Reeb vector field as isometric embedding

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Considering **Sasaki metric** on the unit tangent bundle T_1M, a map $\xi : (M,g) \to (T_1M, G)$, defining by $\xi(x) = (x, \xi(x))$, is **isometric embedding** only if ξ is **parallel**. The rigidity of Sasaki metric motivates many authors consider various deformations of Sasaki metric (see [1, 5, 6, 7, 11]). In particular, Domenico Perrone [8] studied **Reeb vector fields** with respect to a **Riemannian g-natural metrics** on the unit tangent bundle.

A **Riemannian** g**-natural metric** [1, 2] on the unit tangent bundle T_1M is defined by

$$
\begin{aligned}
G_{(x,\xi)}(X^h, Y^h) &= (a+c)g_x(X,Y) + dg_x(X,\xi)g_x(Y,\xi),\\
G_{(x,\xi)}(X^h, Y^v) &= bg_x(X,Y),\\
G_{(x,\xi)}(X^v, Y^v) &= ag_x(X,Y),
\end{aligned}
$$

where $a, b, c, d = const$, $a > 0$.

A **contact metric manifold** is defined by a set (M, g, η, ξ, ϕ), where M is a differential $2n+1$-dimensional manifold, ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is 1-form satisfying

$$
\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi,
$$

$$
g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),
$$

$$
(d\eta)(X,Y) = g(X, \phi Y).
$$

Moreover, ξ is called **Reeb vector field** and η uniquely defines ξ by the conditions

$$
\eta(\xi) = 1, \quad d\eta(\xi, X) = 0.
$$

The Reeb vector field of a contact manifold plays a fundamental role in the study of the Riemannian geometry of a contact metric manifold (see [3]). A contact metric manifold (M, g, η, ξ, ϕ) is called K**-contact manifold** if ξ is **Killing vector field**. Moreover, if

$$
(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X,
$$

then the contact metric manifold (M, g, η, ξ, ϕ) is called **Sasakian manifold**.

Domenico Perrone [8] showed that there are **non-parallel** unit vector fields which define **isometric embeddings** with respect to a family of **Riemannian g-natural metrics** on the unit tangent bundle that depend on two parameters, which does not include the Sasaki metric.

Proposition 1. *Let (M, g, η, ξ, ϕ) be a contact metric manifold, $dimM = 2n+1$, and let G be a Riemannian g-natural metric on T_1M with $c = 1 - 2a$. Then the map $\xi : (M,g) \to (T_1M, G)$ is an isometric embedding if and only if $d = a$ and M is K-contact manifold.*

If $\xi(x)$ is a unit vector field on M, then it defines a map $\xi : M \to T_1M$, defining by $\xi(x) = (x, \xi(x))$. From geometrical viewpoint $\xi(M)$ is explicitly given submanifold in T_1M.

A unit vector field ξ is said to be **harmonic** (see [9]) if it is a critical point of the energy functional defined on the space of all unit vector fields. The corresponding map $\xi : M \to T_1M$ is said to be **harmonic map** if it is a critical point of the energy functional defined on the space of all maps from M to T_1M. Note that a harmonic vector field ξ does not define, in general, a harmonic map from $\xi : M \to T_1M$.

Minimal submanifold is a submanifold with vector of mean curvature zero. A unit vector field ξ on Riemannian manifold M is called **minimal** (see [4]) if the image of (local) embedding $\xi : M \to T_1M$

is minimal submanifold in the unit tangent bundle T_1M. Note that an isometric immersion is minimal if and only if it is a harmonic map.

Domenico Perrone [8] suggested the following theorems.

Theorem 2. *The Reeb vector field ξ of a K-contact manifold (M, g, η, ξ, ϕ) defines a harmonic map $\xi : (M, g) \to (T_1M, G)$ for any Riemannian g-natural metric G on T_1M.*

Theorem 3. *Let (M, g, η, ξ, ϕ) be a K-contact manifold. Let $\mathcal{F}$ be the family of all Riemannian g-natural metrics on T_1M defined by the parameters*

$$0 < a < 1, \quad b^2 < a(1-a), \quad c = 1-2a, \quad d = a.$$

Then, the Reeb vector field determines a minimal isometric immersion $\xi : (M, g) \to (T_1M, G)$ for any $G \in \mathcal{F}$.

Totally geodesic submanifold is a submanifold such that all geodesics in the submanifold are also geodesics of the surrounding manifold. A unit vector field ξ on Riemannian manifold M is called **totally geodesic** (see [10]) if the image of (local) embedding $\xi : M \to T_1M$ is totally geodesic submanifold in the unit tangent bundle T_1M. The corresponding map $\xi : M \to T_1M$ is said to be **totally geodesic map**. Namely, ξ is total geodesic if the second fundamental form of the map $\xi : M \to T_1M$ vanishes. Note that every totally geodesic map $\xi : M \to T_1M$ is harmonic and minimal.

The concept of totally geodesicity arises naturally in connection with the concepts of harmonicity and minimality. As a result, we have the following theorem.

Theorem 4. *Let (M, g, η, ξ, ϕ) be a K-contact metric manifold, $dimM = 2n+1$, and let G be a Riemannian g-natural metric on T_1M with $c = 1-2a$ and $d = a$. Then the Reeb vector field ξ defining the isometric embedding $\xi : (M, g) \to (T_1M, G)$ is totally geodesic if and only if M is Sasakian manifold.*

Thus totally geodesic property of the Reeb vector fields as isometric embeddings is distinguished Sasakian manifold among K-contact metric manifold with the Riemannian g-natural metrics on the unit tangent bundle.

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