Cooperation between Direct Method and Translation Method in Non Classical Logics: Some Results in Propositional S5

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Abstract

The aim of this work is to combine advantageously the two existing approaches for theorem proving in non classical logics: proving in the considered non classical logic (called here the direct approach) and proving in classical logic by way of translation -called here the translation approach. Some results in propositional S5 show evidence of the relevance of this approach. We assume a translation from S5 into first-order logic and then we define a partial inverse formula translation from first-order classical logic into S5. Semantic relations are proved to hold between the backward translated formulas. We answer positively (for S5) to one conjecture stated in a previous work by the authors. An Interpolation Theorem stating a property stronger than refutational completeness is also proved. A plausible conjecture stronger than the Interpolation Theorem is proposed. These results are interpreted in the framework of a slight variant of an existing resolution calculus for S5. We illustrate our method on a simple example. Future work includes applications of the approach to other modal logics.

1 Introduction

The interest in non classical logics is now unanimously accepted in Artificial Intelligence and in Computer Science. Concerning the way to mechanize them, there are two approaches:

• the direct approach: it consists of building (or using existing) specific proof systems for these logics (see for ex. [Fitting, 1983; Enjalbert and Farinas del Cerro, 1989])

• the translation approach: a problem expressed in a non classical logic (from now on called source logic) is translated into classical logic (from now on called target logic). The problem is therefore solved in the target logic (see for ex. [Ohlbach, 1988]).

Each approach has good defenders -see for ex. [Thislewaite et al., 1988; Ohlbach, 1988]. The direct approach naturally arose the first. Relating logics is a technique that has been used in pure logical studies: in correspondence theory and in definability theory. It has been also used in automated deduction for non classical logics as an alternative to implement specific calculi for each non classical logic and it is the theoretical foundations of the second approach. Historically the innovative work of E. Orlowska introduced the notion of resolution-interpretability of a logic into another -see [Orlowska, 1980J. More recently A. Herzig and H-J. Olhbach (see [Heme, 1989; Ohlbach, 1988] and a related work by M. Chan [Chan, 1987]) emphasized the idea of logic morphism, which is implicitly used in the previous work of E. Orlowska. Their works in which unification plays a central role were applied to several classes of modal and temporal logics.

The two approaches have drawbacks -and obviously advantages. The direct approach needs the construction of new theorem provers whereas the translation approach generates proofs from which the relation with the initial problem is difficult to grasp. In this paper we contend that the two approaches are not mutually exclusive but can be profitably combined. We propose to build from the proofs obtained after translation, partial (possibly total) proofs in the specific systems for non classical logics, with the help of inverse translations. So, it becomes possible to add the advantage of the efficiency of theorem provers for classical logics with that of presenting results (in the present case partial proofs) in the source logic. A computer system implementing this hybrid approach would allow the user to formalize his problem in his favorite logic and to get a reasonable solution in the same logic.

We do not consider in this work neither the problems inherent to the translation approach such as the nature of the classical logic into which the translation is done (first-order logic, fragment of second-order logic ...), nor the theoretical limits of translation (definability) or the choice of proof systems for non classical logics -tableaux, resolution, matings and so on.

In order to show how cooperation between the translation approach and the direct one is possible we shall define a partial backward translation from classical first-order logic (from now on abbreviated FOL) into propositional logic S5. Semantic relations hold between the backward formulas and we shall show that some S5-clauses that logically entail some backward translated formulas, can be derived in a variant of the resolution system RS5 [Enjalbert and Farinas del Cerro, 1989]. In that way we shall answer positively (for S5) to one of the conjectures stated in [Caferra et al., 1993]. Actually, we shall study how the proofs in FOL can be useful to build proofs in S5 with resolution methods.

There are different reasons to consider S5. S5 propositional formulas have a reasonable normal form -S5-clauses in [Enjalbert and Farinas del Cerro, 1989]. It can be used as a model of autoepistemic reasoning [Moore, 1985]. Moreover the translation from S5 into FOL we shall use, is quite simple. Finally the problem of deciding S5-satisfiability is only[1] NP-complete [Ladner, 1977].

The paper is structured as follows. The next section recalls the features for S5 we shall work on. In section 3 the inverse formula translation is defined and different semantic results are presented to state the main theorem on partially ordered sets of S5-formulas. Section 4 states a result similar to the "consequence finding theorem" [Lee, 1967] for the inverse formula translation in a variant of RS5. We also propose a plausible conjecture related to this theorem. In section 5, a simple example is fully treated. Finally we propose different ways to continue this work.

2 Preliminaries

We assume familiarity with the syntax and semantics of propositional S5. The standard definitions of satisfiability and validity will be used -see [Hughes and Cresswell, 1968]. The set of well-formed S5-formulas will be noted MFor. We now recall one normal form for the S5-formulas [Enjalbert and Farinas del Cerro, 1989]. In the sequel, by the term 'clause' (resp. 'literal') we shall mean a clause (resp. literal) in the classical logic.

Definition: A S5-formula is said to be in conjunctive normal form iff it is a conjunction of formulas of the form: $C \vee \Box D_1 \vee \ldots \vee \Box D_N \vee \Diamond C_1 \vee \ldots \vee \Diamond C_R$ where C and the D_i's are clauses and the Cj's are conjunctions of clauses. Each conjunct is called a S5-clausc. Δ

Fact 1 Every S5-formula is equivalent to a formula in conjunctive normal form.

2.1 Translation into First-Order Formulas

The proposed translation is standard (see for ex. [Chellas, 1980; Miura, 1983]) and it preserves validity. The translation is based on the naive one where the binary predicate R for the accessibility relation is deleted. If F denotes a S5-formula then $\mathcal{T}(F)$ denotes its translation. We use an auxiliary function Tr with the profile: Term $\times$ *MFor* $\rightarrow$ *FOLFor* -through all this paper 'term' will mean 'first-order term'. The constant 'aw' stands for "actual world" and $\mathcal{T}(F) = \mathrm{Tr}(aw, F)$.

1. Tr(t, P) = P(t) for P propositional variable
2. $\mathrm{Tr}(t, f \wedge g) = \mathrm{Tr}(t, f) \wedge \mathrm{Tr}(t, g)$
3. $\mathrm{Tr}(t, \neg f) = \neg \mathrm{Tr}(t, f)$
4. $\mathrm{Tr}(t, \Box f) = \forall x' \, \mathrm{Tr}(x', f)$, x' is a new variable symbol
5. $\mathrm{Tr}(t, \Diamond f) = \exists x'' \, \mathrm{Tr}(x'', f)$, x'' is a new variable symbol

Fact 2 *A formula f is S5-valid iff $\mathcal{T}(f)$ is FOL-valid.*

[1]The satisfiability problem for usual logics such as S4 has a higher complexity

Notation: **A first-order literal will be noted either Q(l) or sP(l), with $s \in \{\Lambda, \neg\}$ (Λ the empty string), P a predicate symbol, l a list of terms and Q a predicate symbol preceded by one element of $\{\Lambda, \neg\}$.**

2.2 A Resolution Proof System

We recall the rules of the elegant resolution system RS5 defined in [Enjalbert and Fariñas del Cerro, 1989]. D, D', C and C' denote clauses, E a conjunction of clauses and the symbol ',' is a conjunctive operator.

$\Box\Diamond$-rule $\dfrac{C \vee \Box(l \vee D) \qquad C' \vee \Diamond(l' \vee D', E)}{C \vee C' \vee \Diamond(D \vee D', l' \vee D', E)}$ if l and l' are complementary literals

$\Box\Box$-rule $\dfrac{C \vee \Box(p \vee D) \qquad C' \vee \Box(\neg p \vee D')}{C \vee C' \vee \Box(D \vee D')}$,

$\Diamond$-rule $\dfrac{C \vee \Diamond(p \vee D, \neg p \vee D', E)}{C \vee \Diamond(D \vee D', p \vee D, \neg p \vee D', E)}$

$\Box$-rule $\dfrac{C \vee \Box(l \vee D) \qquad C' \vee l'}{C \vee C' \vee D}$ if l and l' are complementary literals

Clas-rule $\dfrac{C \vee p \qquad C' \vee \neg p}{C \vee C'}$ (resolution rule for classical logic),

Fact-rule $\dfrac{E[D \vee D \vee C]}{E[D \vee C]}$ (factoring), $\Box \perp$-rule $\dfrac{C \vee \Box \perp}{C}$,

$\Diamond \perp$-rule $\dfrac{C \vee \Diamond(\perp, E)}{C}$

RS5 is a variant of the system RT [Enjalbert and Fariñas del Cerro, 1989] for the logic T on which the special form of the S5-clauses has been taken into account. Every modal formula without nesting of modal operators is S5-satisfiable iff it is T-satisfiable. RS5 is sound and complete for the refutation [Enjalbert and Fariñas del Cerro, 1989].

3 Partial Inverse Formula Translation

The proposed partial inverse translation transforms FOL-formulas into S5-formulas. It applies to the set of first-order clauses that are deduced from the translation of S5-formulas. We shall note $\mathcal{R}_{FOL}$ the operator giving the set of all the resolvents of two clauses and similarly $\mathcal{F}_{FOL}$ for the factorization rule. The symbol $\vdash_{Res FOL}$ denotes the derivation operator for classical resolution.

3.1 Definitions

Let $S = \{C_1, \ldots, C_N\}$ be a set of S5-clauses. We note TC_i the FOL-formula $\mathcal{T}(C_i)$ and TC_i' the skolemized (in the usual sense) form of TC_i. Only Skolem constants (noted a_i) are introduced. TC_i' has the following form : $K_1(aw) \vee \ldots \vee K_N(aw) \vee L_1^1(x_1) \vee \ldots \vee L_1^{n(1)}(x_1) \vee \ldots \vee L_K^1(x_K) \vee \ldots \vee L_K^{n(K)}(x_K) \vee ((M_{1,1}^1(a_1) \vee \ldots \vee M_{1,1}^{m(1,1)}(a_1)) \wedge \ldots \wedge (M_{1,u(1)}^1(a_1) \vee \ldots \vee M_{1,u(1)}^{m(1,u(1))}(a_1))) \vee \ldots \vee ((M_{L,1}^1(a_L) \vee \ldots \vee M_{L,1}^{m(1,1)}(a_L)) \wedge \ldots \wedge (M_{L,u(L)}^1(a_L) \vee \ldots \vee M_{L,u(L)}^{m(L,u(L))}(a_L)))$ where the a_i's are constants and the x_i's are variables. So from the formula TC_i' we get $\prod_{k=1}^{L} u(k)$ clauses.

Definition: Let S be a set of S5-clauses equivalent to the S5-formula F. We note S' the set of clauses generated from S. Every clause c such that $S' \vdash_{ResFOL} c$ is called a *F-clause.* Δ

A clause c is a *f-clause* (generic term) if there is a S5-formula f' such that c is a f'-clause with the above definition. By definition we restrict the application of the backward translation only to the f-clauses. Every f-clause c has the form: $K_1(\mathrm{aw}) \vee \ldots \vee K_N(\mathrm{aw}) \vee L_1^1(x_1) \vee \ldots \vee L_1^{n(1)}(x_1) \vee \ldots \vee L_K^1(x_K) \vee \ldots \vee L_K^{n(K)}(x_K) \vee M_1^1(a_1) \vee \ldots \vee M_1^{m(1)}(a_1) \vee \ldots \vee M_1^L(a_L) \vee \ldots \vee M_L^{m(L)}(a_L)$. Informally if ϕ is the inverse translation, then $\phi(c) = K_1 \vee \ldots \vee K_N \vee \Box(L_1^1 \vee \ldots \vee L_1^{n(1)}) \vee \ldots \vee \Box(L_1^K \vee \ldots \vee L_K^{n(K)}) \vee \Diamond(M_1^1 \vee \ldots \vee M_1^{m(1)}) \vee \ldots \vee \Diamond(M_L^1 \vee \ldots \vee M_L^{m(L)})$. We now define formally the partial inverse formula translation.

Backward translation for clauses

The principle of the backward translation is to gather the literals that have some *modal context* in common. In the particular case of S5, a modal context is simply a variable, or a constant. For other logics such as S4 a normal form for the clauses must be found -see [Caferra and Demri, 1992]. Indeed, the inverse partial formula translation is basically defined for clauses and then for the conjunction of clauses. We first define auxiliary functions.

- generate-op(t) := **if** t is a variable **then** $\Box$ **else** $\Diamond$
- β(C) := aw % C is a term %

We extend β to literals, clauses and sets of clauses.

Definition of ϕ: Let D be a f-clause $L_1 \vee \ldots \vee L_n$. We define a partition with the literals of D, i.e., we define the set of classes $\{C_i, 1 \leq i \leq k\}$. Moreover the classes satisfy the following properties. For $1 \leq j \leq$ k,
1. For all P,Q $\in C_j$, $t^Q = t^P$ % $L_i = s_i P_{Li}(t^{L_i})$ %
2. For all P $\in C_j$ and $Q \notin C_j$, $t^Q \neq t^P$

It can be easily checked that the *decomposition in classes is unique*. The definition of $\phi(D)$ uses a function α mapping a class S to a S5-formula:
$\alpha(\{l_1, \ldots, l_p\}) :=$ % S = $\{l_1, \ldots, l_p\}$, $l_i = s_i P_i(c_i)$ %
if c_1 = aw **then** $s_1 P_1 \vee \ldots \vee s_p P_p$ **else** generate-op(c_1).$\bigvee\{s_i P_i \mid s_i P_i(c_i) \in \mathrm{S}\}$
$\phi(D) = \alpha(C_1) \vee \ldots \vee \alpha(C_k)$.

Backward translation for conjunctions of clauses

The partial inverse formula translation is extended to conjunctions of clauses. To do so, *superclasses* are defined in so for as sets of classes. In order to compute $\phi(C_1 \wedge \ldots \wedge C_N)$ the partition $\{c_i^1, \ldots, c_i^{u_i}\}$ is associated to each clause C_i and so is $t^{(i,j)}$ for each class c_i^j. The point is now to compute superclasses considered as subsets of the set $\{c_i^j$, for $1 \leq \mathrm{i} \leq \mathrm{N}$, $1 \leq \mathrm{j} \leq u_i\}$. A superclass $\{c_{i1}^{j1}, \ldots, c_{in}^{jn}\}$ satisfies the following properties :
1. for $1 \leq \mathrm{a} < \mathrm{b} \leq \mathrm{n}$, $i_a \neq i_b$
2. for $1 \leq \mathrm{a} \leq \mathrm{b} \leq \mathrm{n}$, $t^{ia,ja} = t^{ib,jb}$ and $t^{i1,j1} \neq$ aw

By definition ϕ'(SC)=generate-op($t^{(i1,j1)}$).$\bigwedge\{\phi(\beta(c_{ik}^{jk}))$, $1 \leq \mathrm{k} \leq \mathrm{n}\}$.

$\phi(C_1 \wedge \ldots \wedge C_N) = \bigvee\{\ \epsilon(\{c_j^{ij}, 1 \leq \mathrm{j} \leq \mathrm{N}\ \}), 1 \leq i_1 \leq u_1, \ldots, 1 \leq i_N \leq u_N\}$ and $\epsilon(\{c_1, \ldots, c_l\}) :=$ % the c_i's are classes % **if** $\{c_1, \ldots, c_l\}$ does not contain any superclass **then** $\alpha(c_1) \wedge \ldots \wedge \alpha(c_l)$ **then let** $\{sc_1, \ldots, sc_k\}$ be the superclasses contained in $\{c_1, \ldots, c_l\}$ %they are disjoint% $\alpha(c_1) \wedge \ldots \wedge \alpha(c_l)[sc_1 \leftarrow \phi'(sc_1), \ldots, sc_k \leftarrow \phi'(sc_k)]$.

For N = 1, we obtain the restricted definition for the clauses. Moreover if no superclass can be found in $\phi(C_1 \wedge \ldots \wedge C_N)$ then $\models_{S5} \phi(C_1 \wedge \ldots \wedge C_N) \Leftrightarrow \phi(C_1) \wedge \ldots \wedge \phi(C_N)$.

Example: A = $Q(x) \vee R(x) \vee M(a) \vee P(a)$;
B = $P'(aw) \vee M'(a)$ (the only variable is 'x')
ϕ(A) = $\Box(Q \vee R) \vee \Diamond(M \vee P)$
ϕ(A $\wedge$ B) = $(\Box(Q \vee R) \wedge P') \vee (\Box(Q \vee R) \wedge \Diamond M') \vee (\Diamond(M \vee P) \wedge P') \vee \Diamond((M \vee P) \wedge M')$
In A, the class {Q(x), R(x)} has been considered as the superclass {{P(a), M(a)}, {M'(a)}} in $A \wedge B$.

The detailed proof of the next lemma as well as the other omitted proofs of this paper can be found in [Caferra and Demri, 1993].

Lemma 1 *Let a and b be two f-clauses. It is decidable whether* $\models_{S5} \phi(a \wedge b) \Leftrightarrow \phi(a) \wedge \phi(b)$ *and* $\phi(a \wedge b) \models_{S5} \phi(a) \wedge \phi(b)$ *holds.*

This lemma can be extended for n clauses. Furthermore, the equivalence holds iff the different clauses do not share any Skolem constants.

3.2 Partially Ordered Sets of S5-formulas

The next lemma states that if the resolution or the factorization rules are used then there exist semantic relations in S5 between the backward translation of the corresponding clauses.

Lemma 2 *Let a, b and c be f-clauses. If* $b \in \mathcal{F}_{FOL}(a)$ *then* $\phi(a) \models_{S5} \phi(b)$*. If* $c \in \mathcal{R}_{FOL}(a,b)$ *then* $\phi(a \wedge b) \models_{S5} \phi(c)$.

The next proposition answers positively (for propositional S5) to conjecture 1 in [Caferra *et al.*, 1993]: the formula f entails the inverse translation of f-clauses.

Proposition 1 *Let f be a S5-formula equivalent to the set S of S5-clauses. Let S' be the set of clauses obtained from the translation of S. For all f-clause c, if* $S' \vdash_{ResFOL} c$ *then* $f \models_{S5} \phi(c)$.

Proof: The proof can be obtained by using the results about *induced model* in [Miura, 1983]. Let SB = (W, R, m) be a substructure of the S5-model (W, R, m, w). In [Miura, 1983], it is shown that an *induction translation* $\mathcal{W}$ can be defined for S5 such that $\forall u \in W$ (SB, u) $\models_{S5}$ f iff $\mathcal{W}$(SB)(aw $\leftarrow$ u) $\models_{FOL}$ Tr(f, aw). In the terminology used in [Miura, 1983], ($\mathcal{W}$(SB), u) is an *induced FOL-model.*

Let I = (W, R, m, w_0) be a S5-model such that I $\models_{S5}$ f (SB = (W, R, m)). We therefore get that $\mathcal{W}$(SB)(aw $\leftarrow w_0$) = $I_{fol} \models_{FOL} T$(f). We note I_{fol} + Sko a model with the interpretation of Skolem constants (the Axiom of Choice is used here) such that I_{fol} + Sko $\models_{FOL} T$(f) and I_{fol} + Sko $\models_{FOL} S'$. Since the resolution is sound, if $S' \vdash_{ResFOL}$ c then I_{fol} + Sko $\models_{FOL}$ c.
We know that the clause c has the format:
$P_1(aw) \vee \ldots \vee P_n(aw) \vee Q_1^1(x_1) \vee \ldots \vee Q_1^{n1}(x_1) \vee \ldots \vee Q_\alpha^1(x_1) \vee \ldots \vee Q_\alpha^{n\alpha}(x_\alpha) \vee R_1^1(a_1) \vee \ldots \vee R_1^{m1}(a_1) \vee \ldots \vee R_\beta^1(a_1) \vee \ldots \vee R_\beta^{m\beta}(a_\beta)$ where the x_i's are variables and the a_i's are (Skolem) constants.
We get $I_{fol} \models_{FOL} (P_1(aw) \vee \ldots \vee P_n(aw)) \vee (\forall x_1 Q_1^1(x_1) \vee$

$\ldots \vee Q_1^{n1}(x_1)) \vee \ldots \vee (\forall x_\alpha\ Q_\alpha^1(x_1) \vee \ldots \vee Q_\alpha^{n\alpha}(x_\alpha)) \vee (\exists y_1 R_1^1(y_1) \vee \ldots \vee R_1^{m1}(y_1)) \vee \ldots \vee (\exists y_\beta R_\beta^1(y_\beta) \vee \ldots \vee R_\beta^{m\beta}(y_\beta))$.
There exists a S5-clause c' such that $\mathcal{T}(c')$ is equal to the above formula modulo the renaming of variables. So by using the equivalence relation of the beginning of this proof, we get I $\models_{S5} c'$. Q.E.D.

From the properties previously stated, we can build from a deduction in FOL, partially ordered sets of S5-formulas semantically related to each other.

Theorem 1 *Let U be a set of S5-clauses equivalent to f. We note S' the set of clauses obtained from U by using the translation $\mathcal{T}$, the introduction of Skolem functions and the clausal transformation. Let $(l_1, \ldots, l_m)$ be a deduction from S' by resolution and factorization.* ***Then*** *there exists a partially ordered set -p.o.s.- (S, I) such that (*conditions SC*)*

- *S is a finite set of S5-formulas with f, $\phi(l_m) \in S$ and I is a partial order on S*
- *If (f I g) then f $\models_{S5}$ g*
- *f is minimal in (S, I) and $\phi(l_m)$ is maximal in (S, I)*
- *The non oriented graph (S, I) connects f and $\phi(l_m)$*

Proof: By induction on the number of steps s used to deduce l_m.
Base Case : $l_m \in S'$ $(s = 0)$
Proposition 1 states that f $\models_{S5} \phi(l_m)$. The p.o.s. ({f, $\phi(l_m)$}, {(f, $\phi(l_m)$)}) satisfies SC -this p.o.s. trivially satisfies the conditions.
Induction Step
Let $(l_1, \ldots, l_m)$ be a deduction such that l_m has been deduced in s+1 steps.
Case 1 $(l_m \in \mathcal{F}(l_j))$: There exists, by the induction hypothesis, (S_j, I_j) satisfying the *semantic conditions* with l_j. We consider the p.o.s. $(S_j \bigcup \{\phi(l_m)\}, I_j \bigcup \{(\phi(l_j), \phi(l_m))\})$. It can be easily shown that this structure satisfies SC for l_m -see Lemma 2.
Case 2 $(l_m \in \mathcal{R}(l_j, l_k))$: There exists (by the induction hypothesis) a p.o.s. (S_j, I_j) (resp. (S_k, I_k)) satisfying SC for l_j -resp. l_k. We consider the p.o.s. $(S_j \bigcup S_k \bigcup \{\phi(l_j) \wedge \phi(l_k), \phi(l_j \wedge l_k), \phi(l_m)\}, I_j \bigcup I_k \bigcup \{(\phi(l_j) \wedge \phi(l_k), \phi(l_j)), (\phi(l_j) \wedge \phi(l_k), \phi(l_k)), (\phi(l_j \wedge l_k), \phi(l_j) \wedge \phi(l_k)), (\phi(l_j \wedge l_k), \phi(l_m))\})$. It can be easily shown that this p.o.s. satisfies SC for l_m -see Lemma 1, 2. Q.E.D.

In the fourth condition the backward translation generates S5-formulas related semantically with the initial formula f. In Corollary 1 we shall present a stronger result.

The relationship between backward translated formulas should be compared with the definition of consequence relation as a *partial ordering* on well-formed formulas in [Scott, 1974].

The partial order may be total so that there exists a sequence $(f_0, \ldots, f_u)$ such that for $0 \leq i \leq u-1$, $f_i \models_{S5} f_{i+1}$ where $f_0 =$ f and $f_u = \phi(l_m)$. The following corollary states a sufficient condition to get such a sequence.

Corollary 1 *(Sequence of Semantic Entailments)* ***If the hypothesis*** *(if $l_i \in \mathcal{R}_{FOL}(l_j, l_k)$ then $\models_{S5} \phi(l_j \wedge l_k) \Leftrightarrow \phi(l_j) \wedge \phi(l_k)$) is added to Theorem 1* ***then*** *there exists a sequence $(f_0, \ldots, f_u)$ such that for $0 \leq i \leq u-1$, $f_i \models_{S5} f_{i+1}$ where $f_0 = f$ and $f_u = \phi(l_m)$.*

The sequence built with the proof of Theorem 1 is not necessarily minimal. Its length may be reduced as well as the size of formulas. The condition $\models_{S5} \phi(l_j \wedge l_k) \Leftrightarrow \phi(l_j) \wedge \phi(l_k)$ is decidable -Lemma 1.

4 Modal Consequence Finding and Interpolation

In Section 3, it has been shown that semantic relations hold between the initial S5-formula and the backward translation of clauses obtained with classical resolution. We shall show that in a variant of the resolution system RS5 -called $RS5'$- a partial "consequence finding theorem" similar to the one in [Lee, 1967] can be found. For a set of S5-clauses S, all the S5-clauses with a given format that are logical consequences of S admit an interpolant S5-clause derivable from S in $RS5'$. Unfortunately it cannot be extended to all the S5-clauses -see the counterexample in this section. A plausible result would be to connect syntactic and semantic properties about the backward translated formulas as it was mentioned in [Caferra *et al.*, 1993]. At that time, we ignored the result in [Lee, 1967] for the classical logic but we consider that a similar result in the systems for the non-classical logics would be a skilled criterion to compare different proof systems.

We shall call $\Box'$-rule the following rule $\frac{C \vee \Box(C')}{C \vee C'}$. We call $RS5'$ the system (RS5\{$\Box$-rule})$\cup${$\Box'$-rule}. In $RS5'$, the $\Box$-rule is a derived inference rule and the $\Box'$-rule is sound so $RS5'$ is sound and complete.

The following theorem gives conditions to ensure the existence of an interpolant clause in the system $RS5'$. Theorem 3 can be seen as a partial modal equivalent of the so called "consequence finding theorem". The *consequence finding problem* is stated in [Inoue, 1991]:

> "Given a set of formulas T and a resolution procedure P, for any logical consequence D of T, can P derive a logical consequence C of T such that C subsumes D"?

The problem for classical logic has been solved in [Lee, 1967]:

Theorem 2 *(Consequence Finding) Given a set S of clauses, if a clause C is a logical consequence of S, then for some $n \leq 0$, there exists a clause T[2] $\in R^n$, such that T implies C.*

The above theorem can be seen as an "extended completeness theorem for resolution", i.e., not only refutational completeness. The following theorem provides conditions to ensure the existence of an interpolant clause in the system $RS5'$.

Theorem 3 *(Interpolation) Let S be a set of S5-clauses. Let c be a S5-clause with the format $d \vee \Box(c_1) \vee \ldots \vee \Box(c_n)$ where d and the c_i's are clauses.* ***If*** *S $\models_{S5}$ c* ***then*** *there exists a S5-clause c' such that $S \vdash_{RS5'} c'$ and $c' \models_{S5} c$.*

The set S can contain any kind of S5-clauses.

[2] *R^n is the n^{th} resolution operator*

Proof (sketch): The (constructive) proof is tedious. It is based on the format of c, on the refutational completeness of RS5 and on the proofs of the completeness of the system RT [Enjalbert and Farinas del Cerro, 1989]. Two base cases are distinguished and the general case combines them. Q.E.D.

Corollary 2 *(Lemma Discovery) Let S be a set of S5-clauses equivalent to f. For all f-clause c such that $\phi(c)$ has the format of Theorem 3, there exists c_{S5} such that $S \vdash_{RS5'} c_{S5}$ and $c_{S5} \models_{S5} \phi(c)$.*

We believe this result can be extended to every f-clause with RS5, but we were not able to completely prove the property stated in the following conjecture.

Conjecture 1 *Let S be a set of S5-clauses equivalent to f and c be a f-clause. Then there exists a S5-clause c' such that $S \vdash_{RS5} c'$ and $c' \models_{S5} \phi(c)$.*

If S is unsatisfiable then the information given by the *Interpolation Theorem* is too weak[3]. However, if S is minimally unsatisfiable, Corollary 2 and Conjecture 1 can be relevant for any proper subset of S.

Moreover RS5 is not strongly complete , i.e., if S $\models_{S5}$ c then S $\vdash_{RS5}$ c. Obviously, if S $= \{\Box(P \vee Q), \Diamond\neg Q\}$ then S $\models_{S5} \Diamond P$ and S $\nvdash_{RS5} \Diamond P$ -but S $\vdash_{RS5} \Diamond(\neg Q \wedge P)$. The natural deduction system in [Corcoran and Weaver, 1969] can be mentioned as an example of strongly complete system, with S5-formulas instead of S5-clauses.

Furthermore Theorem 3 *cannot* be generalized to all the S5-clauses: the following counterexample proves this. If S $= \{Q \vee \Diamond P, \Box R\}$ then S $\models_{S5} Q \vee \Diamond(P \wedge R)$ but there is no S5-clause c_{s5} such that S $\vdash_{RS5} c_{s5}$ and $c_{s5} \models_{S5} Q \vee \Diamond(P \wedge R)$.

So, we cannot build an interpolant between any two formulas f and g by using the system RS5. The alternative syntactic interpolation lemma presented in [Czermak, 1973] cannot be adapted to RS5 -see also the related semantic interpolation lemma given in [Gabbay, 1972]. Conject ure 1 can be seen as an intermediate plausible result between Theorem 3 and the modal equivalent of the Lee's Theorem. Furthermore we conjecture that Proposition 2 can be extended to RS5. We believe that in spite of the relative lack of interest of consequence finding in research of automated theorem proving for classical logics, results about modal consequence finding could generate applications for Artificial Intelligence. That is why we propose the following result with the hope that a modal equivalent may exist for the system RS5 -or *RS5'*.

Proposition 2 *Let S be a minimally unsatisfiable set of propositional clauses. Let c be a non tautological clause ($\neq \perp$). There exists a clause d ($\neq \perp$) such that $S \vdash_{Res,PC}$ d and d $\models_{PC}$ c iff Varprop(c) $\subseteq$ Varprop(S)*[4].

In the case of unsatisfiability, this is stronger than Theorem 3. We conjecture this proposition has a modal equivalent.

5 Example

We illustrate on a simple example the results of the previous sections. We consider the valid S5-formula f $= (\Box(A \Rightarrow \Diamond B) \wedge \Box(C \Rightarrow \Diamond D)) \Rightarrow ((A \vee C) \Rightarrow \Diamond(B \vee D))$. From a proof in FOL, we construct a partially ordered set of S5-formulas with $\neg$ f as a minimal element and we present a deduction in RS5 that enables us to apply with f, Corollary 1 and Conjecture 1. The set of S5-clauses S $= \{A \vee C, \Box\neg B, \Box\neg D, \Diamond B \vee \Box\neg A, \Box\neg C \vee \Diamond D\}$ is equivalent to $\neg$ f. The translation of S into FOL generates the set of clauses $\{A(aw) \vee C(aw), \neg B(x_1), \neg D(x_2), B(a_1) \vee \neg A(x_3), \neg C(x_4) \vee D(a_2)\}$.

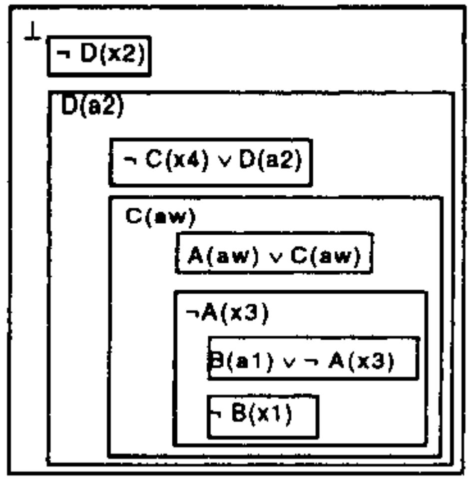

Figure 1: Refutation with classical resolution

From the proof in Figure 1, Lemma 2 and Theorem 1 we get a p.o.s. of S5-clauses ; its construction is based on Theorem 1. Moreover in this case Corollary 1 is applicable so we get the following sequence of semantic entailments. Each formula of the sequence logically entails the next one.

1. $\neg$ f
2. $(A \vee C) \wedge (\Box\neg B) \wedge \Box\neg D \wedge (\Diamond B \vee \Box\neg A) \wedge (\Box\neg C \vee \Diamond D)$
3. $(A \vee C) \wedge \Box\neg D \wedge (\Box\neg C \vee \Diamond D) \wedge \Box\neg A$
4. $\Box\neg D \wedge (\Box\neg C \vee \Diamond D) \wedge C$
5. $\Box\neg D \wedge \Diamond D$
6. $\perp$

The properties of the previous sections allow us to built other p.o.s. of S5-formulas. Furthermore we present a refutation in RS5 -Figure 2. From Conjecture 1, every backward translation of a clause in FOL admits an interpolant in RS5. The proof in Figure 2 contains all the required interpolants that are the backward translations themselves but it is not always the case.

6 Conclusion and Future Work

With respect to the aims stated in Section 1, it has been shown that from a standard translation of S5-formulas into first-order logic it is possible to define a partial inverse formula translation having interesting semantic properties -Theorem 1, Corollary 1. The properties of the inverse translation have been used to prove a theorem concerning consequence finding in S5 using a vari-

[3] Empty clause as interpolant

[4] *Varprop gives the set of propositional variables*

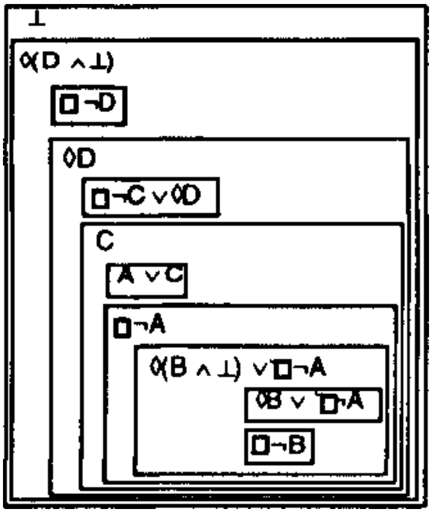

Figure 2: Refutation with the proof system RS5

ant of the resolution system RS5 -Corollary 2. Corollary 2 can be used as a syntactic criterion to guide proofs in RS5'. We also answer positively to one conjecture and partially answer to a second one -see [Caferra et ai, 1993].

The inherent limitations of our work are threefold. The expressive power of propositional logic S5 is obviously limited. The extension of our results to first-order S5 is not straightforward because there is neither Interpolation Lemma for first-order S5 [Fine, 1979], nor reasonable normal form for all the quantificational S5-formulas. We have shown that RS5 is incomplete for consequence-finding. Finally it should be mentioned that the computation of S5 normal forms remains expensive.

The main lines of future work are to prove Conjecture 1 of Section 4, to extend the present results to other modal logics (K, S4 ...) with expectable increasing difficulties and to consider other proof systems -tableaux, matings.

References

[Caferra and Demri, 1992] R. Cafer ra and S. Demri. Semantic entailment in non classical logics based on proofs found in classical logic. In CADE-11, pages 385-399. Springer-Verlag, LNAI 607, June 1992.

[Caferra and Demri, 1993] R. Caferra and S Demri. Cooperation of direct and translation methods in propositional S5 (long version), 1993. Forthcoming.

[Caferra tt ai, 1993] R. Caferra, S. Demri, and M. Herment. A framework for the transfer of proofs and strategies from classical to non classical logics. Studia Logica, 52(2), 1993.

[Chan, 1987] M. C. Chan. The recursive resolution method for modal logics. New Generation Computing, 5:155-183, 1987.

[Chellas, 1980] F. B. Chellas. Modal Logic. Cambridge University Press, 1980.

[Corcoran and Weaver, 1969] J. Corcoran and G. Weaver. Logical consequence in modal logic: natural deduction in S5. Notre Dame Journal of Formal Logic, X(4):370-384, October 1969.

[Czermak, 1973] J. Czermak. Interpolation theorem for some modal logics. In Rose and Sheperdson, editors, Logic Colloquium 75, pages 382-393. North-Holland Publishing Company, 1973.

[Enjalbert and Farinas del Cerro, 1989] P. Enjalbert and L Farinas del Cerro. Modal resolution in clausal form. Theoretical Computer Science, 65:1-33, 1989.

[Fine, 1979] K. Fine. Failures of the interpolation lemma in quantified modal logic. Journal of Symbolic Logic, 44(2):201-206, June 1979.

[Fitting, 1983] M. C. Fitting. Proof methods for modal and tntuitionistic logics. D. Reidel Publishing Co., 1983.

[Gabbay, 1972] D. M. Gabbay. Craig's interpolation theorem for modal logics. In Conference in Mathematical Logic, London "70, pages 111-127. Springer-Verlag, LNM 255, 1972.

[Herzig, 1989] A. Herzig. Raisonnement automatique en logique modale et algorithmes d'unification. PhD thesis, Universite P. Sabatier, Toulouse, 1989.

[Hughes and Cress well, 1968] G. E. Hughes and M. J. Cresswell. An introduction to modal logic. Methuen and Co., 1968.

[inoue, 1991] K. Inoue. Consequence-finding based on ordered linear resolution. In IJCAI-12, Sidney, pages 158-164, August 1991.

[Ladner, 1977] R. E. Ladner. The computational complexity of provability in systems of modal propositional logic. SIAM J. Comp., 6(3):467-480, September 1977.

[Lee, 1967] R. C. Lee. A completeness theorem and a computer program for finding theorems derivable for given axioms. PhD thesis, University of California, Berkeley, 1967.

[Miura, 1983] S. Miura. Embedding of modal predicate logics into lower predicate calculus I. Rassegna Internazionale Di Logica, 28:94-105, 1983.

[Moore, 1985] R. C. Moore. Semantical considerations on nonmonotonic logic. Artificial Intelligence, 25:75-94, 1985.

[Ohlbach, 1988] J.H. Ohlbach. A resolution calculus for modal logics. In CADE-9, pages 500-516. Springer-Verlag, LNCS 310, 1988.

[Orlowska, 1980] E. Orlowska. Resolution systems and their applications II. Fundamenta Informaticae, 3:333-362, 1980.

[Scott, 1974] D.Scott. Completeness and axiomatizability in many-valued logic. In L. Henkin et al., editor, Tarskt Symposium, pages 411-435, 1974.

[Thislewaite et ai, 1988] P. B. Thislewaite, M. A. McRobbie, and R. K. Meyer. Automated theorem-proving in non-classical logics. Pitman, 1988.