The Equivalence of Model-Theoretic and Structural Subsumption in Description Logics

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Abstract

A new approach to the semantics of description logics, using *concept algebras,* was introduced in [Dionne *et* al., 1992b]. In that approach, the terms of a description logic, i.e., concept descriptions, were viewed as elements of a free algebra. In the context of a given knowledge base possibly involving cycles, an intensional semantics was given by mapping every concept description to a possibly non-well-founded set that embodied the abstract structure of the concept description. This is in contrast to an extensional semantics that assigns to a concept description a specific set it describes. These sets that embody the abstract structures of concept descriptions are the elements of the *universal concept algebra.* The novelty of this approach is that one can define an ordering on the terms in these algebras that corresponds directly to the structural subsumption algorithms that most of these logics employ in their implementations. In this paper we prove that structural subsumption in the universal concept algebra is the most abstract with respect to all the greatest fixed point models, i.e., that subsumption defined structurally is equivalent to subsumption defined model-theoretically. This result provides the link between our intensional semantics based on concept algebras and the usual extensional models.

1 Introduction

The nature of extensional semantics for description logics is easily grasped: given a knowledge base, concept terms are interpreted as sets of objects from some universe being modeled by that knowledge base. Thus, there is a particular world view in the form of a fixed universe that is at the root of an extensional semantic model. In contrast, an intensional semantics implies an interpretation uncommitted to the specifics of a particular world view. A related contrasting use of terminology is found when we speak of a function being defined intensionally as some sort of algorithm or formula, as opposed to extensionally as a set of ordered pairs. *Concept algebras,* recently introduced in [Dionne *et* al., 1992b], provide the mathematical underpinnings for a first attempt at an intensional semantics for description logics. An element of the *universal concept algebra* is an abstract object that reflects the structural essence of a term of a description logic. The ordering of the terms in this algebra corresponds to subsumption. In fact, the definition of this ordering is actually an abstract specification of structural subsumption used in description logics like K-REP [Mays *et* al., 1991], LOOM [MacGregor and Bates, 1987], and CLASSIC [Borgida *et* al., 1989], in contrast to KRIS [Baader and Hollunder, 1991], which employs a model-based approach. This algebraic model captures the essence of subsumption as a process of structural comparison, and is defined without any use of extensional models.

Bill Woods has argued eloquently for taking a more operational or intensional view of subsumption [Woods, 1991], with several examples that exhibit the confusions that arise through implicit use of quantifiers. His work has motivated our development of concept algebras. Though certainly a broader approach to intensional semantics is called for by Woods, concept algebras arc an initial step in that direction. Another motivation for the algebraic approach was the problem of cycles. We've shown that, in elementary description logics supporting conjunctions of concepts, recursive definitions can be handled. This model has also been extended to handle disjunctions, while still handling cycles [Dionne *et* a/., 1992a]. Since we have not yet extended our proof of the main theorem of this paper to the model supporting disjunction, we will restrict ourselves to the more elementary case.

The class of concept algebras is defined using universal algebra [jacobson, 1989]. The signature is specified using the operators for term formation in the description logic, and axioms are specified that must be satisfied by algebras in the class. Three kinds of concept algebras are of particular interest. First, there are free algebras, i.e, algebras of syntactic terms, that are generated by a set of variables that may be defined by a knowledge base, where a knowledge base is simply a set of equations equating variables with terms of the free algebra. Second, we consider quotients of free algebras by congruences generated by knowledge bases. Finally we consider the *universal concept algebra,* which is given by the solution of a certain set-theoretic equation. The carrier of this algebra is a collection of non-well-founded sets, each representing the structure of a concept description. For each given knowledge base satisfying a certain weak condition, there is a unique homomorphism from the corresponding quotient algebra to the universal concept algebra. The universal concept algebra can be thought of as the collection of all possible concepts. In [Dionne *et a*/., 1992b], we conjectured that this algebra was the most abstract of all the greatest fixed point extensional models. In this paper, the conjecture is stated formally and proved as a theorem, thus linking this new approach with existing model theories. We claim the novelty of this algebraic approach is that the quotient algebra corresponding to a given knowledge base, together with its mapping to the universal concept algebra, elucidates the distinctions between descriptive and greatest fixed point semantics. This coincides with Bernhard Nebel's comment [Nebel, 1990] that in descriptive fixed point models, concept names are a distinguishing feature. In fact, in a new implementation of K-REP two spaces of objects are maintained, a definition space corresponding to the quotient algebra, and a semantic space corresponding to the universal concept algebra. Two concepts might be defined with different names that classify to the same object. Their separate definitions arc kept in the definition space, yet they point to the same object in the semantic space. We feel this is a first step towards an intensional semantics that will enhance our understanding of how to implement correctly the effect of definitional changes on subsumption relations.

In the next section we discuss some related works. This is followed by a brief section describing the description logic we are discussing. Then, there is a section reviewing [Dionne *et a*/., 1992b] covering the formal definitions of concept algebras and of homomorphisms between them. In Section 5 we outline the construction of extensional greatest fixed point models. Next is Section 6, which contains the main result: a theorem relating the universal concept algebra to the class of extensional greatest fixed point models. The paper ends with remarks on intensional semantics and prospects for handling negations.

2 Related Work

Various fixed point models have been investigated in [Nebel, 1990]. The problem of cyclic definitions, in roughly the same subset of description logics that we discuss here, was first solved in [Baader, 1990]. Baader views concept descriptions as automata, and couches subsumption questions in terms of language acceptance.

A use of non-well-founded sets similar to ours is made in [Rounds, 1991]. There the emphasis is on merging relational data bases, situation theory, and feature logics. Our main emphasis is on subsumption of terms and the incremental maintenance of a network of terms. Our work is restricted to description logics, with emphasis placed on issues particular to them. The "complex objects" appearing in [Rounds, 1991] have both intensional and extensional aspects, and both least upper bounds and greatest lower bounds of complex objects are examined, including a detailed explanation of how Aczel's Solution Lemma is used to compute the least upper bound. In our implementation we are never required actually to compute least upper bounds, since subsumption is computed directly. Although our situation is, on the surface, less complex, it does not seem to be a special case of the results presented in [Rounds, 1991]. Nevertheless, the use of non-well-founded set theory to handle recursive definitions is similar.

Another algebraic approach [Brink and Schmidt, 1992], involves an equational algebra of relations acting on a boolean algebra of sets, similar to the notion of a module. Though more general than concept algebras, since roles are treated as first class citizens, the details of subsumption, which amounts to equational reasoning in their algebra, are not yet worked out. Even though cycles are not considered, this work is similar to concept algebras, in that the view is algebraic as opposed to model-theoretic.

3 The Representation Language

As in [Dionne *et a*/., 1992b], we are only considering a small subset of K-REP [Mays *et al.;* 1991], a description logic based knowledge representation language. Table 1 shows the constructs in this language, together with their abstract form and set-theoretic semantics. Though we have developed concept algebras for a larger model that includes disjunctions, and are currently pursuing negation, we restrict ourselves here to the original model. This is the model for which we prove the conjecture originally stated in [Dionne *et al.y* 1992b]. Note that our "allsonse" operator combines two operators that are normally kept separate in most description logics. The "some" operator by itself would act as a join homomorphism (see [Brink and Schmidt, 1992]), and one attempt at extending concept algebras to handle negations involves separating the two out to make use of the classical relationship between negation and existential and universal quantification.

We assume the reader is familiar with description logics. The point we wish to emphasize is that concepts are organized into an ordered structure (in this model a meet-semilattice, in [Dionne *et a*/., 1992a] a distributive lattice), by a process called classification, whose main component is structural subsumption. One concept term subsumes another if: 1) all of its primitive components are contained in the other, 2) its roles are a subset of the roles of the other, and 3) the value restriction of each of its roles subsumes the value restriction of the other concept's role. Later this subsumption ordering will be rigorously stated in terms of the ordering of elements in the universal concept algebra.

4 Concept Algebras

Consider the signature E containing a set *P* of constants, a set !R of unary operators, a binary operator A, and a constant T. Let *E* be the following set of axioms with respect to this signature:

Table 1: A subset of the K-REP language

Concrete Form	Abstract Form	Semantics
Concept Forming Operators		
top	$\top$	$\mathcal{U}$
(and $C_1 \ldots C_n$)	$C_1 \wedge \ldots \wedge C_n$	$C_1^{\mathcal{I}} \cap \ldots \cap C_n^{\mathcal{I}}$
(allsome R C)	$\forall\exists R : C$	$\{d \in \mathcal{U} \mid R^{\mathcal{I}}(d) \subseteq C^{\mathcal{I}} \wedge R^{\mathcal{I}}(d) \neq \emptyset\}$ where $R^{\mathcal{I}} \subseteq \mathcal{U} \times \mathcal{U}$
Terminological Axioms		
(defconcept N C)	$N \equiv C$	$N^{\mathcal{I}} = C^{\mathcal{I}}$
(defprimconcept N C)	$N \sqsubseteq C$	$N^{\mathcal{I}} \subseteq C^{\mathcal{I}}$

$$
\begin{array}{rcll}
x \wedge x & = & x & (\textit{idempotence}) \\
x \wedge y & = & y \wedge x & (\textit{commutativity}) \\
(x \wedge y) \wedge z & = & x \wedge (y \wedge z) & (\textit{associativity}) \\
x \wedge \top & = & x & (\top \text{ is a unit}) \\
R(x \wedge y) & = & R(x) \wedge R(y) & \forall R \in \Re
\end{array}
$$

We call the class of algebras for this signature Σ that satisfy the axioms E, *concept algebras.* Using $\wedge$, a partial order $\geq$ can be imposed on a concept algebra ($p \geq q$ iff $p \wedge q = q$). The constants in P are meet-irreducible primitives and each $R \in \Re$ defines a meet homomorphism on the algebra.

Given $X = \{x_1, \ldots, x_n\}$, $A[X] = A[x_1, \ldots, x_n]$ is the free concept algebra generated by X. $A[\,] = A[\emptyset]$ is the initial concept algebra.

A KB is a set of n possibly mutually recursive definitions

$$\Delta = \{x_1 \equiv t_1, \ldots, x_n \equiv t_n\}$$

where $t_i \in A[x_1, \ldots, x_n]$. We can then let $\equiv_\Delta$ be the least congruence on $A[X]$ that contains Δ, and then define $A_\Delta[X]$ to be the quotient algebra $A[X]/\equiv_\Delta$. This is a standard construction (see [Jacobson, 1989] and [Dionne *et al.*, 1992b] for details). Let $\pi_\Delta : A[X] \rightarrow A_\Delta[X]$ be the canonical projection homomorphism that sends each term to its congruence class.

For each KB, Δ, we will be interested in three relations on $A[X]$:

$$s_1 \sqsupseteq_\Delta s_2 \text{ (descriptive subsumption)}$$

$$s_1 \succeq_\Delta s_2 \text{ (structural subsumption)}$$

$$s_1 \supseteq_\Delta s_2 \text{ (extensional subsumption)}$$

The algebra $A_\Delta[X]$ captures descriptive semantics in the sense that names with structurally similar definitions are not necessarilly identified.

Definition 4.1 *Given two terms $s_1, s_2 \in A[X]$, we say s_1 descriptively subsumes s_2, written $s_1 \sqsupseteq_\Delta s_2$, iff $\pi_\Delta s_1 \geq_{A_\Delta[X]} \pi_\Delta s_2$.*

We next wish to construct an algebra that provides an intensional semantics for terms and that is used to define structural subsumption. There will be no variables in this algebra, and the interesting sets of equations with cycles in them have unique solutions in this semantic algebra. The *universal concept algebra* is the greatest fixed-point solution of the equation

$$C = (\mathcal{P}_{<\omega} P) \times (\Re \xrightarrow[<\omega]{\cdot} C)$$

where $(\mathcal{P}_{<\omega} P)$ is the collection of finite subsets of P, and $(\Re \xrightarrow[<\omega]{\cdot} C)$ is the collection of partial functions from $\Re$ to C with finite domain.

A concept definition is composed of a collection of primitives conjoined with a collection of role definitions. Each element of C is an ordered pair whose first component is a set of primitives from P, and whose second component is a set of ordered pairs, each arising from one of the role definitions. This set can just be represented as a partial function on $\Re$, defined for each role in the concept's definition. Note that if we were working within standard Zermelo-Fraenkel set theory (ZFC) the only solution to the above equation would be $C = \emptyset$! However, a nontrivial solution C that corresponds to our intuitions exists in a consistent extension of ZFC, namely, Aczel's theory of non-well-founded sets (roughly sets that can contain themselves as members). This is how circularity of concept definitions can be supported.

To see that C is a concept algebra, we must interpret the operators of Σ:

$$\top_C = \langle \emptyset, \emptyset \rangle$$

$$p_C = \langle \{p\}, \emptyset \rangle$$

$$R_C(x) = \langle \emptyset, \{\langle R, x \rangle\} \rangle$$

Given $C_1 = \langle Q_1, f_{C_1} \rangle$ where $Q_1 \subseteq P$ and $f_{C_1} \in (\Re \xrightarrow[<\omega]{\cdot} C)$ and similarly for $C_2 = \langle Q_2, f_{C_2} \rangle$ we define:

$$C_1 \wedge C_2 \equiv \langle Q_1 \cup Q_2, f_{C_1 \wedge C_2} \rangle$$

where $f_{C_1 \wedge C_2} \in (\Re \xrightarrow[<\omega]{\cdot} C)$ is defined as $f_{C_1 \wedge C_2}(R) =$

$$
\begin{cases}
f_{C_1}(R) & \text{if } R \in dom(f_{C_1}) - dom(f_{C_2}) \\
f_{C_2}(R) & \text{if } R \in dom(f_{C_2}) - dom(f_{C_1}) \\
f_{C_1}(R) \wedge f_{C_2}(R) & \text{if } R \in dom(f_{C_1}) \cap dom(f_{C_2})
\end{cases}
$$

Technically, to define the meet operation on C, one must do a bit more than write down this recursive definition, because it is not entirely clear that this definition leads to a well defined function. However, the recursive definition can readily be translated into a simultaneous system of equations, to which the Solution Lemma [Acsel, 1988] can be applied, and this allows us to assert the existence and uniqueness of the meet operation (see [Rounds, 1991]). With respect to Δ, let us define what is meant by a *good* set of definitions.

Definition 4.2 *A set of definitions Δ is* good *if each x_i appears only once on the left hand side, and each equation $x_i = t_i$ is of the form $x_i = (\bigwedge_j P_{i,j}) \wedge (\bigwedge_k (R_{i,k}\ s_{i,k}))$ where $P_{i,j} \in P$ and $s_{i,k} \in A[X]\ \forall i, j, k$. In other words each equation is composed of a conjunction of a conjunction of primitives, and a conjunction of role terms that may or may not contain variables.*

Theorem 4.3 *There exists a unique concept algebra homomorphism $A_\Delta[x_1, \ldots, x_n] \xrightarrow{\varphi_\Delta} C$ if Δ is a* good set *of definitions*

The proof appears in [Dionne *et al.*, 1992b]. It uses Acsel's Solution Lemma, along with some basic theorems of universal algebra (see [Acsel, 1988; Jacobson, 1989]).

Thus, we have the following picture:

$$A[X] \xrightarrow{\pi_\Delta} A_\Delta[X] \xrightarrow{\varphi_\Delta} C$$

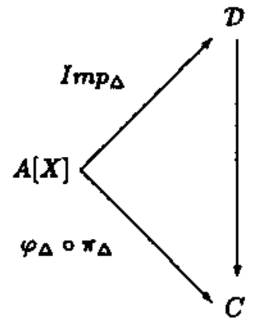
Now we can define *structural subsumption*:

Definition 4.4 *Given two terms $s_1, s_2 \in A[X]$, we say s_1 structurally subsumes s_2, written $s_1 \succeq_\Delta s_2$, iff $(\varphi_\Delta \circ \pi_\Delta)s_1 \geq_C (\varphi_\Delta \circ \pi_\Delta)s_2$.*

Thus, structural subsumption is given by the ordering on C. We now explicitly state the ordering on C, to both relate it to $\wedge$ and the actual implementation in K-REP. Given $C_1 = \langle Q_1, f_{C_1} \rangle$ where $Q_1 \subseteq P$ and $f_{C_1} \in (\Re \xrightarrow[<\omega]{} C)$ and similarly for $C_2 = \langle Q_2, f_{C_2} \rangle$, we say that $C_1 \geq_C C_2$, if $Q_1 \subseteq Q_2$, and $dom(f_{C_1}) \subseteq dom(f_{C_2})$, and $\forall R \in dom(f_{C_1})$, $f_{C_1}(R) \geq_C f_{C_2}(R)$. This is exactly the test for subsumption of concepts stated earlier in the section on K-REP. Notice also that when $C_1 \geq_C C_2$, if one inspects $C_1 \wedge C_2$, that $Q_1 \cup Q_2 = Q_2$ and since $dom(f_{C_1}) \subseteq dom(f_{C_2})$, that $f_{C_1 \wedge C_2}$ reduces to the second and third cases. The third case corresponds to $f_{C_1}(R) \geq_C f_{C_2}(R)$, and thus $C_1 \geq_C C_2$ iff $C_1 \wedge C_2 = C_2$.

The algorithm for computing subsumption, essentially constructs apg-like objects[1] for each term of $A[X]$ in which we are interested. Let $\mathcal{D}$ be the collection of these objects. This construction corresponds to the map Imp_Δ in the diagram below. Each apg in $\mathcal{D}$ then describes a unique set in C via the unnamed arrow. The testing of subsumption is actually testing the presence of a Hoare simulation between the objects in C. With respect to our implementation we now have the following commutative diagram.

[1]apgs (accessible pointed graphs) are pictures of non-well-founded sets (see [Acsel, 1988])



(Our new implementation of K-REP creates two spaces of objects representing concepts. The *definition space* corresponds to $A_\Delta[X]$, and the *semantic space* corresponds to C. Thus the definition space allows for multiple definitions that might map to the same object in the semantic space).

The fact that φ_Δ preserves order proves the following theorem:

Theorem 4.5 *Descriptive subsumption implies structural subsumption.*

This shows us that descriptive subsumption is weaker than structural subsumption, and agrees with Proposition 5.2 (page 133) in [Nebel, 1990].

5 Extensional Greatest Fixed Point Models

In this section we will define *extensional*, i.e., model-theoretic, subsumption. We need to do this carefully in order to have the necessary notation to carry out the proof to be given in the next section.

If B is a concept algebra and $\alpha: X \to B$, then $\hat{\alpha}: A[X] \to B$ is the unique homomorphism that extends the definition of α on X to the free algebra $A[X]$. The effect of applying $\hat{\alpha}$ to a term $s \in A[X]$ is to substitute in s for occurences of variables corresponding to values from B specified by $\alpha: X \to B$.

From a formal point of view, a knowledge base

$$\Delta = \{x_1 \equiv t_1, \ldots, x_n \equiv t_n\}$$

is a function $\Delta: X \to A[X]$, where $\Delta x_i = t_i$. The function Δ induces a transformation $K_{\Delta,B}: B^X \to B^X$ defined by $K_{\Delta,B}\alpha = \hat{\alpha} \circ \Delta$, for all $\alpha: X \to B$. A fixed point of $K_{\Delta,B}$ is exactly what is meant by a solution in the concept algebra B to the system of equations

$$\Delta = \{x_1 \equiv t_1, \ldots, x_n \equiv t_n\}.$$

As a general reference on ordered sets, we suggest [Davey and Priestley, 1990]. Let L and M be complete lattices. A function from L to M is *Scott continuous* if it preserves sup's of directed subsets of L. If $f: L \to L$ is Scott continuous, then the Tarski-Scott Fixed Point Theorem asserts that f has a least fixed point, and this least fixed point is given by $\bigvee_{n \geq 0}(f^n \bot)$. However, extensional subsumption is defined using greatest fixed points, so we need a dual notion of continuity. A function from L to M is *dually Scott continuous* if it preserves inf's of filtered subsets of L. The Tarski-Scott Fixed Point Theorem implies that, if $f: L \to L$ is dually Scott continuous, then f has a greatest fixed point, and this greatest fixed point is given by $\bigwedge_{n \geq 0}(f^n \top)$.

To build an extensional greatest fixed point model in the context of a knowledge base $\Delta: X \to A[X]$, we start with a universe $\mathcal{U}$. Then we interpret the primitives as subsets of $\mathcal{U}$ and the roles as binary relations on $\mathcal{U}$. Using these interpretations, and guided by Table 1, we endow the power set $\mathcal{B} = 2^{\mathcal{U}}$ with a concept algebra structure. Of course, $\mathcal{B}$ is a complete lattice under is set inclusion. Furthermore, each of the concept algebra operators turns out then to have a dually Scott continuous interpretation as an operation on $\mathcal{B}$. Since $\mathcal{B}$ is a complete lattice, so also is the set $\mathcal{B}^X$, ordered pointwise. Since the operations on $\mathcal{B}$ are dually Scott continuous, it follows that $K_{\Delta,\mathcal{B}}: \mathcal{B}^X \to \mathcal{B}^X$ is also dually Scott continuous. Hence, $K_{\Delta,\mathcal{B}}$ has a greatest fixed point $\beta: X \to \mathcal{B}$. The extensional greatest fixed point model for $A[X]$ is then the homomorphism

$$\mathcal{M} = \hat{\beta}: A[X] \to \mathcal{B}.$$

It is conventional to denote by $s^{\mathcal{M}} = \hat{\beta}s$ the subset of $\mathcal{U}$ assigned to a term s by an extensional greatest fixed point model $\mathcal{M}$.

Definition 5.1 *Given two terms $s_1, s_2 \in A[X]$, we say s_1 extensionally subsumes s_2 (written $s_1 \sqsupseteq_\Delta s_2$) iff, for all extensional greatest fixed point models $\mathcal{M}$, $s_1^{\mathcal{M}} \supseteq s_2^{\mathcal{M}}$.*

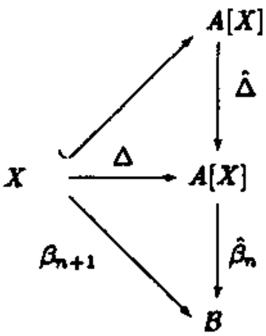
The remainder of this section is devoted to a closer look at a fixed extensional greatest fixed point model $\hat{\beta}: A[X] \to \mathcal{B}$. Note that the maximum element of $\mathcal{B}^X$ is the function $\beta_0: X \to \mathcal{B}$ given by $\beta_0 x = \mathcal{U}$, for all $x \in X$. If we inductively define $\beta_n: X \to \mathcal{B}$ by $\beta_{n+1} = K_{\Delta,\mathcal{B}}\beta_n$ for each nonnegative integer n, it follows that $\beta_0 \geq \beta_1 \geq \beta_2 \geq \cdots$, and $\beta = \bigwedge_{n \in \omega} \beta_n$. Upon realising that $\mathcal{B}^{A[X]}$, ordered pointwise, is a complete lattice, it is not hard to prove the following proposition.

Proposition 5.2 *$\hat{\beta}_0 \geq \hat{\beta}_1 \geq \hat{\beta}_2 \geq \ldots$ and $\hat{\beta} = \bigwedge_{n \in \omega} \hat{\beta}_n : A[X] \to \mathcal{B}$.*

In keeping with our conventions, the unique homomorphism extending $\Delta: X \to A[X]$ is $\hat{\Delta}: A[X] \to A[X]$.

Proposition 5.3 *For all nonnegative integers n, $\hat{\beta}_{n+1} = \hat{\beta}_n \circ \hat{\Delta}$.*

Proof. This follows easily by considering the commutative diagram



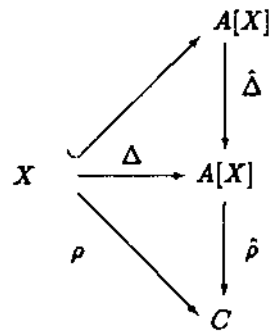
□

Corollary 5.4 *For all nonnegative integers k, $\hat{\beta} = \bigwedge_{n \in \omega}(\hat{\beta}_n \circ \hat{\Delta}^k)$.*

6 Concept Algebras and Greatest Fixed Point Models

In this section we will prove Theorem 6.3, which asserts that structural subsumption is equivalent to model-theoretic subsumption. This theorem was first posed as a conjecture in [Dionne *et al.*, 1992b]. Roughly speaking, it states that structural subsumption in the algebra C is the most abstract of all the greatest fixed point models. By "most abstract" we mean that if a subsumption holds in C, then it holds in all greatest fixed point models, and conversely. In fact it appears that a stronger statement can be made about finite models, which we will discuss later as a new conjecture.

Let $\Delta: X \to A[X]$ be some fixed knowledge base that is a *good* set of definitions. To simplify notation, let $\hat{\rho}$ be the intensional model described in Section 4 that corresponds to Δ. Thus, $\hat{\rho} = \varphi_\Delta \circ \pi_\Delta : A[X] \to C$. When the homomorphism $\hat{\rho}$ is restricted to the set X of knowledge base variables, we obtain $\rho: X \to C$, which gives the *unique solution* in C to the system of equations $\Delta = \{x_1 \equiv t_1, \ldots, x_n \equiv t_n\}$. Since ρ is the unique fixed point of $K_{\Delta,C}$, it follows that $\rho = \hat{\rho} \circ \Delta$. By considering the diagram



we see that $\hat{\rho} = \hat{\rho} \circ \hat{\Delta}$.

For each $t \in A[X]$, let *depth* t be the length of the longest role chain in t. (Note that *depth* t is a purely syntactic notion that is independent of Δ. One could not similarly define the depth of an element of $A_\Delta[X]$ because in that algebra there may be infinite role chains.)

Most of this section is devoted to the proof of the next proposition, which is fairly technical. Its proof can be skipped on a first reading. The proof of the proposition is based on the following algebraic fact: since $A[X]$ is a free algebra, each term in it has a *unique decomposition* in the sense that, for each $t \in A[X]$, there exist uniquely determined sets $Q_t \subseteq P$ and $V_t \subseteq X$ and a uniquely determined finite partial function $F_t \in \Re \xrightarrow[<\omega]{} A[X]$ such that

$$t = \bigwedge Q_t \wedge \bigwedge V_t \wedge \bigwedge\{R\,(F_t R) \mid R \in dom F_t\}.$$

Thus, the fact that Δ is a good set of definitions can be stated succinctly as $(\forall x)(V_{\Delta x} = \emptyset)$.

Proposition 6.1 *Let $\mathcal{M} = \hat{\beta}: A[X] \to \mathcal{B}$ be an extensional greatest fixed point model, where $\mathcal{B} = 2^{\mathcal{U}}$. For*

each integer n, let the homomorphism $\hat{\beta}_n: A[X] \to B$ be defined as in Section 5. For all nonnegative integers n and k, if $t, t' \in A[X]$ satisfy $\hat{\rho}t = \hat{\rho}t'$ and $k = \max\{depth\ t', \max\{depth\ \Delta x \mid x \in X\}\}$, then $\hat{\beta}_{n+k+1}t \leq \hat{\beta}_n t'$.

Proof. We will use a double induction on n and k.

Case 1: $n = k = 0$. Given $t, t' \in A[X]$, we need to show that $\hat{\beta}_1 t \leq \hat{\beta}_0 t'$. Since depth $t' = 0$, it must be that

$$t' = \bigwedge Q_{t'} \wedge \bigwedge V_{t'}.$$

Note that $\hat{\beta}_0 t' = \bigwedge Q_{t'}$ since $\hat{\beta}_0 x = \top$, and $\hat{\rho}t = \hat{\rho}t'$ implies $\hat{\rho}(\hat{\Delta}t) = \hat{\rho}(\hat{\Delta}t')$. Since $k = 0$,

$$\hat{\rho}(\hat{\Delta}t') = \hat{\rho}(\wedge Q_{t'}) \wedge \hat{\rho}(\bigwedge_{x' \in V_{t'}} (\wedge Q_{\hat{\Delta}x'})).$$

Now we have $\hat{\rho}(\hat{\Delta}t) =$

$$\begin{aligned}&\hat{\rho}(\wedge Q_t)\\ &\wedge\hat{\rho}(\textstyle\bigwedge_{x\in V_t}[\wedge Q_{\Delta x} \wedge \bigwedge\{R\ (F_{\Delta x}\ R) \mid R \in domF_{\Delta x}\}])\\ &\wedge\hat{\rho}(\textstyle\bigwedge\{R\ (\hat{\Delta}(F_t\ R)) \mid R \in domF_t\}),\end{aligned}$$

from which it follows that $\hat{\rho}(\hat{\Delta}t) =$

$$\begin{aligned}&\hat{\rho}(\wedge Q_t)\\ &\wedge\hat{\rho}(\textstyle\bigwedge_{x\in V_t}(\wedge Q_{\Delta x})\\ &\wedge[\textstyle\bigwedge\{R\ (G_t\ R) \mid R \in domF_t \cup \bigcup_{x\in V_t} domF_{\Delta x}\}),\end{aligned}$$

where $G_t(R) =$

$$\begin{cases} \bigwedge_x (F_{\Delta x} R \wedge (\hat{\Delta}(F_t R))) & R \in domF_{\Delta x} \cap domF_t \\ \bigwedge_x F_{\Delta x} R & R \in domF_{\Delta x} - domF_t \\ F_t R & R \in domF_t - domF_{\Delta x} \end{cases}$$

So, $\hat{\rho}(\hat{\Delta}t) = \hat{\rho}(\hat{\Delta}t')$ implies

$$\hat{\rho}(\wedge Q_t \wedge \bigwedge_{x\in V_t} (\wedge Q_{\hat{\Delta}x})) = \hat{\rho}(\wedge Q_{t'} \wedge \bigwedge_{x'\in V_{t'}} (\wedge Q_{\hat{\Delta}x'})).$$

Therefore $\hat{\beta}_1 t = \hat{\beta}_0(\hat{\Delta}t) = \hat{\beta}_0[\wedge Q_t \wedge \bigwedge_{x\in V_t}(\wedge Q_{\Delta x}) \cdots] \leq \hat{\beta}_0(\wedge Q_{t'}) = \hat{\beta}_0 t'$

Case 2: Suppose the proposition is true if $n = 0$ and $\forall k < j$. Consider the case $k = j, n = 0$. We need to show that $\hat{\beta}_{j+1} t \leq \hat{\beta}_0 t'$. As in the base case,

$$t = \bigwedge Q_t \wedge \bigwedge V_t \wedge \bigwedge\{R\ (F_t R) \mid R \in domF_t\}$$

and similarly for t'. Again, we have that $\hat{\rho}t = \hat{\rho}t'$ implies $\hat{\rho}(\hat{\Delta}t) = \hat{\rho}(\hat{\Delta}t')$. So, $\hat{\rho}(\hat{\Delta}t) =$

$$\begin{aligned}&\hat{\rho}(\wedge Q_t)\\ &\wedge\hat{\rho}(\textstyle\bigwedge_{x\in V_t}(\wedge Q_{\Delta x})\\ &\wedge\textstyle\bigwedge\{R\ (G_t R) \mid R \in domF_t \cup \bigcup_{x\in V_t} domF_{\Delta x}\})\end{aligned}$$

and $\hat{\rho}(\hat{\Delta}t') =$

$$\begin{aligned}&\hat{\rho}(\wedge Q_{t'})\\ &\wedge\hat{\rho}(\textstyle\bigwedge_{x\in V_{t'}}(\wedge Q_{\Delta x'})\\ &\wedge\textstyle\bigwedge\{R\ (G_{t'} R) \mid R \in domF_{t'} \cup \bigcup_{x'\in V_{t'}} domF_{\Delta x'}\}),\end{aligned}$$

where $G_{t'}$ is defined the same way as G_t. Note that:

$$\max\{depth\ G_{t'}(R), \max\{depth\ \Delta x' \mid x' \in V_{t'}\}\} < j$$

Thus, separating the action of $\hat{\rho}$ on t, t' into its primitive and role components allows us to use the induction hypothesis, so that $\hat{\beta}_{j+1} t =$

$$\begin{aligned}&\hat{\beta}_j((\wedge Q_t)\\ &\wedge \textstyle\bigwedge_{x\in V_t}(\wedge Q_{\Delta x})\\ &\wedge \textstyle\bigwedge\{R(G_t R) \mid R \in domF_t \cup \bigcup_{x\in V_t} domF_{\Delta x}\})\\ \leq\ &\hat{\beta}_0((\wedge Q_{t'})\\ &\quad\wedge \textstyle\bigwedge_{x'\in V_{t'}}(\wedge Q_{\Delta x'})\\ &\quad\wedge \textstyle\bigwedge\{R(G_{t'} R) \mid R \in domF_{t'} \cup \bigcup_{x'\in V_{t'}} domF_{\Delta x'}\})\\ =\ &\hat{\beta}_0(\hat{\Delta}t')\\ \leq\ &\hat{\beta}_0 t'.\end{aligned}$$

Case 3: $n = m, k = 0$ where the proposition is true for all $n < m$. We need to show that $\hat{\beta}_{m+1} \leq \hat{\beta}_m t'$. The action of $\hat{\rho}$ on t and t' are the same as in the previous cases, so we'll just sketch the main argument. Observe that $\hat{\beta}_{m+1} t =$

$$\begin{aligned}&\hat{\beta}_m((\wedge Q_t)\\ &\wedge \textstyle\bigwedge_{x\in V_t}(\wedge Q_{\Delta x})\\ &\wedge \textstyle\bigwedge\{R\ (G_t R) \mid R \in domF_t \cup \bigcup_{x\in V_t} domF_{\Delta x}\}).\\ \leq\ &\hat{\beta}_m(\wedge Q_t \wedge \textstyle\bigwedge_{x\in V_t}(\wedge Q_{\Delta x}))\end{aligned}$$

Since $k = 0$, application of the induction hypothesis yields

$$\begin{aligned}\hat{\beta}_{m+1} t &\leq \hat{\beta}_{m-1}(\wedge Q_{t'} \wedge \textstyle\bigwedge_{x'\in V_{t'}}(\wedge Q_{\Delta x'}))\\ &= \hat{\beta}_{m-1}(\hat{\Delta}[\wedge Q_{t'} \wedge \textstyle\bigwedge V_{t'}])\\ &= \hat{\beta}_m t'.\end{aligned}$$

Case 4: $n = m, k = j$ where the proposition is true for all $n < m$ and $k < j$. We need to see that $\hat{\beta}_{m+j+1} t \leq \hat{\beta}_m t'$. We have $\hat{\beta}_{m+j+1} t =$

$$\begin{aligned}&\hat{\beta}_{m+j}((\wedge Q_t)\\ &\wedge \textstyle\bigwedge_{x\in V_t}(\wedge Q_{\Delta x})\\ &\wedge \textstyle\bigwedge\{R(G_t R) \mid R \in domF_t \cup \bigcup_{x\in V_t} domF_{\Delta x}\})\\ =\ &\hat{\beta}_{m+j-1}((\wedge Q_t)\\ &\quad\wedge \textstyle\bigwedge_{x\in V_t}(\wedge Q_{\Delta x})\\ &\quad\wedge \textstyle\bigwedge\{R(\hat{\Delta}(G_t R)) \mid R \in domF_t \cup \bigcup_{x\in V_t} domF_{\Delta x}\}).\end{aligned}$$

Therefore, by the induction hypothesis, $\hat{\beta}_{m+j+1} t \leq$

$$\begin{aligned}&\hat{\beta}_{m-1}((\wedge Q_{t'})\\ &\wedge \textstyle\bigwedge_{x'\in V_{t'}}(\wedge Q_{\Delta x'})\\ &\wedge \textstyle\bigwedge\{R(\hat{\Delta}(G_{t'} R)) \mid R \in domF_{t'} \cup \bigcup_{x'\in V_{t'}} domF_{\Delta x'}\})\\ =\ &\hat{\beta}_m t'.\end{aligned}$$

This proves Proposition 6.1. □

Lemma 6.2 *Let β be the greatest fixed point of $K_{\Delta,B}$ for an arbitrary algebra B of sets corresponding to some model. For all $t, t' \in A[X]$ $\hat{\rho}t = \hat{\rho}t'$ implies $\hat{\beta}t = \hat{\beta}t'$.*

Proof. Suppose that $\hat{\rho}t = \hat{\rho}t'$. By the preceding proposition,

$$\hat{\beta}t = \bigwedge_{n\in\omega} \hat{\beta}_n(\hat{\Delta}^k t) \leq \bigwedge_{n\in\omega} \hat{\beta}_n t' = \hat{\beta}t'$$

The reverse inequality is proved similarly. □

Theorem 6.3 *Given two terms $s_1, s_2 \in A[X]$*

$$s_1 \succeq_\Delta s_2 \quad \textit{iff} \quad s_1 \supseteq_\Delta s_2.$$

Proof. The proof in the "if" direction is a direct consequence of a Stone-like representation theorem, that every meet-semilattice is isomorphic to a meet-semilattice of sets (see pp. 82 of [Davey and Priestley, 1990]). It is easy to see that this implies that C is isomorphic to an extensional greatest fixed point model.

For the "only if" direction, suppose we have an arbitrary greatest fixed point model $\mathcal{M} = \hat{\beta}: A[X] \rightarrow B$, where $B = 2^{\mathcal{U}}$. To prove that $s_1 \succeq_\Delta s_2 \Rightarrow s_1 \supseteq_\Delta s_2$, $\forall s_1, s_2 \in A[X]$, we argue as follows:

$$s_1 \succeq_\Delta s_2 \Rightarrow \hat{\rho}(s_1 \wedge s_2) = \hat{\rho}(s_2)$$

$$\Rightarrow \hat{\beta}(s_1 \wedge s_2) = \hat{\beta}(s_2) \Rightarrow s_1^{\mathcal{M}} \supseteq s_2^{\mathcal{M}}$$

Since $\mathcal{M}$ is arbitrary, this implies $s_1 \supseteq_\Delta s_2$, and this proves Theorem 6.3. □

It appears that more can be said about the "only if" direction of this proof. Namely that if two terms do not subsume one another in the universal concept algebra C, then it appears that one can construct a finite model that exhibits this failure. The intuition comes from observing that if one concepts fails to subsume another, then it is for one of three reasons. Either they disagree on primitives, or the subsuming concepts roles are not a subset of the subsuming concepts roles, or they disagree on some value restriction. The third case eventually reduces to either the first or the second, and even in the cyclic cases, that clearly give rise to infinite chains, some finite piece is sufficient. However this reasoning is hardly a proof and since we've proved one conjecture already we'll finish with another.

Conjecture 6.4 *Given two terms $s_1, s_2 \in A[X]$*

$$s_1 \succeq_\Delta s_2 \quad \textit{iff} \quad s_1 \supseteq_\Delta s_2 \quad \textit{iff} \quad s_1^{\mathcal{M}} \supseteq s_2^{\mathcal{M}} \ \forall \textit{ finite models } \mathcal{M}.$$

7 Conclusion

This paper has shown that structural subsumption as computed in the universal concept algebra, and subsumption in all greatest fixed point models, are essentially the same. Therefore, for purposes of subsumption testing, one need not appeal to extensional models. This appears to be a first step towards intensional semantics, in the larger sense, as advocated in [Woods, 1991]. Concepts are seen as just descriptions. Their relationships, via subsumption, to other concepts is a structural comparison process. Each concept in K-REP gives rise to two objects: a definitional one in $A_\Delta[X]$ and a semantic one in C.

Currently we are working on extending concept algebras, which now include disjunctions (see [Dionne et al., 1992a]), to include negation while still handling cycles. One approach might make use of the fact that a finite distributive lattice is a Heyting algebra, in which one can define negation in terms of disjunction. Another approach would be to use boolean algebras, in which case normalization in the presence of cycles is problematic.

References

[Aczel, 1988] Peter Aczel. Non-Well-Founded Sets, volume 14 of CSLI Lectures Notes, CSLI/Standford, 1988.

[Baader and Hollunder, 1991] Franz Baader and Bernhard Hollunder. KRIS: Knowledge representation and inference system. SIGART Bulletin, 2(3), June 1991.

[Baader, 1990] Franz Baader. Terminological cycles in KL-ONE-based knowledge representation languages. In Proceedings of AAAI, Boston, Mass., June 1990.

[Borgida et al, 1989] Alexander Borgida, Ronald J. Brachman, Deborah L. McGuinness, and Lori Alperin Resnick. CLASSIC: a structural data model for objects. In Proceedings of the 1989 ACM SIGMOD International Conference on Management of Data, Portland, Oregon, May-June 1989.

[Brink and Schmidt, 1992] Chris Brink and Renate A. Schmidt. Subsumption computed algebraically. Computers Math, Applications., 23(2-5):329-342, 1992.

[Davey and Priestley, 1990] B. A. Davey and H. A. Priestley. Introduction to Lattices and Order. Cambridge University Press, 1990.

[Dionne et aL, 1992a] Robert Dionne, Eric Mays, and Frank J. Oles. Disjunctive concept algebras. Technical Report RC 18458, IBM, 1992.

[Dionne et aL, 1992b] Robert Dionne, Eric Mays, and Frank J. Oles. A non-well-founded approach to terminological cycles. In Proceedings of AA AI 92, San Jose, Cal., July 1992.

[Jacobson, 1989] Nathan Jacobson. Basic Algebra II. W.H. Freeman and Company, 1989.

[MacGregor and Bates, 1987] Robert MacGregor and Raymond Bates. The LOOM knowledge representation language. Technical Report ISI/RS-87-188, University of Southern California, Information Science Institute, Marina del Rey, Cal., 1987.

[Mays et al, 1991] E. Mays, R. Dionne, and R. Weida. K-REP system overview. SIGART Bulletin, 2(3), June 1991.

[Nebel, 1990] Bernhard Nebel. Reasoning and Revision in Hybrid Representation Systems, volume 422 of Lecture Notes in Artificial Intelligence. Springer-Verlag, 1990.

[Rounds, 1991] William C. Rounds. Situation-theoretic aspects of databases. In Barwise, Gawron, Plotkin and Tutiya, editor, Situation Theory and its Applications, volume 26. CSLI/Stanford, 1991.

[Woods, 1991] William A. Woods. Understanding subsumption and taxonomy: A framework for progress. In J. F. Sowa, editor, Principles of Semantic Networks. Morgan Kaufman, 1991.