i \cup \{e\}) = v_i(A_i)+1$. Let B be the allocation that is obtained from A by moving item e from j's bundle to i's bundle. As $|A_i| < |A_j|$ and A Lorenz dominates B, we have that $|B_i| = |A_i|+1 = |A_j| = |B_j|+1$. Hence, B is also a clean Lorenz dominating allocation. Thus, $\max_{C\in\mathbf{cLD}} |C_i| = |A_i| + 1$ and $\min_{C\in\mathbf{cLD}} |C_j| = |A_j| - 1$, which implies $p_i = 1$ and $p_j = 0$ by Proposition **??**. This contradicts the assumption that i envies j because $v_i(A_i) + p_i = |A_i| + 1 = |A_j| = v_i(A_j) + p_j$.

Case 3. Suppose that $v_i(A_i) = v_i(A_j)$. Note that $|A_i| = v_i(A_i) = v_i(A_j) \le |A_j|$. As $v_i(A_i)+p_i < v_i(A_j)+p_j$, it must be that $p_i = 0$ and $p_j = 1$. Then $|A_j| = \min_{A'\in\mathbf{cLD}} |A'_j| < \max_{k\in N} |A_k|$ because j gets subsidized. Also, $\min_{A'\in\mathbf{cLD}} |A'_i| = |A_i| - 1$ or $|A_i| = \max_{k\in N} |A_k|$ because i gets no subsidy. We observe that $\min_{A'\in\mathbf{cLD}} |A'_i| = |A_i|-1$, because otherwise $\min_{A'\in\mathbf{cLD}} |A'_i| = |A_i| = \max_{k\in N} |A_k| > |A_j|$, and hence $v_i(A_i) = |A_i| > |A_j| \ge v_i(A_j)$, which is a contradiction. As $\{\mathrm{sv}(A') \mid A' \in \mathbf{cLD}\}$ is an M-convex set, there is a clean Lorenz dominating allocation B such that $\mathrm{sv}(B) = \mathrm{sv}(A) - \chi_i + \chi_k$ for some $k \in N$. As A and B are both in **cLD** and hence $\mathrm{sv}^{\uparrow}(A) = \mathrm{sv}^{\uparrow}(B)$, we have that $|B_i| + 1 = |A_i| = |B_k| = |A_k| + 1$. Note that $k \ne j$ because $|A_i| \le |A_j|$ by $v_i(A_i) = v_i(A_j)$.

By applying Lemma 3.3 to B and A (note that the roles are interchanged), we obtain a sequence of clean allocations $C^0, C^1, \ldots, C^r$ with $k^0, k^1, \ldots, k^r$ and $e^1, \ldots, e^r$ where $C^0 = A$, $k^0 = k$, $k^r = i$, and $\mathrm{sv}(C^r) = \mathrm{sv}(C^0) + \chi_{k^0} - \chi_{k^r} = \mathrm{sv}(B)$. If $k^t = j$ for some t, then $\mathrm{sv}(C^t) = \mathrm{sv}(A) + \chi_k - \chi_j$ and $|A_k|+1 = |A_i| \le |A_j|$, and hence C^t is a clean Lorenz dominating allocation with $|C^t_j| < |A_j|$. This implies $p_j = 0$, which is a contradiction. Otherwise (i.e., $k^t \ne j$ for all t), we have $C^r_j = A_j$. Then, there exists an element $e \in C^r_j$ such that $v_i(C^r_i \cup \{e\}) = |A_i|$ by $v_i(C^r_i) = |A_i|-1 < |A_i| = v_i(C^r_j)$ and the matroid augmentation property. Thus, the allocation that is obtained from C^r by transferring e from j to i is a clean Lorenz dominating allocation. This also implies that $p_j = 0$, a contradiction. □

3.3 Truthfulness of the SE mechanism

Finally, we prove that the SE mechanism is truthful. In a setting without money, Babaioff et al. [6] proved that a mechanism is truthful if it satisfies *strong faithfulness* and *monotonicity*. We introduce two similar properties that can be applied to a setting with subsidies: subsidized-monotone and subsidized-faithful.

First, the subsidized-faithfulness requires that, if agent i changes her report from v_i to $v_i|_X$, either (a) i receives X and her subsidy

[3] For a given set of vectors, an *increasingly maximal element* is an element such that the smallest entry is as large as possible; within this, the next smallest entry is as large as possible; and so on.

[4] For matroidal valuations, a clean Lorenz dominating allocation is equivalent to a clean utilitarian optimal allocation A such that the smallest entry of $\mathrm{sv}^{\uparrow}(A)$ is as large as possible; within this, the next smallest entry is as large as possible; and so on [5, 11]. This certifies the equivalence between an increasingly maximal element of S_2 and a clean Lorenz dominating allocation.

does not decrease or (b) i receives a proper subset of X and her subsidy strictly increases. We say that a mechanism is *subsidized-faithful* if

$$v_i(X) + p_i \leq v_i(A'_i) + p'_i \tag{1}$$

for any valuation function $(v_1, \dots, v_n)$, agent $i \in N$, and subset $X \subseteq A_i$, where (A, p) and (A', p') are the allocations with subsidies returned by the mechanism when agents report $P = (v_1, \dots, v_i, \dots, v_n)$ and $P' = (v_1, \dots, v_i|_X, \dots, v_n)$, respectively. Note that the strong faithfulness (i.e., $A'_i = X$ instead of (1)) does not hold for the SE mechanism because $A \in \mathbf{cLD}$ is chosen arbitrarily.

Next, the subsidized-monotonicity means that the (true) utility of an agent is monotone with respect to restriction of her report. Formally, we say that a mechanism is *subsidized-monotone* if the utility of an agent is monotone with respect to the restriction, i.e.,

$$v_i(A_i) + p_i \leq v_i(A'_i) + p'_i$$

for any valuation function $(v_1, \dots, v_n)$, agent $i \in N$, and subsets $X \subseteq Y \subseteq M$, where (A, p) and (A', p') are the allocations with subsidies returned by the mechanism when agents report $P = (v_1, \dots, v_i|_X, \dots, v_n)$ and $P' = (v_1, \dots, v_i|_Y, \dots, v_n)$, respectively. The two properties of subsidized-faithfulness and subsidized-monotonicity ensure the truthfulness of a mechanism.

Lemma 3.10. *A mechanism is truthful if it is subsidized-faithful and subsidized-monotone.*

We show that the SE mechanism is subsidized-monotone and subsidized-faithful by the matroidal M-convex structure of **cLD** and the way to distribute subsidies.

Lemma 3.11. *The SE mechanism is subsidized-faithful and subsidized-monotone.*

By combining Lemmas 3.10 and 3.11, we obtain the truthfulness of the SE mechanism.

3.4 Without the free-disposal assumption

In Theorem 3.1, we presented the so-called SE mechanism, which attains truthfulness, utilitarian optimality, and envy-freeness with each agent receiving a subsidy of 0 or 1. In the mechanism's output, however, the allocation may not be complete (i.e., some items may be left unallocated). In some situations, this disposal of items is not ideal. For example, consider a shift scheduling at a call center or a production factory; all shifts must be allocated to employees in order not to stop the operation, even if no one may find that time slot valuable. Another example is the allocation of research papers to reviewers; every paper must be reviewed by a certain number of reviewers irrespective of whether the paper is attractive or not. Unfortunately, the following theorem shows that no mechanism outputs a complete allocation while attaining all the properties of the SE mechanism (i.e., truthfulness, Lorenz domination, and envy-freeness with each agent receiving a subsidy of at most 1).

Theorem 3.12. *If a truthful mechanism is envy-free, and returns a complete Lorenz dominating allocation, it requires a subsidy of $\Omega(m)$, even when there are two agents with binary additive valuations.*

For matroidal valuations, we provide an algorithm that returns a Lorenz dominating allocation and simultaneously attains completeness and envy-freeness with each agent receiving a subsidy of at most 1 while tolerating a violation of truthfulness.

Theorem 3.13. *For matroidal valuations, there is a polynomial-time algorithm for computing an allocation with a subsidy that is complete, utilitarian optimal, and envy-free, with each agent receiving a subsidy of* 0 *or* 1 *and the total subsidy being at most* $n - 1$.

We construct the allocation required in the theorem by extending an arbitrary clean Lorenz dominating allocation $A = (A_1, A_2, \dots, A_n)$; that is, we initialize A to be the one computed in Step 1 of the SE mechanism. By Theorem 3.1, A then maximizes the utilitarian social welfare $\sum_{i \in N} v_i(A_i)$ and is envy-freeable with a subsidy of at most 1 for each agent. Therefore, we can obtain a desired allocation if we can allocate items in $M \setminus \bigcup_{i \in N} A_i$ while preserving the utilitarian optimality and the bound 1 of the subsidy for each agent. Note that, for binary additive valuations, this task is trivial because an item unallocated in A has a value of 0 for all agents by the utilitarian optimality; hence allocating it to any agent does not cause envy. However, a similar argument does not apply to matroidal valuations, as shown by the following example.

Example 3.14. Let $N = \{1, 2, 3\}$ and $M = \{e_1, e_2, e_3, e_4, e_5\}$ and define the matroidal valuations v_1, v_2, v_3 by $v_1(X) = |X \cap \{e_1, e_2\}|$, $v_2(X) = |X \cap \{e_1, e_2, e_3\}|$, and $v_3(X) = |X \cap \{e_1, e_2, e_3\}| + \min\{1, |X \cap \{e_4, e_5\}|\}$. Then $A = (A_1, A_2, A_3) = \{\{e_1, e_2\}, \{e_3\}, \{e_4\}\}$ is a clean Lorenz dominating allocation. It is not difficult to see that we cannot increase the utility of any agent by allocating e_5, which is currently unallocated. However, if we allocate e_5 to agent 2, the amount $v_3(A_2) - v_3(A_3)$ of envy agent 3 has towards 2 changes from 0 to 1. To eliminate envy for the resultant allocation $A' = (A'_1, A'_2, A'_3) = \{\{e_1, e_2\}, \{e_3, e_5\}, \{e_4\}\}$, we need to pay at least one dollar to agent 2 because her envy towards agent 1 is $v_2(A'_1) - v_2(A'_2) = 1$. Then $v_3(A'_2) + p_2 \geq 3$ while $v_3(A'_3) = 1$, and to eliminate the envy of agent 3 towards agent 2, we must pay at least 2 dollars to agent 3.

We present the *subsidized egalitarian with completion* (SEC) algorithm, which extends any clean Lorenz dominating allocation to a complete allocation while preserving the property that each agent requires at most 1 subsidy.

To describe our algorithm, we introduce the notion of envy graphs. For an allocation A, its envy graph G_A is the complete weighted directed graph whose node set is the agent set N; for each $i, j \in N$, the arc (i, j) has weight $w(i, j) = v_i(A_j) - v_i(A_i)$, which represents the amount of envy of i towards j. This value can be negative if i prefers her bundle to j's bundle. A *walk* Q in G_A is a sequence of nodes $(i_1, i_2, \dots, i_k)$, and its weight is defined as $w(Q) = \sum_{t=1}^{k-1} w(i_t, i_{t+1})$. A walk is a *path* if all nodes are distinct, and a *cycle* if $i_1, i_2, \dots, i_{k-1}$ are all distinct and $i_1 = i_k$. The following theorem is a combination of Theorems 1 and 2 in [28].

Theorem 3.15 (Halpern and Shah [28]). *For any allocation $A = (A_1, \dots, A_n)$ and any nonnegative real $q \in \mathbb{R}_+$, the following two conditions are equivalent:*

- *A is envy-freeable with a subsidy of at most q for each agent.*
- *G_A has neither a positive-weight cycle nor a path with a weight larger than q.*

When these conditions hold, if we set p_i as the maximum weight of any path starting at i in G_A for each $i \in N$, then (A, p) is envy-free.

Note that Theorem 3.15 is shown for general valuations. In the case of matroidal valuations, which are integer-valued, each arc in G_A has an integer weight.

Subsidized Egalitarian with Completion

1. Allocate items according to an arbitrarily chosen $A \in$ **cLD**.
2. For each unallocated item $e \in M \setminus \bigcup_{i \in N} A_i$, do as follows:
   (a) Take an agent i arbitrarily.
   (b) Let $A^{i,e} := (A_1, \dots, A_i \cup \{e\}, \dots, A_n)$. If $G_{A^{i,e}}$ has a positive-weight path ending at i, then take such a path P_i arbitrarily, update i by the initial agent of P_i, and repeat (b). Else, go to (c).
   (c) Update $A \leftarrow A^{i,e}$ (i.e., $A_i \leftarrow A_i \cup \{e\}$).
3. Give 1 subsidy to each agent $i \in N$ such that the envy graph G_A has a path of weight 1 starting at i.

Lemma 3.16. *The following conditions hold throughout the SEC algorithm: (i) $A = (A_1, \dots, A_n)$ is utilitarian optimal, and (ii) G_A has neither a path of weight more than 1 nor a positive-weight cycle.*

By condition (ii) in Lemma 3.16 and Theorem 3.15, the allocation with a subsidy returned by the SEC algorithm is envy-free, with each agent receiving a subsidy of 0 or 1. Furthermore, there is at least one agent $i \in N$ such that G_A has no path of weight 1 starting at i (since otherwise there exists a positive-weight cycle in G_A, which contradicts (ii)). Thus, the total subsidy is at most $n - 1$. By the algorithm and condition (i) in Lemma 3.16, the allocation is complete and utilitarian optimal. To complete the proof of Theorem 3.13, we now estimate the time complexity. The following claim guarantees that the algorithm does not fall into an infinite loop at Step 2 (b).

Lemma 3.17. *In Step 2, for each item e, any agent is chosen as i in (b) at most once, and hence (b) is repeated at most n times.*

Note that Step 1 is the same as that of the SE mechanism. Steps 2 and 3 can be computed by the method used by Halpern and Shah [27], i.e., by applying the Floyd–Warshall algorithm [21, 43]. Thus, the algorithm runs in polynomial time.

4 SUPERADDITIVE VALUATIONS

In this section, we consider a class of valuations that do not possess the substitution property, namely, a class of superadditive valuations. We provide a truthful mechanism that achieves envy-freeness and utilitarian optimality, with each agent receiving a subsidy of at most m. Although the upper bound of the subsidy seems too large, we show that this amount of subsidy is essentially required.

Holmström [29] proved that when the set V of valuations satisfies the *convexity* condition, the Groves mechanisms are the only utilitarian optimal and truthful mechanisms. This result is carried to superadditive valuations, which satisfy convexity. Moreover, for superadditive valuations, some rules of the Groves mechanisms, including the VCG mechanism, satisfy envy-freeness [36].

We require that the subsidy for each agent must be non-negative; to fulfill this goal, we can use the following mechanism:

VCG with an upfront subsidy m

1. Allocate items according to an arbitrarily chosen A^* in $\arg\max_A \sum_{j \in N} v_j(A_j)$.
2. Give $m - \left(\max_A \sum_{j \neq i} v_j(A_j) - \sum_{j \neq i} v_j(A^*_j)\right)$ subsidy to each $i \in N$.

Note that the second term of the subsidy (i.e., $\max_A \sum_{j \neq i} v_j(A_j) - \sum_{j \neq i} v_j(A^*_j)$) is equal to the standard VCG payment. Thus, this mechanism is equivalent to the following mechanism; first, each agent obtains an upfront subsidy m (where $m \geq v_i(M)$ holds $\forall i \in N$). Then, items are allocated using the standard VCG, where each agent pays the VCG payment from the upfront subsidy.

Theorem 4.1. *For superadditive valuations, the VCG with an upfront subsidy m is truthful, utilitarian optimal, and envy-free, and each subsidy is in $[0, m]$.*

For additive valuations, a utilitarian optimal allocation can be computed in polynomial time by allocating each item to the agent who likes the most. Hence, the above mechanism is polynomial-time implementable for a class of additive valuations. However, generally, the problem is NP-hard for superadditive valuations (see, e.g., [34, Proposition 11.5]).

We discuss in Section 4 of the full version [25] that unlike the SE mechanism, Groves mechanisms cannot achieve a limited amount of subsidy even when valuations are binary additive. Here, we define a *Groves mechanism* to be a generalization of a VCG with an upfront subsidy m, where we replace the right term of the subsidy rule in Step 2 with an arbitrary function h that only depends on valuations of the other agents $j \neq i$.

Now, is there any other mechanism that can reduce the amount of subsidy while achieving envy-freeness and utilitarian optimality? The next theorem shows that we need the amount of m for each agent to achieve such objectives, even when valuations are additive.

Theorem 4.2. *For any $\epsilon > 0$, if a mechanism is envy-free and utilitarian optimal, it requires a subsidy of $m(n-1) - \epsilon$ in total, even when n agents have additive valuations such that the value of each item is at most 1.*

5 CONCLUDING REMARKS

We studied the mechanism design for allocating an indivisible resource with limited subsidy. Although our work is concerned with utilitarian optimality, studying the compatibility of truthfulness and fairness with other efficiency requirements, such as completeness and non-wastefulness, is a natural direction. Specifically, the mechanism in Section 4 can allocate all items and achieves the bound of m for additive valuations; it would be interesting to see whether the amount of m is necessary to achieve a truthful, envy-free, and complete mechanism for additive valuations.

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