THE ORDERS OF MAGNITUDE MODELS AS QUALITATIVE ALGEBRAS

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Abstract

This paper provides a unifying mathematical framework for orders of magnitude models used in Qualitative Physics. An axiomatic of the qualitative equality is provided and a general algebraic structure called qualitative algebra is defined. It is shown that the usual model (+,-,0,?) and the extended model recently introduced by Dubois and Prade are particular cases in the class of models that are generated from a partition of the real line. Any of these models can be structured as qualitative algebra. On the other hand, we characterize those qualitative algebras that are isomorphous, in a qualitative sense. Besides, it is shown that all these models can be embedded into one another as qualitative subalgebras.

Key words. Qualitative Physics, Qualitative Algebras, Order of Magnitude Reasoning.

1. Introduction

Economists made a handsome contribution to the interest of reasoning about systems behaviour in a qualitative way. In the sixties, they showed that qualitative models could provide a good representation of some economic systems and that meaningful conclusions could be drawn from pure qualitative data. Without elaborating on a well-defined mathematical framework, such concepts as qualitative vectors and matrices, qualitative linear systems, qualitative solutions, were introduced. A number of methods like comparative statics were then proposed to solve qualitative-model-based problems [Lancaster, 1962, Lancaster, 1966, Maybee, 1980, Quirk, 1981, Ritschard, 1983].

After sometime, the qualitative approach reemerged at the beginning of the eighties in such varied areas as Control Theory [Caloud, 1987, Gentil *et al.,* 1987, Trave and Kaszkurewicz, 1986, Trave, 1988, Trave-Massuyes, 1989] and Artificial Intelligence [De Kleer and Brown, 1984, Dormoy, 1987, Dormoy, 1988, Forbus, 1984, Kuipers, 1984, Raiman, 1986, Piera and Trave-Massuyes, 1989, Trave-Massuyes et *al.*, 1989]. The interest of the Artificial Intelligence community in qualitative methods was motivated by the need for a representation of the physical world close to human patterns. Indeed, Qualitative Physics [De Kleer and Brown, 1984, Forbus, 1984, Kuipers, 1984] originate from the idea of modeling human understanding and reasoning about the physical systems in order to implement computer modules including this specific intelligence. As a matter of fact, when they examine a device, humans do not solve differential equations Instead, reasoning relies on structural decompositions, common sense and basic physical laws. These basic "reasoning pieces" are then combined by some qualitative calculus so that, "mysteriously", such questions as: "What is the use of the device?", "How does it work?", "Why does not it work normally?"... can be answered. Humans are therefore able to provide a scheme of the functionality and behaviour of systems in a very efficient way without requiring formal physics. Qualitative Physics try to reproduce this procedure.

Although new approaches, such as order of magnitude reasoning [Dubois and Prade, 1988, Raiman, 1986], are currently presented, the word "qualitative" still remains associated with a qualitative calculus based on signs. People using qualitative methods in different areas of application have provided models, tools, and methods for handling the qualitative space {+,-,0,?} and a set of mathematical qualitative concepts are now available for solving qualitative-model-based problems. However, they were introduced without connection, whenever practical problems required new theoretical investigations.

To our knowledge, no pure theoretical work has been developed towards a well-defined qualitative algebraic structure. Even the mathematical properties of {+,-,0,?} are not accurately known. It is the authors opinion that henceforth, advances in applied investigations will closely depend on theoretical knowledge.

It therefore seems natural in this paper to first present the axiomatic of the qualitative equality. Then, these properties of the qualitative equality necessary for a good understanding of the main results of the paper are given.

On the other hand, we consider the interesting extension of the sign-based model, involving order

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of magnitudes, recently introduced in [Dubois and Prade, 1988]. The major aim of this paper is to show that both models can be unified under a general algebraic structure called *qualitative algebra* Moreover, they can be viewed as two particular cases of those models which are generated from a real line partition. The finer the partition, the sharper the degree of refinment of order of magnitude becomes. Thus any of these models can be structured as a qualitative algebra (Theorem 5.1).

Further, it is shown that these models are embedded into one another as qualitative subalgebras (Theorem 5.2). This has a very important practical implication since during a calculation, one can move from one model to the next, depending on the degree of refinment locally required.

2. Qualitative equality

In Qualitative Physics, reasoning about magnitudes of objects has often been addressed by considering a model based on three labels: positive, negative, and zero. This has not, however, been sufficient to satisfy some elementary mathematical properties, e.g. closeness of addition. A good model was shown to require a lower level of specification given by the label "indeterminate",that is, either positive, negative or zero. Put together, these four labels make up what is referred to as the *Universe of Description*[1] of the qualitative model (Q-model). Indeed, this particular case illustrates a general rule of major importance "the same object can have different qualitative descriptions (Q-descriptions) according to the level of specification considered". The higher a Q-description, the more quantitative it tends to become implying thus the highest possible level of specification.

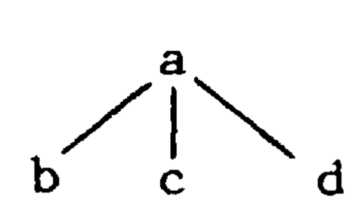
Within this scheme, it may be questioned whether two objects described qualitatively *are equal.* It seems natural to consider them as qualitatively equal if they are equal *at some level of specification* of our universe of description. This relation will be termed qualitative equality and can be formalized as follows.

Consider a non empty set S (Universe of description), and an order relation < defined on S. *Qualitative equality* (Q-equality) refers to the binary relation induced by < on S and defined by:

$a \approx b$ if there exists $x \in S$ such that $x<a$ and $x<b$.

It can be noticed that this definition agrees with the above discussion.

The Q-equality is *reflexive and symmetric*. However, it is not usually transitive. A simple example showing the absence of transitivity is as follows. Consider S-(a,b,c,d) and the order relation given by the following figure:



[1] Also called the *Quantity Space* [De Kleer and Brown, 1984, Kuipers, 1984].

then, it clearly appears that $b \approx a$, $a \approx c$ and $b \approx c$.

On the other hand, the order relation may be such that all the elements of S are Q-equal, implying that the Q-equality is trivially transitive. This corresponds to what we call *degenerated Q-equality*. Notice however that, in all cases, transitivity is preserved if the middle element is < to the two others.

Characterizing the order relations which induce transitive Q-equalities on one side and degenerated Q-equalities on the other side is particularly interesting. A detailed study can be found in [Piera and Trave-Massuyes, 1989].

3. Basic elements

Consider a set S and a Q-equality, then given $a \in S$ we call S_a and I the following subsets of S:

$$S_a = \{x \in S;\ x \approx a) \textit{ and } 1_a = \{x \in S;\ x < a).$$

It can be stated that $(S,\approx)$ is *irreducible* if given $a,b \in S$ such that $S_a=S_b$, then a=b.

An element $a \in S$ is called a *basic element of* $\approx$ if $\approx$ restricted to S_a is degenerated, i.e., if for all $x,y \in S_a$, $x \approx y$. From these definitions, the following properties can be deduced :

(i) If x<y, then $S_x \subset S_y$.

(ii) a is a basic element of $\approx$ if, and only if, $\approx$ is degenerated on I_a .

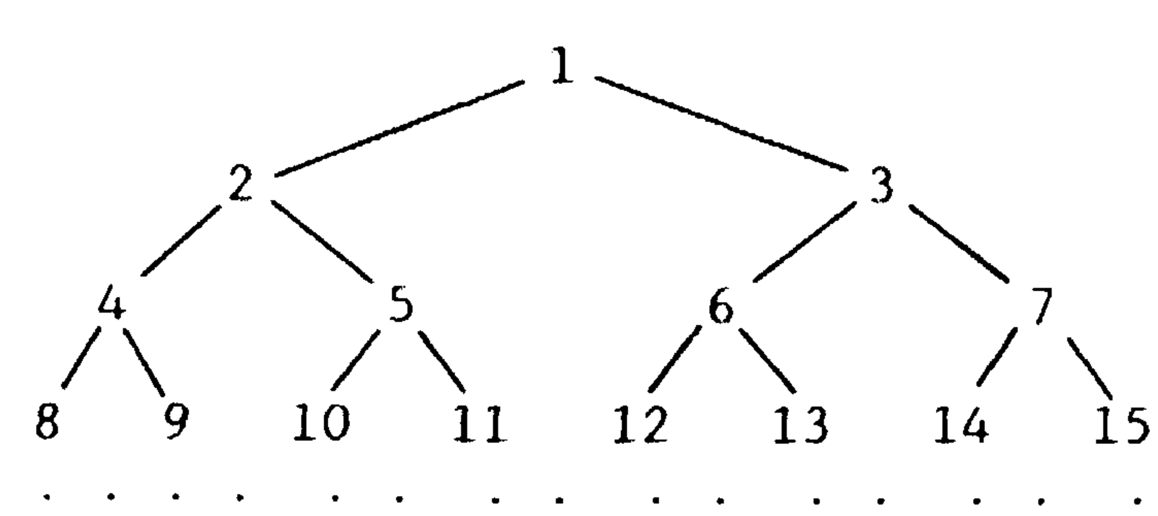
(iii) If a is a basic, element and $S_b \subset S_a$ then b is also basic and $S_b=S_a$. In particular, if a is basic, for all $b<a$, b is basic.

(iv) If $x \approx a$, and a is basic, then there exists a basic element b such that b<x.

If S is finite, the basic elements are the minimal elements of <. However, not all Q-equalities have basic elements. For example, given the set of natural numbers N, $x \in N$ is said to be on the k level, $k \in N \cup \{0)$, if $2^{k<}x<2^{k+1}$. We define an order relation on N, denoted by $<_N$, such that for any x,y£N on the levels s and r respectively, we have :

$$x \leqslant_N y \textit{ if } r \leqslant s \textit{ and } x=2^{s-r}+q,$$
$$\textit{where } q \in N \cup \{0\} \textit{ and } 0 \leqslant q \leqslant 2^{s-r}-1.$$

$\leqslant_N$ can be graphically depicted as follows :



The Q-equality induced by $<_N$ has no basic elements since for any n^N, we have 2n, 2n+l£l_n and 2n#2n+l

Thus (S,~) is said to be *basically complete* if for all x£S, there exists a basic element a^S such that x~a. In particular, if S is finite, then S is basically complete.

Consider that $(S,\approx)$ is basically complete and denote $\mathcal{B}=\{a\in S;$ a is basic$\}$. $B\subset\mathcal{B}$ is said to be a *qualitative base* (Q-base) of S if B fulfils the following conditions:

(i) For any $x\in S$, there exist $a\in B$ such that $x\approx a$;
(ii) $(B,\approx)$ is irreducible.

It is clear that any basically complete set S has a base. If B and B' are two bases of S, then they share the same cardinality. The number of elements of a base is therefore a qualitative invariant of S. Consequently, this number can be defined as the *dimension* of S [Piera and Trave-Massuyes, 1989].

If $(S,\approx)$ is irreducible and basically complete, then S has a unique base B, and for any $x\in S$ there exists a unique subset B_x of B such that $S_x=\bigcup_{a\in B_x} S_a$.

The Q-models which are useful in Qualitative Physics belong to this category.

Finally, given $(S,\approx)$, $(T,\approx)$, and an application f from S to T, f is said to be a *qualitative morphism* (Q-morphism) if it maintains Q-equality, i.e., for all $a,b\in S$ such that $a\approx b$, we have $f(a)\approx f(b)$.

4. Qualitative algebras

First, consider a problem arising from the Q-equality concept. When defining operations between objects of a given set, they are of course always compatible with the classical equality. This is not true of the Q-equality and attention must be paid to the operations which must satisfy the compatibility property. Therefore, given a set S, $S\neq\emptyset$, a Q-equality defined on S and an internal operation denoted *, * must be *compatible with* , i.e., for all $a,b,c,d\in S$ such that $a\approx b$ and $c\approx d$, we have $a*c\approx b*d$.

Naturally if * is compatible with the order relation which induces the Q-equality ≈, then * is also compatible with ≈. The converse is not true. Indeed, if≈~ is degenerated, then any operation defined on S is compatible with ≈ whereas it cannot be assured that it is so with the order relation.

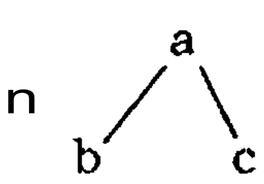
Consider a set S, $S\neq\emptyset$, on which we define a Q-equality ≈ and two internal operations, denoted + and *, which are compatible with ≈ and satisfy the following properties:

(i) + and * are *qualitatively associative* (Q-associative), i.e., for all $a,b,c\in S$, we have $a+(b+c)\approx(a+b)+c$ and $a*(b*c)\approx(a*b)*c$.
(ii) +, * are *qualitatively commutative* (Q-commutative), i.e., for all $a,b\in S$, $a+b\approx b+a$ and $a*b\approx b*a$.
(iii) the product * is *qualitatively distributive* (Q-distributive) with respect to the sum +, i.e., for all $a,b,c\in S$, we have $a*(b+c)\approx(a*b)+(a*c)$.

Then, $(S,\approx,+,*)$ is said to be a *qualitative algebra* (Q-algebra).

If the Q-equality ≈ is degenerated, then $(S,\approx,+,*)$ is called a *degenerated Q-algebra.* Besides, if the commutativity, associativity, and distributivity of the two internal operations of a Q-algebra are satisfied by using - instead of ≈, $(S,\approx,+,*)$ is then said to be a *strict Q-algebra.*

Notice that the definition of Q-algebra does not require the existence of relevant elements with respect to the operations. In particlar, neither neutral nor zero elements are required. This fulfils our objective of defining the most general structure which could simultaneously capture the essential properties for calculation.

A Q-algebra $(S,\approx,4,*)$ is said to have neutral (zero) element with respect to the sum (product) if there exists $0_+\in S$ $(e_+\in S)$ such that for all $a\in S$, $a+0_+=0_++a=a$ $(e_++a=a+e_+=e_+)$. In the definitions of zero and neutral elements, the strict equality is imposed. This is because it seems natural to require them to be "qualitatively unique", in the sense that if two non strictly equal elements are zero (neutral), they should be Q-equal. This is not satisfied when using the Q-equality in the definitions. For example, consider A={a,b,c) and the order relation . We define two internal operations + and * such that $(A,\approx,+,*)$ i a Q-algebra and e+=a. Then, we have $b\approx e_+$, $c\approx e_+$ and b#c. This does not occur if the strict equality is used since strict uniqueness is even guaranteed.

Consider a Q-algebra $(S,\leqslant,+,*)$ and $T\subset C$, $T\neq\emptyset$. T is said to be a *qualitative subalgebra* (Q-subalgebra) of S if $(T,\leqslant,+,*)$ is a Q-algebra.

It is interesting to point out that all the Q-subalgebras of a strict Q-algebra are also strict. On the other hand, a non strict Q-algebra may have strict and non strict Q-subalgebras.

Given A and B, two Q-algebras and f, a mapping from A on to B. f is said to be a *Q-morphism of algebras* if f preserves the qualitative equality and satisfies the two following conditions:

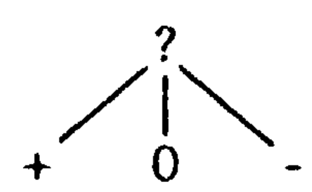
(i) $f(a+b)\approx f(a)+f(b)$;
(ii) $f(a*b)\approx f(a)*f(b)$.

If f is a one-to-one mapping, then f is called a *qualitative isomorphism of algebras* (Q-isomorphism).

5. Qualitative algebras of orders of magnitude

In Qualitative Physics, the word "qualitative" remains associated with a qualitative sign-based calculus. Working with signs was shown to provide an aggregation level similar to the one involved in human reasoning about the physical world. Indeed, this reasoning is mainly based on the variations of significant quantities, which can be formally represented by the signs of the derivatives. On the other hand, this approach was shown to be an alternative for managing unprecise knowledge. Thus the knowledge of parameter signs in a model sometimes suffices to draw significant conclusions on the modelled system.

The qualitative procedure consists of partitioning the real line in three classes by considering the equivalence relation given by the same sign on the set of reals R. Labels +, 0, and - are chosen to represent the classes of positive, zero, and negative real numbers, respectively. These three labels thus constitute the highest level of specification of our Q-model. Now, a lower level of specification is necessary to handle real numbers with indeterminate sign. It is represented by label ?. The universe of description of this model is thus S={+,0,-,?}. An order relation < is defined, which should convey the idea that the higher the level of specification, the lower the possibility of Q-equality. < is represented below:



Notice that Q-equality ≈ induced by < is reflexive, symmetric but not transitive. But, in accordance with section 2, transitivity exists if the middle element is < to the two others, i.e., if the middle element is different from ?.

Two internal operations ⊕ and ⊗ are defined by:

⊕	+	-	0	?
+	+	?	+	?
-	?	-	-	?
0	+	-	0	?
?	?	?	?	?

⊗	+	-	0	?
+	+	-	0	?
-	-	+	0	?
0	0	0	0	0
?	?	?	0	?

These operations are compatible with the Q-equality ≈ induced by <. The tables also show that ⊕ and ⊗ are associative and commutative in the strict sense. Distributivity of ⊕ with respect to ⊗ is also strictly verified. These conditions provide (S,≈,⊕,⊗) with a structure of strict Q-algebra. (S,≈,⊕,⊗) is referred to as the *Q-algebra of signs.*

Notice that the Q-algebra of signs has neutral and zero elements for ⊕ and ⊗. Put together, this property and the fact that it is a *strict* Q-algebra make it very suitable for qualitative calculus purposes [Trave-Massuyes *et al.*, 1989].

An interesting extension of this model was recently proposed in [Dubois and Prade, 1988]. The main idea was to include orders of magnitude, advantageously setting total indetermination (specification level ?) further away by creating new levels of specification. The real line is now partitioned into seven classes with associated labels: negative large (NL), negative medium (NM), negative small (NS), zero (0), positive small (PS), positive medium (PM), positive large (PL).

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    NL       NM       NS   0   PS       PM       PL
--------+-------+--------+--------+-------+-------
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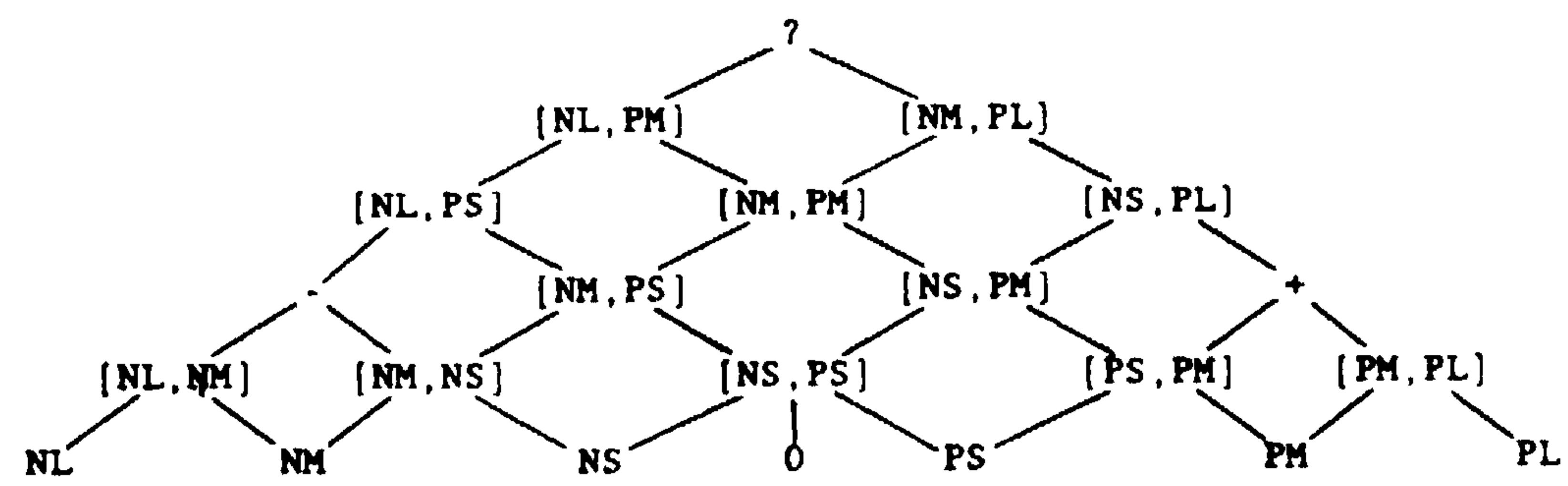
The set S_1 = {NL,NM,NS,0,PS,PM,PL}, ordered by NL<NM<NS<0<PS<PM<PL, constitutes the highest level of specification. Between this level and the lowest one given by ?, the actual partitioning induces four new ordered levels of specification. Interpreting the labels as intervals, the four levels correspond to the union of two, three, four, and five adjacent intervals. Labels within these levels are denoted [a,b], where a,b∈S_1-{0} and a<b. [a,b] represents the interval obtained from the union of the intervals associated with a and b and the one in between. Our universe of description is S= S_1∪{[NL.NM],[NM,NS], ...,[NL,PM],[NM,PL],?} and the order relation < can be represented as follows:



It is easy to see that S is basically complete (S_1 is a base and is unique) and irreducible.

When defining the symbolic tables for the internal operations ⊕ and ⊗, we must of remain consistent with the real line operations. Here, a few problems arise since the qualitative result may differ from the boundaries of the intervals. For example, if 10 is the upper boundary of PS, then PS+PS-[PS,PM] if the upper boundary of PM is greater than or equal to 20. If it is less than 20, then PS+PS=+. Similar situations occur for the product, in particular with respect to the position of numbers 1 and -1. This shows a fundamental difference between this model and the one based on signs. Indeed, here tables for ⊕ and ⊗ are not unique. Their number is finite however and two specific tables for ⊕ and ⊗ will be suitable for a whole class of practical problems.

Nevertheless, the essential point is that, irrespective of the specific pair of operations, they always provide (S,≈) with a structure of Q-algebra. This result will come as a particular case of the Theorem 5.1 below. The Q-model so far considered is indeed a particular case within the following general framework. Let us show that any finite interval partition S_1, of the real line such that {0}∈S_1 gives rise to a Q-model.[2] Given A,B∈S_1 A is said to be *anterior* to B, denoted A<B, if for any a∈A, there exists b∈B such that a<b. Of course the binary relation "be anterior to" is a total order relation on S_1. m_s and m_s denote the minimal and maximal element respectively. The *complete universe of description induced by* S_1, denoted S, is built from S_1 in the following way: A∈S if A-{0} or A-[B_1,B_2], such that B_1,B_2∈S_1-{{0}}, B_1≤B_2, and for all B∈S_1 such that B_1≤B≤B_2, then B∈A. Now, the following order relation is defined on S:

2 More generally, this can be extended to arbitrary infinite interval partitions of the real line [Trave-Massuyes *et al.*, 1989] but this is beyond the scope of this paper.

$A,C \in S$, $A \leq C$, *if for all* $B \in S_1$; $B \in A$, *then* $B \in C$.

Graphically, this order relation can be represented by a tree such that the elements of S_1 constitute the base and the number of levels is equal to the cardinal of S_1 minus 1. With the above construction of S, $(S,\approx)$ is basically complete and irreducible and S_1 is the base of S.

The qualitative sum $\oplus$ is said to be *consistent* with the usual real line sum + if, for all $A,B \in S$, $A \oplus B \in S$ corresponds to the smallest set - in the sence of inclusion - containing A+B. Consistency for the qualitative product is defined in a similar way. The partition S_1 determines thus one and only one pair of operations $(\oplus,\otimes)$ on $(S,\approx)$ which are consistent with the usual real line sum and product. The pair $(\oplus,\otimes)$ is then called *R-consistent.*

Theorem 5.1. Given a finite interval partition of the real line S_1, such that $\{0\} \in S_1$, consider the complete universe of description S induced by S_1. Then, the pair of IR-consistent internal operations $(\oplus,\otimes)$ defined on S provides $(S,\sim)$ with a structure of Q-algebra.

Proof. Let us show that $\oplus$ is Q-associative. Consider $X,Y,Z \in S$ and let $A = X \oplus (Y \oplus Z)$ and $C = (X \oplus Y) \oplus Z$. Consider $x,y,z \in IR$ such that $x \in X$, $y \in Y$, and $z \in Z$. Let be $x+(y+z) = a \in IR$, there exists $B \in S_1$ such that $a \in B$, then we have $B \approx A$. Because + is associative on (R, $a = (x+y)+z \in C$ and since $a \in B$, we also have B ^. B is a basic element, therefore $A \approx C$.

The remaining properties of $\oplus$ and $\otimes$ can be proved in a similar way,

Notice that only the qualitative properties are guaranteed. For example, consider S_1, the following finite interval partition of IR:

```
  NL      NM      NS  0  PS      PM      PL
--------+-------+-------+-------+-------+--------
       -100    -10      0      10     100
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Then $A = NM+(PS+PM) = NM+[PM,PL] = [NM,PL]$ and $C = (NM+PS)+PM = [NM,NL]+PM = [NM,PM]$. So $A \approx C$ but $A \neq C$ ■

The following result has a major practical implication. It is shown that the models obtained from a sequence of finer and finer partitions are embedded into one another as Q-subalgebras. In other words, variables described at different levels of precision can be considered together by moving from one model to the next.

Theorem 5.2. Given two finite interval partitions T_1 and S., of the real line such that {0} belongs to both and denote by S and T the corresponding complete universe of description, $(\oplus_S,\otimes_S)$, $(\oplus_T,\otimes_T)$ denote the IR-consistent pairs of operations defined on S and T, respectively. Under these assumptions, if T_1 is a partition finer than S, , then $(S,\approx_T,\oplus_T,\otimes_T)$ is a Q-subalgebra of $(T,\approx_T,\oplus_T,\otimes_T)$ Further, relation $\sim_T$ and operators $\oplus_T$ and $\otimes_T$ restricted to set S are identical to $\sim_S$, $\oplus_S$, and $\otimes_S$, respectively.

Proof. If T_1 is finer than S,, then any element of S is also element of T. Let us show that $\approx_T$ restricted to S is $\approx_S$. Given $A,C \in S$ such that $A \approx_T C$, which means that there exists $B \in T_1$ such that $B \approx_T A$ an<* $B \approx_T C$ (1). Since T, is finer than S_1, there exists $B \in S_1$ such that $B \approx_T B$. It is clear that $A \approx_T B$ and $C \approx_T B$ (since (1)). But $A,C,\bar{B} \in S$ and T_1 is finer than S_1, therefore $A \approx_S B$ a n $C \approx_S B$. S i n c $B \in S_1$ we get $A \approx_S C$.

It can be similarly proved that $\oplus_T$ and $\otimes_T$ restricted to S are equal to $\oplus_S$ and $\otimes_S$. Then, S is a Q-subalgebra of T_m

In the Q-model proposed in [Dubois and Prade, 1989] and described at the begining of this section, it was noted that different Q-algebras could be built depending on the position of the boundaries of the intervals. The following result shows that all these Q-algebras are Q-isomorphous and can therefore be considered as identical from the qualitative point of view.

Theorem 5.3. Consider S_1 and T_1 two finite interval partitions of the real line such that {0} belongs to both, and S and T the corresponding complete universes of description induced by S_1 and T_1, respectively. The Q-algebras built on S and T are Q-isomorphous if, and only if, there exists a one-to-one mapping f from S_1 on to T_1 such that f((0})-(0) and that f maintains the relation <.

Proof. Due to space limitations, the proof is omitted (see [18])=

6. Conclusion

This paper represents the first attempt at providing a unifying mathematical framework to qualitative models so far used in Qualitative Physics. An axiomatic of the qualitative equality is indeed determined and a general algebraic structure called qualitative algebra is defined. Rather than going into deep theoretical developments, it seemed better to confine this paper to two crucial points. First, the usual model {+,-,0,?) as well as the extended one proposed in [Dubois and Prade, 1989] belong to the same class of models. Second, any of these models present the structure of qualitative algebra. The imbrications of these models as subalgebras and the characterization of qualitatively isomorphous models are then considered.

The basic point is that as research progresses, rumerous points become clearer on the specific models we already used. "Evident" relationships and overlappings occur which reinforce our belief that, from now on, advances in application fields will closely depend on theoretical results. Investigations are therefore underway to determine the underlying properties of qualitative equality and further, of qualitative algebras [Piera and Trave-Massuyes, 1989, Trave-Massuyes *et al.*, 1989]. Qualitative vectorial spaces structures will be considered in the near future. This will provide us with the necessary tools enabling us to approach qualitative calculus.

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