Aggregation Operators on Bounded Partially Ordered Sets, Aggregative Spaces and Their Duality

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Abstract. The present paper introduces aggregative spaces and their category **AGS**, and then establishes a dual adjunction between **AGS** and the category **Agop** of aggregation operators on bounded partially ordered sets. Spatial aggregation operators and sober aggregative spaces, enabling us to restrict the dual adjunction between **AGS** and **Agop** to a dual equivalence between the full subcategory of **Agop** consisting of spatial aggregation operators and the full subcategory of **AGS** consisting of sober aggregative spaces, will also be subjects of this paper.

Keywords: Aggregation operator, Aggregative space, Q-space, Generalized topological space, Category theory, Adjoint situation, Duality, Spatiality, Sobriety.

1 Introduction

There is a considerable interest in the studies on (n-ary) aggregation operators (agops for short) for replacing the particular bounded partially ordered set (poset for short) $([0,1],\leq)$ by other reasonable bounded posets, e.g. agops on $([a,b],\leq)$ in [3,10,14], agops on $(I[0,1],\preceq_w)$ (the so-called an interval-valued agops) in [12], agops on $(L^*,\leq_{L^*})$ in [8], triangular norms on a general bounded poset in [4], a general bounded lattice in [15,16], pseudo-uninorms on a general complete lattice in [17,18]. As is shown in [5,11,13], agops on general bounded posets and their category **Agop** provide a useful and an abstract framework for such studies.

The dualities between certain ordered algebraic structures and certain spaces have been an important issue in many branches of mathematics (see [6,7] and the references therein). The famous duality between the full subcategory **SobTop** of **Top** of all sober topological spaces and the full subcategory **SpatFrm** of **Frm** of all spatial frames [9] is one of such dualities. In an analogous manner to this duality, our aim in this paper is to find out an appropriate notion of space providing a categorical duality for agops. For this purpose, after the next preliminary section, we introduce the aggregative spaces and their category **AGS**, and establish a dual adjunction between **AGS** and **Agop** in Section 3. Section 4 provides the notions of spatial agops and sober aggregative spaces, and proves

A. Laurent et al. (Eds.): IPMU 2014, Part I, CCIS 442, pp. 457–465, 2014.

a dual equivalence between the full subcategory **SobAGS** of **AGS** of all sober aggregative spaces and the full subcategory **SpatAgop** of **Agop** of all spatial agops. Furthermore, the presented dual adjunction and dual equivalence have been also discussed for some full subcategories of **Agop**.

2 Preliminaries

2.1 Categorical Tools

Adjoint situations and equivalences in the category theory are essential tools for formulating the main results of this paper. By definition, an adjoint situation $(\varrho, \phi) : F \dashv G : \mathbf{C} \to \mathbf{D}$ consists of functors $G : \mathbf{C} \to \mathbf{D}$, $F : \mathbf{D} \to \mathbf{C}$, and natural transformations $id_{\mathbf{D}} \overset{\varrho}{\to} GF$ (called the unit) and $FG \overset{\phi}{\to} id_{\mathbf{C}}$ (called the co-unit) satisfying the adjunction identities $G(\phi_A) \circ \varrho_{G(A)} = id_{G(A)}$ and $\phi_{F(B)} \circ F(\varrho_B) = id_{F(B)}$ for all A in $\mathbf{C}$ and B in $\mathbf{D}$. If $(\varrho, \phi) : F \dashv G : \mathbf{C} \to \mathbf{D}$ is an adjoint situation for some ϱ and ϕ, then F is said to be a left adjoint to G, $F \dashv G$ in symbols. A functor $G : \mathbf{C} \to \mathbf{D}$ is called an equivalence if it is full, faithful and isomorphism-dense. In this case, $\mathbf{C}$ and $\mathbf{D}$ are called equivalent categories, denoted by $\mathbf{C} \sim \mathbf{D}$.

Proposition 1. *[2,7] Given an adjoint situation $(\varrho, \phi) : F \dashv G : \mathbf{C}^{op} \to \mathbf{D}$, let $Fix\,(\phi)$ denote the full subcategory of $\mathbf{C}$ of all $\mathbf{C}$-objects A such that $\phi_A^{op} : A \to FGA$ is a $\mathbf{C}$-isomorphism, and $Fix\,(\varrho)$ the full subcategory of $\mathbf{D}$ of all $\mathbf{D}$-objects B such that $\varrho_B : B \to GFB$ is a $\mathbf{D}$-isomorphism. Then the following statements are true:*

(i) The restriction of $F \dashv G$ to $[Fix\,(\phi)]^{op}$ and $Fix\,(\varrho)$ induces an equivalence $[Fix\,(\phi)]^{op} \sim Fix\,(\varrho)$.

(ii) If ϕ_A^{op} is an epimorphism in $\mathbf{C}$ for each $\mathbf{C}$-object A, then both $Fix\,(\phi)$ and $Fix\,(\varrho)$ are reflective in their respective categories with the reflectors $F^{op}G^{op}$ and GF, and reflection arrows ϕ_A^{op} and ϱ_B, resp.

For more information about adjoint situations and equivalences, we refer to [1].

2.2 Aggregation Operators and Their Categories

Let $(L, \leq)$ be a bounded poset with the least element $\bot$ and the greatest element $\top$. An aggregation operator on L is defined to be a function $A : \bigcup_{n \in \mathbb{N}^+} L^n \to L$ satisfying the following conditions:

(AG.1) If $\alpha_1 \leq \beta_1$, $\alpha_2 \leq \beta_2$, ... and $\alpha_n \leq \beta_n$ for all α_i, $\beta_i \in L$, $i = 1, 2, ..., n$ ($n \in \mathbb{N}^+$), then $A(\alpha_1, ..., \alpha_n) \leq A(\beta_1, ..., \beta_n)$.

(AG.2) $A(\alpha) = \alpha$ for all $\alpha \in L$.

(AG.3) $A(\bot, ..., \bot) = \bot$ and $A(\top, ..., \top) = \top$.

For $n \geq 2$, a function $B : L^n \to L$ is called an n-ary aggregation operator on L iff the conditions (AG.1) and (AG.3) are satisfied. A 1-ary aggregation operator $B : L \to L$ is the identity map id_L on L. Every aggregation operator A on L

uniquely determines a family of n-ary aggregation operators $\{A_n \mid n \in \mathbb{N}^+\}$ by $A_n(\alpha_1, ..., \alpha_n) = A(\alpha_1, ..., \alpha_n)$.

With regard to the special cases of $(L, \leq)$, an aggregation operator on L produces an aggregation process for fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets, type 2 fuzzy sets and probabilistic metrics [5]. We further remark that whereas (AG.2) is proposed as a convention by some authors (e.g., see [3,11,13]), this condition is used to set up many interesting properties of aggregation operators such as their close connection with partially ordered groupoids in [5].

Definition 1. *[5] The category* ***Agop*** *of aggregation operators has as objects all triples* $(L, \leq, A)$*, where* $(L, \leq)$ *is a bounded poset and* A *is an aggregation operator on* L*, and as morphisms all* $(L, \leq, A) \xrightarrow{u} (M, \leq, B)$*, where* $u : (L, \leq) \to (M, \leq)$ *is an order-preserving function such that* $u(\bot) = \bot$*,* $u(\top) = \top$ *and the following diagram commutes for all* $n \in \mathbb{N}^+$*:*

$$\begin{array}{ccc} L^n & \xrightarrow{u^n} & M^n \\ A_n \downarrow & & \downarrow B_n \\ L & \xrightarrow{u} & M, \end{array}$$

i.e. $u(A(\alpha_1, ..., \alpha_n)) = B(u(\alpha_1), ..., u(\alpha_n))$ *for all* $\alpha_1, ..., \alpha_n \in L$*. Composition and identities in* ***Agop*** *are taken from the category* ***Set*** *of sets and functions.*

Definition 2. *[5] (i)* ***Asagop*** *is the full subcategory of* ***Agop*** *of all* $(L, \leq, A)$ *such that* A *is associative, i.e.*

$$A(\alpha_1, ..., \alpha_k, ...\alpha_n) = A_2(A_k(\alpha_1, ..., \alpha_k), A_{n-k}(\alpha_{k+1}, ..., \alpha_n))$$

for all $n \geq 2$*,* $k = 1, .., n-1$ *and* $\alpha_1, ..., \alpha_n \in L$*.*

(ii) ***Smasagop*** *is the full subcategory of* ***Asagop*** *of all* $(L, \leq, A)$ *such that* A *is symmetric, i.e.*

$$A(\alpha_1, ..., \alpha_n) = A(\alpha_{\pi(1)}, ..., \alpha_{\pi(n)})$$

for all $n \in \mathbb{N}^+$*,* $\alpha_1, ..., \alpha_n \in L$ *and for all permutations* $\pi(1),...,\pi(n)$ *of* $\{1, .., n\}$*.*

3 Aggregative Spaces and Their Relations with Aggregation Operators

3.1 Definition of Aggregative Spaces and Their Category

For a given set X, we call a subset τ of the power set $\mathcal{P}(X)$ of X an aggregative system on X if $\emptyset \in \tau$, $X \in \tau$, and $G_1, G_2 \in \tau$ implies $G_1 \cap G_2 \in \tau$ for all $G_1, G_2 \in \mathcal{P}(X)$. By an aggregative space, we mean a pair (X, τ) of a set X and an aggregative system τ on X. To formulate the category of aggregative spaces, we need to recall that every function $f : X \to Y$ determines a function $f^{\leftarrow} : \mathcal{P}(Y) \to \mathcal{P}(X)$, sending each subset G of Y to the preimage of G under f.

Definition 3. *The category of aggregative spaces, denoted by* $\boldsymbol{AGS}$*, is a category whose objects are aggregative spaces, and whose morphisms* $f : (X, \tau) \to (Y, \nu)$ *are functions* $f : X \to Y$ *having the property that for every* $G \in \nu$*,* $f^{\leftarrow}(G) \in \tau$*. Composition and identities in* $\boldsymbol{AGS}$ *are the same as those in* ***Set****.*

It is remarkable to mention that **AGS** is a special kind of the category $(\mathcal{Z}_1, \mathcal{Z}_2)\mathbf{S}$ of $(\mathcal{Z}_1, \mathcal{Z}_2)$-spaces, which has been developed in [6]. More clearly, **AGS** is exactly the same as the category $(\mathcal{V}_{\perp}, \mathcal{F})\mathbf{S}$, where $\mathcal{V}_{\perp}$ and $\mathcal{F}$ are the subset systems, defined by $\mathcal{V}_{\perp}(P) = \{\emptyset\}$ and $\mathcal{F}(P) = \{G \subseteq P \mid G \text{ is finite}\}$ for every poset P in [6].

3.2 Dual Adjunction between AGS and Agop

Our aim in this section is to show that there exists an adjoint situation $(\eta, \varepsilon) : \Omega_{AG} \dashv Pt_{AG} : \mathbf{Agop}^{op} \to \mathbf{AGS}$. For this purpose, we first establish the functors $\Omega_{AG} : \mathbf{AGS} \to \mathbf{Agop}^{op}$and $Pt_{AG} : \mathbf{Agop}^{op} \to \mathbf{AGS}$.

Every aggregative space (X, τ) induces an **Agop**-object $(\tau, \subseteq, X(\cap)_{\tau})$, where $X(\cap)$ is the aggregation operator on $\mathcal{P}(X)$, defined by $X(\cap)(G_1, ..., G_n) = G_1 \cap ... \cap G_n$ and $X(\cap)(G) = G$ for every $G, G_1, ..., G_n \in \mathcal{P}(X)$ $(n \geq 2)$, and $X(\cap)_{\tau}$ is the restriction of $X(\cap)$ to the set $\bigcup_{n \in \mathbb{N}^+} \tau^n$. This means that the assignment of the **Agop**-object $(\tau, \subseteq, X(\cap)_{\tau})$ to every aggregative space (X, τ) is a function Ω_{AG} from the objects of **AGS** to the objects of **Agop**. On the other hand, for a given **AGS**-morphism $f : (X, \tau) \to (Y, \nu)$, the restriction $f^{\leftarrow}_{|\nu}$ of $f^{\leftarrow}$ to ν is an **Agop**-morphism $f^{\leftarrow}_{|\nu} : (\nu, \subseteq, Y(\cap)_{\nu}) \to (\tau, \subseteq, X(\cap)_{\tau})$. Thus, Ω_{AG} can be extended to a functor from **AGS** to $\mathbf{Agop}^{op}$:

Proposition 2. *The map* $\Omega_{AG} : \boldsymbol{AGS} \to \boldsymbol{Agop}^{op}$*, defined by*

$$\Omega_{AG}(X, \tau) = (\tau, \subseteq, X(\cap)_{\tau}) \text{ and } \Omega_{AG}(f) = \left(f^{\leftarrow}_{|\nu}\right)^{op},$$

is a functor.

In the formulation of the functor $Pt_{AG} : \mathbf{Agop}^{op} \to \mathbf{AGS}$, we will use the notion of filter defined as follows.

Definition 4. *Let* $(L, \leq, A)$ *be an object of* ***Agop****. A subset* G *of* L *is called a filter of* $(L, \leq, A)$ *iff* G *satisfies the next conditions:*

(F1) G *is an upper set of* $(L, \leq)$*, i.e. for all* $\alpha, \beta \in L$*,* $\alpha \in G$ *and* $\alpha \leq \beta$ *imply* $\beta \in G$*,*

(F2) $\perp \notin G$*,*

(F3) $\top \in G$*,*

(F4) For all $\alpha_1, ..., \alpha_n \in L$*,* $\alpha_1, ..., \alpha_n \in G$ *iff* $A(\alpha_1, ..., \alpha_n) \in G$*.*

Lemma 1. *Given an* ***Agop****-object* $(L, \leq, A)$*, let* $\mathfrak{F}(L)$ *denote the set of all filters of* $(L, \leq, A)$*. For each* $a \in L$*, let* $\Psi_a = \{G \in \mathfrak{F}(L) \mid a \in G\}$ *and* $\Psi(L) = \{\Psi_a \mid a \in L\}$*. Then,* $Pt_{AG}(L, \leq, A) = (\mathfrak{F}(L), \Psi(L))$ *is an aggregative space.*

Proof. For each $G \in \mathfrak{F}(L)$, by (F2) and (F3) in Definition 4, $\mathfrak{F}(L) = \Psi_\top \in \Psi(L)$ and $\emptyset = \Psi_\perp \in \Psi(L)$. Furthermore, we obtain from (F4) that for all $a, b \in L$, $\Psi_a \cap \Psi_b = \Psi_{A(a,b)}$. Therefore, $\Psi(L)$ is an aggregative system on $\mathfrak{F}(L)$.

Proposition 3. *The map $Pt_{AG} : \boldsymbol{Agop}^{op} \to \boldsymbol{AGS}$, defined by*

$$Pt_{AG}\left((L, \leq, A) \stackrel{u}{\to} (M, \leq, B)\right) = Pt_{AG}(L, \leq, A) \stackrel{Pt_{AG}(u)}{\to} Pt_{AG}(M, \leq, B),$$

where $[Pt_{AG}(u)](G) = (u^{op})^{\leftarrow}(G)$ for all $G \in \mathfrak{F}(L)$, is a functor.

Proof. Lemma 1 shows that Pt_{AG} maps the objects of $\mathbf{Agop}^{op}$ to the objects of **AGS**. Let $(L, \leq, A) \stackrel{u}{\to} (M, \leq, B)$ be an $\mathbf{Agop}^{op}$-morphism, i.e. $(M, \leq, B) \stackrel{u^{op}}{\to} (L, \leq, A)$ is an **Agop**-morphism. For every $G \in \mathfrak{F}(L)$, since $(u^{op})^{\leftarrow}(G) \in \mathfrak{F}(M)$, $Pt_{AG}(u) : \mathfrak{F}(L) \to \mathfrak{F}(M)$ is a set map. In addition to this, we easily see that for every $b \in M$, $[Pt_{AG}(u)]^{\leftarrow}(\Psi_b) = \Psi_{u^{op}(b)}$, i.e. $[Pt_{AG}(u)]^{\leftarrow}(V) \in \Psi(L)$ for every $V \in \Psi(M)$. This proves that $Pt_{AG}(u) : Pt_{AG}(L, \leq, A) \to Pt_{AG}(M, \leq, B)$ is an **AGS**-morphism. Hence, the assertion follows from the fact that Pt_{AG} preserves composition and identities.

To accomplish our task in this section, we now consider two natural transformations-the unit and co-unit of the asked adjunction-given in the next two lemmas.

Lemma 2. *For every $\boldsymbol{AGS}$-object (X, τ), the map $\eta_{(X,\tau)} : X \to \mathfrak{F}(\tau)$, defined by $\eta_{(X,\tau)}(x) = \tau(x) = \{G \in \tau \mid x \in G\}$, is an $\boldsymbol{AGS}$-morphism $(X, \tau) \to Pt_{AG}\Omega_{AG}(X, \tau)$. Moreover, $\eta = \left(\eta_{(X,\tau)}\right)_{(X,\tau) \in Ob(\mathbf{AGS})} : id_{\mathbf{AGS}} \to Pt_{AG}\Omega_{AG}$ is a natural transformation.*

Proof. It is obvious that for every $x \in X$, $\tau(x) \in \mathfrak{F}(\tau)$, and so $\eta_{(X,\tau)} : X \to \mathfrak{F}(\tau)$ is indeed a map. To see that $\eta_{(X,\tau)} : (X, \tau) \to Pt_{AG}\Omega_{AG}(X, \tau)$ is an **AGS**-morphism, note first that

$$Pt_{AG}\Omega_{AG}(X, \tau) = (\mathfrak{F}(\tau), \Psi(\tau)).$$

Then, since $\eta^{\leftarrow}_{(X,\tau)}(\Psi_G) = G$ for every $G \in \tau$, we obtain that $\eta^{\leftarrow}_{(X,\tau)}(V) \in \tau$ for every $V \in \Psi(\tau)$, i.e. $\eta_{(X,\tau)} : (X, \tau) \to Pt_{AG}\Omega_{AG}(X, \tau)$ is an **AGS**-morphism. The proof of the second part requires only the naturality of η which means the commutativity of the rectangle

$$\begin{array}{ccc} (X,\tau) & \stackrel{\eta_{(X,\tau)}}{\longrightarrow} & Pt_{AG}\Omega_{AG}(X,\tau) \\ f \quad \downarrow & & \downarrow \qquad Pt_{AG}\Omega_{AG}(f) \\ (Y,\nu) & \stackrel{\eta_{(Y,\nu)}}{\longrightarrow} & Pt_{AG}\Omega_{AG}(Y,\nu) \end{array}$$

for every **AGS**-morphism $(X, \tau) \stackrel{f}{\to} (Y, \nu)$. Since

$$[Pt_{AG}\Omega_{AG}(f)]\left(\eta_{(X,\tau)}(x)\right) = \left(f^{\leftarrow}_{|\nu}\right)^{\leftarrow}\left(\eta_{(X,\tau)}(x)\right),$$

the commutativity of the rectangle above follows from the following observation: For every $x \in X$ and every $H \in \nu$,

$$\begin{aligned} H \in \eta_{(Y,\nu)}(f(x)) &\Leftrightarrow H \in \nu \text{ and } f(x) \in H \Leftrightarrow f^{\leftarrow}_{|_\nu}(H) \in \tau \text{ and } x \in f^{\leftarrow}_{|_\nu}(H) \\ &\Leftrightarrow f^{\leftarrow}_{|_\nu}(H) \in \eta_{(X,\tau)}(x) \Leftrightarrow H \in \left(f^{\leftarrow}_{|_\nu}\right)^{\leftarrow}\left(\eta_{(X,\tau)}(x)\right) \\ &\Leftrightarrow H \in [Pt_{AG}\Omega_{AG}(f)]\left(\eta_{(X,\tau)}(x)\right). \end{aligned}$$

Lemma 3. *For every **Agop**-object $(L, \leq, A)$, the map $e_{(L,\leq,A)} : L \to \Psi(L)$, defined by $e_{(L,\leq,A)}(a) = \Psi_a$, is an **Agop**-morphism $(L, \leq, A) \to \Omega_{AG}Pt_{AG}(L, \leq, A)$. Moreover, $\varepsilon = \left(e^{op}_{(L,\leq,A)}\right)_{(L,\leq,A)\in Ob(\mathbf{Agop})} : \Omega_{AG}Pt_{AG} \to id_{\mathbf{Agop}^{op}}$ is a natural transformation.*

Proof. Consider first that $\Omega_{AG}Pt_{AG}(L, \leq, A) = \left(\Psi(L), \subseteq, \mathfrak{F}(L)(\cap)_{\Psi(L)}\right)$. In order to prove that $e_{(L,\leq,A)} : (L, \leq, A) \to \Omega_{AG}Pt_{AG}(L, \leq, A)$ is an **Agop**-morphism, we proceed as follows. For all $a, b \in L$ with $a \leq b$, since $\Psi_a \subseteq \Psi_b$ (by (F1) in Definition 4), $e_{(L,\leq,A)}$ is order-preserving. We also obtain from (F2) and (F3) in Definition 4 that $e_{(L,\leq,A)}$ preserves $\perp$ and $\top$. Furthermore, by making use of (F4) in Definition 4, we see that $\Psi_{a_1} \cap ... \cap \Psi_{a_n} = \Psi_{A(a_1,...,a_n)}$ for all $a_1, ..., a_n \in L$, and so

$$\begin{aligned} e_{(L,\leq,A)}(A(a_1, ..., a_n)) &= \Psi_{A(a_1,...,a_n)} = \Psi_{a_1} \cap ... \cap \Psi_{a_n} \\ &= e_{(L,\leq,A)}(a_1) \cap ... \cap e_{(L,\leq,A)}(a_n) \\ &= \mathfrak{F}(L)(\cap)_{\Psi(L)}\left(e_{(L,\leq,A)}(a_1), ..., e_{(L,\leq,A)}(a_n)\right). \end{aligned}$$

This completes the proof of the first part of the assertion. For the second part, the only property of ε that we have to verify is its naturality, i.e. the commutativity of the diagram

$$\begin{array}{ccccc} & \Omega_{AG}Pt_{AG}(L, \leq, A) & \overset{\varepsilon_{(L,\leq,A)}}{\longrightarrow} & (L, \leq, A) & \\ \Omega_{AG}Pt_{AG}(u) & \downarrow & & \downarrow & u \\ & \Omega_{AG}Pt_{AG}(M, \leq, B) & \overset{\varepsilon_{(M,\leq,B)}}{\longrightarrow} & (M, \leq, B) & \end{array} \tag{1}$$

for each $\mathbf{Agop}^{op}$-morphism $(L, \leq, A) \overset{u}{\to} (M, \leq, B)$. Since all arrows and all compositions in (1) are taken in $\mathbf{Agop}^{op}$, it can be simplified to be a rectangle

$$\begin{array}{ccccc} & (M, \leq, B) & \overset{e_{(M,\leq,B)}}{\longrightarrow} & \left(\Psi(M), \subseteq, \mathfrak{F}(M)(\cap)_{\Psi(M)}\right) & \\ u^{op} & \downarrow & & \downarrow & Pt_{AG}(u)^{\leftarrow}_{|_{\Psi(M)}} \\ & (L, \leq, A) & \overset{e_{(L,\leq,A)}}{\longrightarrow} & \left(\Psi(L), \subseteq, \mathfrak{F}(L)(\cap)_{\Psi(L)}\right), & \end{array} \tag{2}$$

where all arrows and all compositions are performed in **Agop**. The commutativity of (2) is obtained as

$$\begin{aligned} \left[Pt_{AG}(u)^{\leftarrow}_{|_{\Psi(M)}} \circ e_{(M,\leq,B)}\right](b) &= [Pt_{AG}(u)]^{\leftarrow}(\Psi_b) = \Psi_{u^{op}(b)} \\ &= \left[e_{(L,\leq,A)} \circ u^{op}\right](b) \end{aligned}$$

for all $b \in M$.

Theorem 1. $(\eta, \varepsilon) : \Omega_{AG} \dashv Pt_{AG} : \boldsymbol{Agop}^{op} \to \boldsymbol{AGS}$ *is an adjoint situation.*

Proof. It is not difficult to check the adjunction identities

$$Pt_{AG}\left(\varepsilon_{(L,\leq,A)}\right) \circ \eta_{Pt_{AG}(L,\leq,A)} = id_{Pt_{AG}(L,\leq,A)},$$
$$\varepsilon_{\Omega_{AG}(X,\tau)} \circ \Omega_{AG}\left(\eta_{(X,\tau)}\right) = id_{\Omega_{AG}(X,\tau)}$$

for every **Agop**-object $(L, \leq, A)$ and every **AGS**-object (X, τ). Then, the required result follows immediately from Lemma 2 and Lemma 3.

Remark 1. Since $\Omega_{AG}(X, \tau) = (\tau, \subseteq, X(\cap)_\tau)$ is an object of **Smasagop** for every **AGS**-object (X, τ), the adjoint situation in Theorem 1 can be restricted to an adjoint situation $(\eta, \varepsilon^r) : \Omega^r_{AG} \dashv Pt^r_{AG} : \mathbf{Smasagop}^{op} \to \mathbf{AGS}$, where Ω^r_{AG} (Pt^r_{AG}) is the co-domain (the domain) restriction of Ω_{AG} (Pt_{AG}) and $\varepsilon^r_{(L,\leq,A)} = \varepsilon_{(L,\leq,A)}$ for every **Smasagop**-object $(L, \leq, A)$. An analogous adjoint situation can also be written for the category **Asagop** instead of **Smasagop**.

4 Spatial Aggregation Operators, Sober Aggregative Spaces and Their Duality

Spatiality and sobriety are two important notions that enable us to restrict the adjunction $\Omega_{AG} \dashv Pt_{AG}$ to an equivalence. To clarify this fact, we first start with their definitions:

Definition 5. *(i) An* ***Agop****-object* $(L, \leq, A)$ *is called spatial iff for all* $a, b \in L$ *with* $a \nleq b$*, there exists a* $G \in \mathfrak{F}(L)$ *such that* $a \in G$ *and* $b \notin G$*.*

(ii) An ***AGS****-object* (X, τ) *is called sober iff for all* $\mathcal{U} \in \mathfrak{F}(\tau)$*, there exists a unique* $x \in X$ *such that* $\mathcal{U} = \tau(x)$*.*

Proposition 4. *Let* $(L, \leq, A)$ *be an* ***Agop****-object, and* (X, τ) *an* ***AGS****-object.*

(i) $(L, \leq, A)$ *is spatial iff* $e_{(L,\leq,A)} : (L, \leq, A) \to \Omega_{AG}Pt_{AG}(L, \leq, A)$ *is an* ***Agop****-isomorphism.*

(ii) $(L, \leq, A)$ *is spatial iff* $(L, \leq, A)$ *is isomorphic to* $(\nu, \subseteq, Y(\cap)_\nu)$ *for some aggregative space* (Y, ν)*.*

(iii) (X, τ) *is sober iff* $\eta_{(X,\tau)} : (X, \tau) \to Pt_{AG}\Omega_{AG}(X, \tau)$ *is an* ***AGS****-isomorphism.*

Proof. (i) Note first that $(L, \leq, A)$ is spatial iff for all $a, b \in L$, $\Psi_a \subseteq \Psi_b$ implies $a \leq b$. Now, by assuming spatiality of $(L, \leq, A)$, this equivalence directly gives the injectivity of the underlying set map of $e_{(L,\leq,A)}$, and so does its bijectivity. It is easy to check that $e^{-1}_{(L,\leq,A)} : \Omega_{AG}Pt_{AG}(L, \leq, A) \to (L, \leq, A)$ is an **Agop**-morphism, and so $e_{(L,\leq,A)} : (L, \leq, A) \to \Omega_{AG}Pt_{AG}(L, \leq, A)$ is an **Agop**-isomorphism. Conversely, if $e_{(L,\leq,A)} : (L, \leq, A) \to \Omega_{AG}Pt_{AG}(L, \leq, A)$ is an **Agop**-isomorphism, then since $e^{-1}_{(L,\leq,A)} : \Omega_{AG}Pt_{AG}(L, \leq, A) \to (L, \leq, A)$ is an **Agop**-morphism, $\Psi_a \subseteq \Psi_b$ implies $a = e^{-1}_{(L,\leq,A)}(\Psi_a) \leq e^{-1}_{(L,\leq,A)}(\Psi_b) = b$ for all $a, b \in L$, so $(L, \leq, A)$ is spatial.

(ii) If $(L,\leq,A)$ is spatial, then we have from (i) that $(L,\leq,A)$ is isomorphic to $(\nu,\subseteq,Y\,(\cap)_{\nu})$ for the aggreative space $(Y,\nu)=Pt_{AG}(L,\leq,A)$. Conversely, suppose $(L,\leq,A)$ is isomorphic to $(\nu,\subseteq,Y\,(\cap)_{\nu})$ for some aggregative space (Y,ν), i.e. there exists an **Agop**-isomorphism $u:(L,\leq,A)\to(\nu,\subseteq,Y\,(\cap)_{\nu})$. Then, for $a,b\in L$ with $a\nleq b$, since $u(a)\nsubseteq u(b)$, there exists at least one $z\in Y$ such that $z\in u(a)$ and $z\notin u(b)$. It is clear that $u(a)\in\nu\,(z)$ and $u(b)\notin\nu\,(z)$, and so $a\in u^{\leftarrow}\,(\nu\,(z))$ and $b\notin u^{\leftarrow}\,(\nu\,(z))$. Hence, we obtain the spatiality of $(L,\leq,A)$ from the fact that $u^{\leftarrow}\,(\nu\,(z))$ is a filter of $(L,\leq,A)$.

(iii) follows from that for a given **AGS**-object (X,τ), (X,τ) is sober iff the underlying set map of $\eta_{(X,\tau)}$ is a bijection iff $\eta_{(X,\tau)}:(X,\tau)\to Pt_{AG}\Omega_{AG}\,(X,\tau)$ is an **AGS**-isomorphism.

Corollary 1. *The full subcategory **SpatAgop** of **Agop** of all spatial objects is dually equivalent to the full subcategory **SobAGS** of **AGS** of all sober objects.*

Proof. Since Proposition 4 (i) and (iii) verify that $Fix\,(\varepsilon)=$ **SpatAgop** and $Fix\,(\eta)=$ **SobAGS**, the assertion follows from Theorem 1 and Proposition 1 (i).

Proposition 5. ***SpatAgop** and **SobAGS** are reflective subcategories of **Agop** and of **AGS** with reflectors $\Omega_{AG}^{op}Pt_{AG}^{op}$ and $Pt_{AG}\Omega_{AG}$, and the reflection arrows $e_{(L,\leq,A)}$ and $\eta_{(X,\tau)}$, respectively.*

Proof. Since $e_{(L,\leq,A)}=\varepsilon_{(L,\leq,A)}^{op}$ is obviously an epimorphism in **Agop**, and $Fix\,(\varepsilon)=$ **SpatAgop** and $Fix\,(\eta)=$ **SobAGS**, Proposition 1 (ii) directly yields the claimed result.

Proposition 6. *Let $(L,\leq,A)$ be an **Agop**-object, and (X,τ) an **AGS**-object.*
(i) $\Omega_{AG}\,(X,\tau)$ is spatial, (ii) $Pt_{AG}(L,\leq,A)$ is sober.

Proof. (i) is immediate from Proposition 4 (ii). To see (ii), let us first consider the fact that $Pt_{AG}(L,\leq,A)=(\mathfrak{F}(L),\Psi(L))$. Then, the sobriety of $Pt_{AG}(L,\leq,A)$ follows from the observation that for all $\mathcal{U}\in\mathfrak{F}\,(\Psi(L))$, $G=\{a\in L\mid\Psi_a\in\mathcal{U}\}$ is the unique element of $\mathfrak{F}(L)$ with the property that $\mathcal{U}=[\Psi(L)]\,(G)$.

Proposition 7. *The full subcategory **SpatAsagop** of **Asagop** of all spatial objects, the full subcategory **SpatSmasagop** of **Smasagop** of all spatial objects and **SpatAgop** are equivalent to each others.*

Proof. Since **SpatSmasagop** is a full subcategory of **SpatAgop**, the inclusion functor **SpatSmasagop** $\hookrightarrow$ **SpatAgop** is a full and faithful functor. For every **SpatAgop**-object $(L,\leq,A)$, by Proposition 4 (i) and Proposition 6 (i), $\Omega_{AG}Pt_{AG}(L,\leq,A)$ is a **SpatSmasagop**-object, and $e_{(L,\leq,A)}:(L,\leq,A)\to\Omega_{AG}Pt_{AG}(L,\leq,A)$ is a **SpatAgop**-isomorphism. This proves that the inclusion functor **SpatSmasagop** $\hookrightarrow$ **SpatAgop** is isomorphism-dense, and hence an equivalence. Similarly, the inclusion functor **SpatAsagop** $\hookrightarrow$ **SpatAgop** is an equivalence, which completes the proof.

Corollary 2. ***SpatAgop***^{op} $\sim$ ***SpatAsagop***^{op} $\sim$ ***SpatSmasagop***^{op} $\sim$ ***SobAGS***.

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