Vertex-partitions of graphs into cographs and stars

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Abstract

A *cograph* is a graph that contains no path on four vertices as an induced subgraph. A *cograph k-partition* of a graph $G = (V, E)$ is a vertex-partition of G into k sets $V_1, \ldots, V_k \subset V$ so that the graph induced by V_i is a cograph for $1 \leq i \leq k$. Gimbel and Nešetřil [5] studied the complexity aspects of the cograph k-partitions and raised the following questions: Does there exist a triangle-free planar graph that is not cograph 2-partitionable? If the answer is yes, what is the complexity of the associated decision problem? In this paper, we prove that such an example exists and that deciding whether a triangle-free planar graph admits a cograph 2-partition is NP-complete. We also show that every graph with maximum average degree at most $\frac{14}{5}$ admits a cograph 2-partition such that each component is a star on at most three vertices.

1 Introduction

In this paper we focus on vertex-partitions such that each partite set induces a graph with a given structure. Cographs form the minimal family of graphs containing K_1 that is closed with respect to complementation and disjoint union. Cographs are also characterized as the graphs containing no induced copy of P_4, the path on four vertices (see for example [9]). A *star k-partition* (resp. *cograph k-partition*) of G is a vertex-partition of G in k sets $V_1, \ldots, V_k$ such that the graph induced by each V_i is a star forest (resp. a cograph). Moreover we call a *d-star k-partition* a star k-partition where every induced component has order at most d. A 1-star k-partition is a proper k-coloring.

Deciding whether a graph is cograph k-partitionable is linear time solvable when $k = 1$ [4] and is NP-complete for $k \geq 2$ [1]. In [5] Gimbel and Nešetřil focused on planar graphs and proved:

Theorem 1 (Gimbel, Nešetřil [5])

1. *Deciding whether a planar graph is cograph 3-partitionable is NP-complete.*
2. *Deciding whether a planar graph with maximum degree at most 6 is cograph 2-partitionable is NP-complete.*

Also, the following questions are implicit in the same paper:

Question 2 (Gimbel, Nešetřil [5]) *Does there exist a triangle-free planar graph that is not cograph 2-partitionable? If the answer is yes, what is the complexity of the associated decision problem?*

Let $\mathcal{C}_y$ be the class of graphs admitting a 3-star 2-partition. Let $\mathcal{C}_n$ be the class of graphs admitting no vertex-partition into two cographs. Notice that $\mathcal{C}_y \cap \mathcal{C}_n = \emptyset$. In Section 2, we provide an example of a non cograph 2-partitionable triangle-free planar graph, and prove:

Theorem 3

1. *It is NP-complete to determine whether a triangle-free planar graph in $\mathcal{C}_y \cup \mathcal{C}_n$ belongs to $\mathcal{C}_y$.*
2. *It is NP-complete to determine whether a planar graph with no 4-cycle and with maximum degree 4 in $\mathcal{C}_y \cup \mathcal{C}_n$ belongs to $\mathcal{C}_y$.*

This answers Question 2 ; moreover this improves the hypothesis on the maximum degree from 6 to 4 in Theorem 1, which is best possible since graphs with maximum degree 3 admit a vertex-partition into two subgraphs of maximum degree 1.

Many studies on vertex partitions use the *maximum average degree* as a parameter, see for example [2, 3]. The maximum average degree of a graph G is defined by

$$\mathrm{mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|}, H \subseteq G\right\}$$

This parameter can be computed in polynomial time as proved by Jensen and Toft in [7]. It is also well known that every planar graph G with girth at least g satisfies $\mathrm{mad}(G) < \frac{2g}{g-2}$. In regards to the previous studies, it seems natural to consider the following problem:

Problem 4 *Given an integer $k \geq 1$, does there exist $f(k)$ such that every graph with $\mathrm{mad}(G) < f(k)$ is k-star 2-partitionable?*

Graphs that are 1-star 2-partitionable correspond to 2-colorable graphs; hence, every graph G with $\mathrm{mad}(G) < 2$ is 1-star 2-partitionable. By Havet and Sereni [6], every graph with $\mathrm{mad}(G) < \frac{8}{3}$ is 2-star 2-partitionable. Studying list strong linear 2-arboricity of sparse graphs, Borodin and Ivanova proved that every graph with $\mathrm{mad}(G) < \frac{14}{5}$ and girth at least 7 is 3-star 2-partitionable [3]. In Section 3, we show that we can drop the assumption on the girth and prove:

Theorem 5 *Every graph G with* $\mathrm{mad(G)} < \frac{14}{5}$ *is 3-star 2-partitionable.*

For $k \geq 4$, Problem 4 remains open. By [3], every planar graph of girth at least 7 is 3-star 2-partitionable. Moreover, there exist planar graphs with girth 4 which are not cograph 2-partitionable, and therefore not k-star 2-partitionable for any k (see Section 2.1). We thus conclude with the following two questions:

Question 6

1. *Does there exist an integer s_6 so that every planar graph with girth at least 6 is s_6-star 2-partitionable?*

2. *Does there exist an integer s_5 so that every planar graph with girth at least 5 is s_5-star 2-partitionable?*

2 NP-completeness

This section is dedicated to the proof of Theorem 3.

We recall that a 2-coloring of a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is a partition of its vertex set $\mathcal{V}$ into two color classes such that no edge in $\mathcal{E}$ is monochromatic. We reduce our problem to the NP-complete problem of deciding the 2-colorability of 3-uniform hypergraphs [8].

2.1 Triangle-free planar graphs

Our reduction is based on some gadget graphs $S_{x,y}, F_{x,y,z}$, and U_{x_1,x_2,y_1,y_2} that have some nice properties.

(C01) *The graph $S_{x,y}$ in Figure 1 has no cograph 2-partition such that x and y are in the same partite set.*
PROOF. Suppose for the sake of contradiction that $S_{x,y}$ has a cograph 2-partition $V = A\dot{\cup}B$ such that x and y are in the same partite set A.

(O1) For $1 \leq i \leq 2$ and $1 \leq j \leq 3$, $v_{i,j}$ and $v_{i,j+1}$ cannot be both in A. Otherwise, this forces the four vertices "between" them to be in B, creating a P_4. Similarly, $v_{1,i}, v_{1,i+1}, v_{2,i+1}$ cannot be all in the same partite set, and so it is for $v_{2,i}, v_{2,i+1}, v_{1,i}$.

(O2) Two non-adjacent vertices $v_{1,i}, v_{2,j}$ cannot be both in A. Otherwise $v_{1,i}xyv_{2,j}$ is a P_4 in A, a contradiction.

First assume that $v_{1,1}$ is in A, then $v_{1,2}$ is in B by O1 and $v_{2,3}$ is in B by O2. Now $v_{1,3}$ and $v_{2,2}$ must be in A by O1. This contradicts O2.

By symmetry, we can assume that $v_{1,1}$ and $v_{2,4}$ are in B. So suppose now that $v_{1,2}$ is in A. By O1, $v_{1,3}$ is in B. Again by O1, $v_{1,4}$ and $v_{2,3}$ are in A. This contradicts O2.

So we have $v_{1,1}, v_{1,2}, v_{2,3}, v_{2,4}$ in B. By O1, it follows that $v_{1,3}$ and $v_{2,2}$ are in A. This contradicts O2.

□

The graph $S_{x,y}$ can be seen as a *switcher*: if x is in A, then y is in B and vice versa. Two copies of $S_{x,y}$, say S_{x_1,y_1}, S_{x_2,y_2} where $y_1 = x_2$ can be seen as an *extender*:

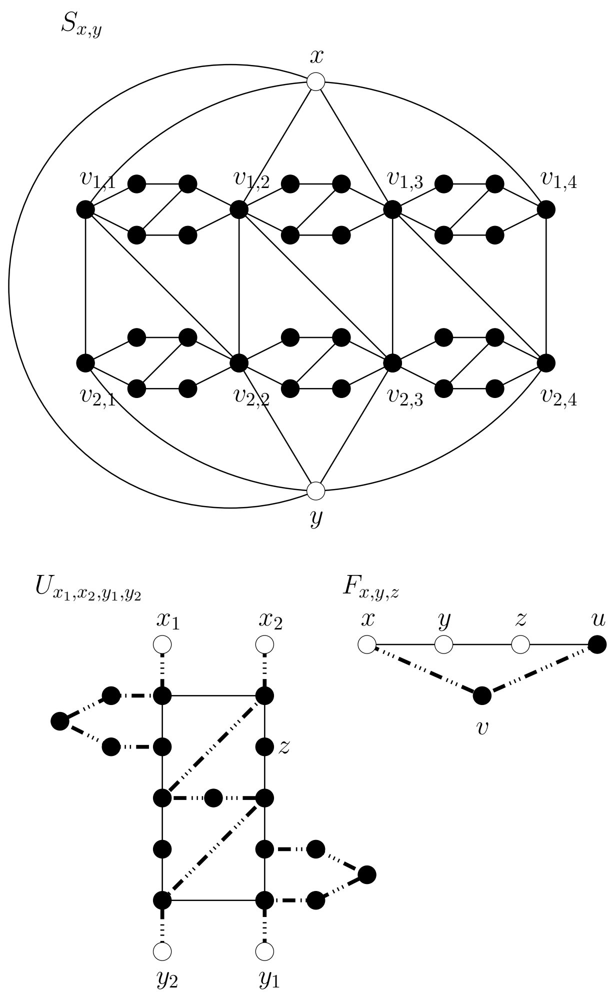

Figure 1: The graphs $S_{x,y}$, U_{x_1,x_2,y_1,y_2}, and $F_{x,y,z}$.

vertices x_1 and y_2 must belong to the same partite set. One can construct a triangle-free planar graph that does not admit a cograph 2-partition by taking a cycle of length 5 and replacing each of its edges xy by $S_{x,y}$. This answers the first part of Question 2.

The graphs $F_{x,y,z}$ and U_{x_1,x_2,y_1,y_2} are the graphs depicted in Figure 1 where each dashed edge is a copy of $S_{x,y}$.

(C02) *The graph $F_{x,y,z}$ has no cograph 2-partition such that x, y, and z are in the same partite set.*
PROOF. Suppose, by way of contradiction, that $F_{x,y,z}$ has a cograph 2-partition $V = A\dot{\cup}B$ such that x, y, and z are in the same partite set, say A. The two switchers between xv and vu force u to be in A ; this produces a $P_4 = xyzu$ in A, a contradiction. □

(C03) *Let $A\dot{\cup}B$ be a cograph 2-partition of U_{x_1,x_2,y_1,y_2}. Then x_1, y_1 (resp. x_2, y_2) must be in the same partite set.*
PROOF. By the path of switchers between x_2 and y_2, necessarily x_2 and y_2 must be in the same partite set, say A. Suppose now that x_1 and y_1 are in different partite sets, say x_1 is in A and y_1 is in B. Propagating the partition from x_1 and y_1 using the switchers creates two paths P', P'' of length 3 ending in z such that the first three vertices of P' (resp. P'') are in B (resp. A). It follows that putting z in A or B creates a P_4, a contradiction. □

The graph U_{x_1,x_2,y_1,y_2} can be seen as an *uncrosser*: if x_1 is in a partite set, then y_1 must be in the same partite set, and so it is for x_2, y_2.

We can now present the reduction. We transform an instance $\mathcal{H}$ of 2-colorability of 3-uniform hypergraphs into an instance G of our problem. For each vertex in $\mathcal{H}$, we associate a vertex in G. For each edge in $\mathcal{H}$, we associate a copy of the graph $F_{x,y,z}$. Now for each incidence between a vertex v and an edge e, we link the vertex associated to v to one of the vertices x, y, z of the copy of $F_{x,y,z}$ associated to e. We construct such links using extenders in series. The obtained graph is not necessarily planar: we handle each crossing with an uncrosser. Finally we obtain an instance G that is planar and triangle-free. See Figure 2.

By the properties of the switchers, extenders, and uncrossers, the graph G admits a 3-star 2-partition if $\mathcal{H}$ is 2-colorable and it admits no cograph 2-partition otherwise.

If $\mathcal{H}$ is not 2-colorable, then this implies that in any vertex-partition $A\dot{\cup}B$ of G there is a copy of $F_{x,y,z}$ whose all vertices x, y, z are in the same partite set; by C02, G admits no cograph 2-partition.

Assume $\mathcal{H}$ is 2-colorable. Let $\mathcal{A}\dot{\cup}\mathcal{B}$ be a 2-colouring of $\mathcal{H}$. We construct a 3-star 2-partition of G as follows. We put each vertex of G corresponding to a vertex of $\mathcal{H}$ in $\mathcal{A}$ (resp. $\mathcal{B}$) in A (resp. B). We then extend this partition to all the vertices of extenders (composed of two switchers) and uncrossers as depicted in Figure 3. Observe that in the 2-partition of $S_{x,y}$, the endvertices x and y have no neighbour in their own partite set. This yields a 3-star 2-partition of G.

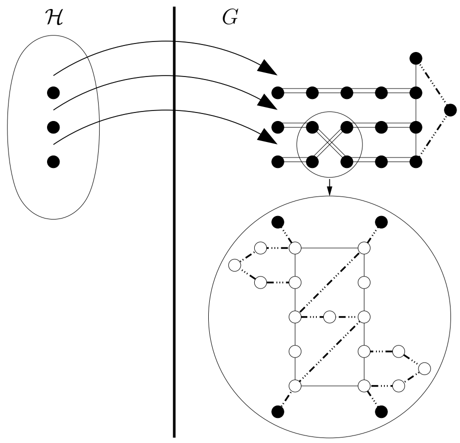

Figure 2: Reduction from $\mathcal{H}$ to G. Double edges, dashed edges represent extenders, switchers, respectively. Thin edges are usual edges. Each crossing will be handled by an uncrosser.

2.2 Planar graphs with maximum degree four

The proof of Theorem 3.2 is similar to the proof of Theorem 3.1, but is based on a new switcher $E_{x,y}$.

(C01) *Let P be the graph depicted in Figure 4. Let $x_1x_2x_3x_4x_5$ be the inner cycle of length 5 of P. Up to permutation, in any cograph 2-partition $A\dot{\cup}B$ of P, we have necessarily $x_1, x_2, x_4 \in A$ and $x_3, x_5 \in B$.*

(C02) *Let H_x be the graph depicted in Figure 4. Let $A\dot{\cup}B$ be a cograph 2-partition of H_x. Suppose $x \in A$. Then at least one of a_1, a_2 is in A.*
PROOF. Suppose by way of contradiction that none of a_1, a_2 is in A. By (C01), it follows that $a_1, a_2, u \in B$ and $c_1, c_2 \in A$. At least one of b_1, b_2 must be in A (otherwise $b_1a_1a_2b_2$ is a P_4 in B), say $b_1 \in A$. By (C01), we have $i_1 \in A$ and $f_1, g_1 \in B$. As well $j_2 \in B$ and $j_1, g_2 \in A$ by (C01), and similarly $f_2 \in A$ and $b_2, i_2 \in B$. Now, by symmetry, assume v is in A ; it follows that e_1, e_2, h_1 are in B, creating a $P_4 = e_2e_1h_1f_1$ in B, a contradiction. □

The graph G_x is obtained from a triangle xx_1x_2 and two copies H_{x_1}, H_{x_2} glued on x_1 and x_2 (see Figure 4).

(C03) *Let $A\dot{\cup}B$ be a cograph 2-partition of G_x. Suppose $x \in A$. Then vertex x is an end of a path of length 3 in A.*
PROOF. If none of x_1, x_2 is in A, then by (C02) there is a P_4 in B. Suppose w.l.o.g. that x_1 is in A. By (C02), a neighbor of x_1 in H_{x_1} is in A. This completes the proof of (C03). □

Finally the graph $E_{x,y}$ is constructed as follows: Take a path of length 4 xz_1z_2y and identify z_1 (resp. z_2) with the vertex x of a copy of G_x. By (C03) we have:

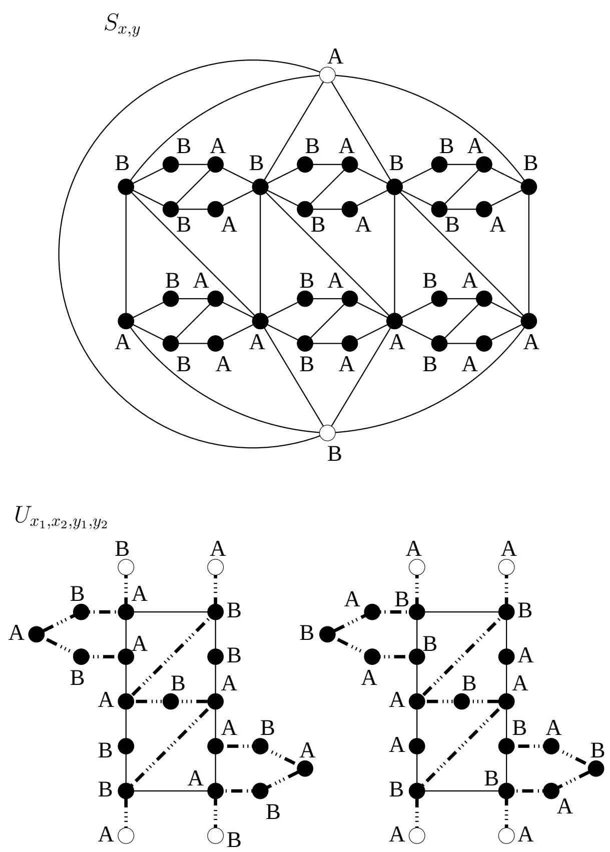

Figure 3: 3-star 2-partitions of $S_{x,y}$ and U_{x_1,x_2,y_1,y_2}.

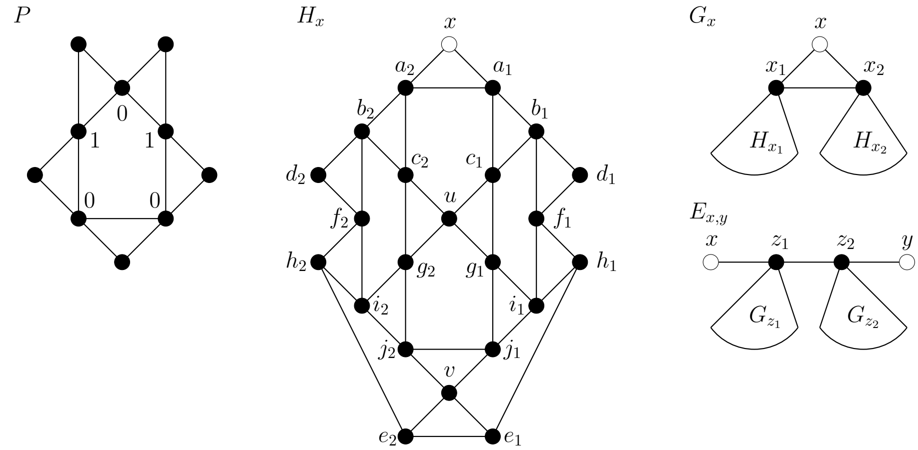

Figure 4: The graphs $P, H_x, G_x, E_{x,y}$.

(C04) *Let $A\dot{\cup}B$ be a cograph 2-partition of $E_{x,y}$. Then x and y are in different partite sets.*

The proof of Theorem 3.2 is the same as the proof of Theorem 3.1 where the switcher $S_{x,y}$ is replaced by the switcher $E_{x,y}$. A 3-star 2-partition of $E_{x,y}$ is given in Figure 5.

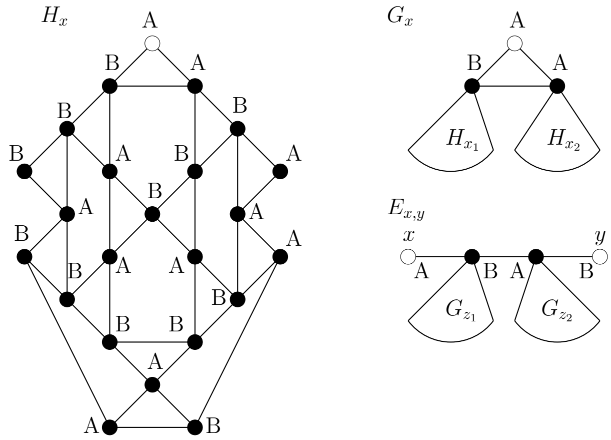

Figure 5: 3-star 2-partition of $E_{x,y}$.

Observe that the value 4 in Theorem 3.2 is best possible, since every graph with maximum degree at most 3 is $(1,1)$-colorable, *i.e.* admits a vertex-partition into two subsets, each of them inducing a subgraph with maximum degree at most 1 (and so is

3-star 2-partitionable). To see this, consider ϕ a coloring of the vertices with two colors 0 and 1, so that $\sigma(\phi) = E_{00}(\phi) + E_{11}(\phi)$ is minimum, where E_{ii} denotes the number of edges whose both ends are colored with color i. Then ϕ is a $(1,1)$-coloring (if not, we can recolor a vertex and obtain a coloring ϕ' with $\sigma(\phi') < \sigma(\phi)$, contradicting the choice of ϕ).

3 3-star 2-partition of graphs with $\mathrm{mad} < \frac{14}{5}$

The first part of the proof, namely Lemma 7, is similar to the lemma proposed by Borodin and Ivanova in [3]. It is given for sake of completeness.

For simplicity, we use the following nomenclature. A *k-vertex* (resp. *k^+-vertex, k^--vertex*) is a vertex of degree k (resp. at least k, at most k).

A *light 3-vertex* is a 3-vertex adjacent to a 2-vertex. A *weary 3-vertex* is a light 3-vertex adjacent to another light 3-vertex. An *exhausted 3-vertex* is a 3-vertex adjacent to a light 3-vertex and a weary 3-vertex. Note that all of them are always 3-vertices, we may omit to precise it (and say light vertex, weary vertex and exhausted vertex) in the following. Examples of such vertices are provided in Figure 6.

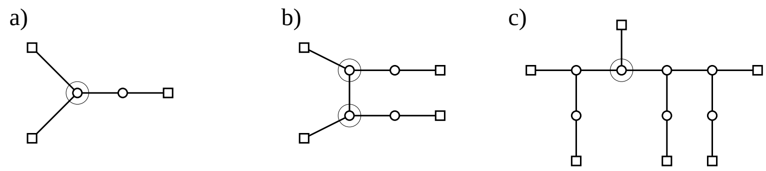

Figure 6: Examples of a) light, b) weary, and c) exhausted vertices

Lemma 7 *If a graph G satisfies $\mathrm{mad}(G) < \frac{14}{5}$, then it contains one of the following configurations C1-C12, examples of which are depicted in Figure 7:*

C1. A 1^--vertex.

C2. Two adjacent 2-vertices.

C3. A 3-vertex adjacent to two 2-vertices.

C4. A light 3-vertex adjacent to two light 3-vertices.

C5. A 3-vertex adjacent to three light 3-vertices.

C6. Two adjacent 3-vertices, each of them adjacent to two light 3-vertices.

C7. A 3-vertex adjacent to two weary 3-vertices.

C8. A 3-vertex adjacent to a weary 3-vertex and an exhausted 3-vertex.

C9. A 3-vertex adjacent to two exhausted 3-vertices.

C10. A 4-vertex adjacent to four 2-vertices.

C11. A 4-vertex adjacent to three 2-vertices and a light 3-vertex.

C12. A 4-vertex adjacent to three 2-vertices and an exhausted 3-vertex.

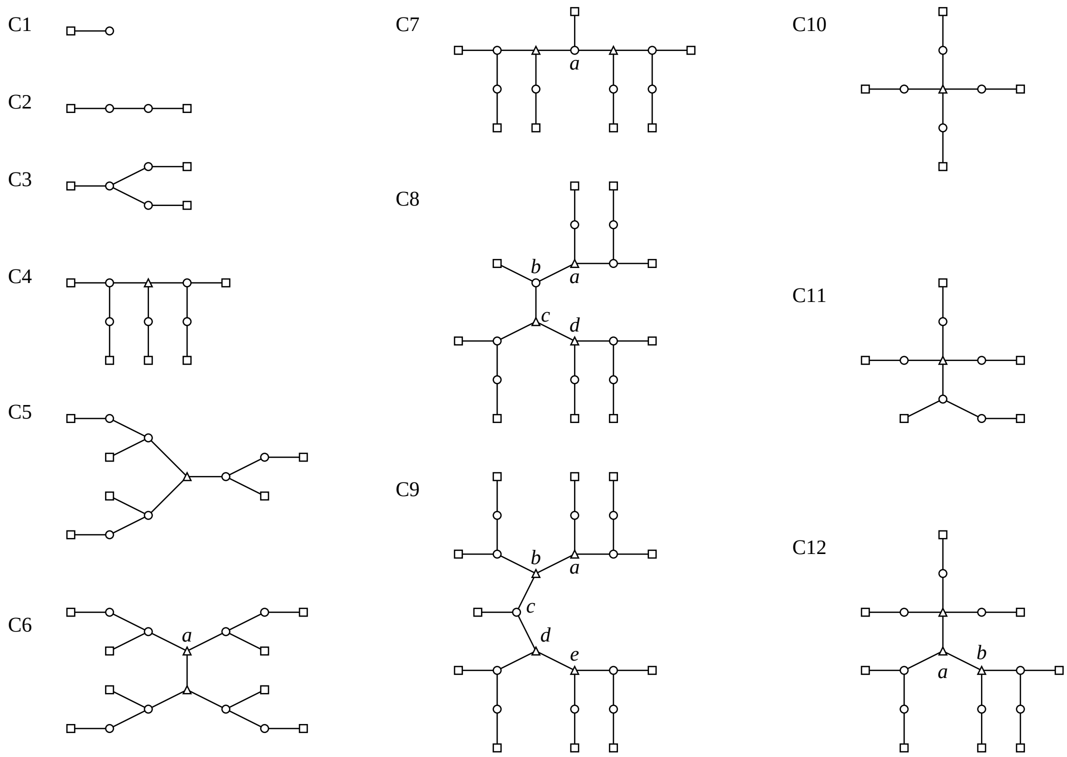

Figure 7: The smallest trees containing configurations C1-C12.

PROOF. We prove Lemma 7 using a discharging procedure. Suppose G is a counterexample to the lemma, namely a graph with $\mathrm{mad}(G) < \frac{14}{5}$, containing none of the configurations C1 to C12. We first assign to each vertex v a charge $\omega(v)$ equal to its degree: $\forall v \in V(G), \omega(v) = d(v)$. By hypothesis, $\sum_{v \in V(G)} \omega(v) < \frac{14}{5}|V(G)|$. We then redistribute the charges according to the rules R1 to R4, illustrated in Figure 8:

R1. Every 3^+-vertex gives $\frac{2}{5}$ to each adjacent 2-vertex.

R2. Every 4^+-vertex or non-light (non weary) 3-vertex gives $\frac{1}{10}$ to each adjacent non-weary light vertex.

R3. Every 4^+-vertex or non-light (non-weary) 3-vertex gives $\frac{1}{5}$ to each adjacent weary vertex.

R4. Every 4^+-vertex or non-light (non-weary) non-exhausted 3-vertex gives $\frac{1}{10}$ to each adjacent exhausted vertex.

Once the rules have been applied, each vertex v ends with a new charge $\omega^*(v)$. During the procedure, no new charge appears and no charge disappears; hence $\sum_{v\in V(G)} \omega(v) = \sum_{v\in V(G)} \omega^*(v)$. However we will prove that every vertex ends with a new charge $\omega^*(v)$ at least $\frac{14}{5}$ which will lead to the following contradiction, that completes the proof:

$$\frac{14}{5}|V(G)| > \sum_{v\in V(G)} \omega(v) = \sum_{v\in V(G)} \omega^*(v) \geq \frac{14}{5}|V(G)|$$

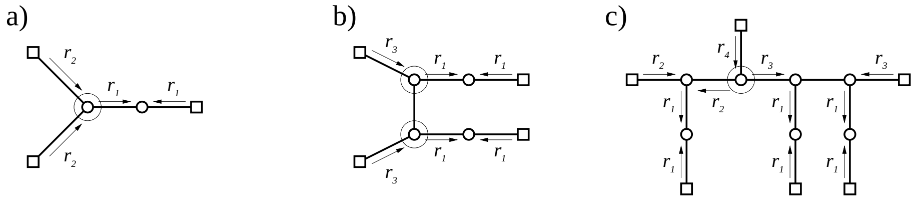

Figure 8: Rules R1 to R4 applied to the a) light, b) weary, and c) exhausted 3-vertices.

$$\left(r_1 = \tfrac{2}{5}, r_2 = r_4 = \tfrac{1}{10}, r_3 = \tfrac{1}{5}\right)$$

Let us now prove that $\forall v \in V(G), \omega^*(v) \geq \frac{14}{5}$. By exclusion of configuration C1, each vertex v of our graph G has degree at least 2. We consider the following cases according to the degree of v.

Case $d(v) = 2$. Initially, $\omega(v) = 2$. By exclusion of configuration C2, v is adjacent to two 3^+-vertices, and thus receives a charge $\frac{2}{5}$ from each of them, by application of rule R1. It follows that $\omega^*(v) = 2 + 2 \cdot \frac{2}{5} = \frac{14}{5}$.

Case $d(v) = 3$. Initially, $\omega(v) = 3$. By exclusion of C3, v is adjacent to at most one 2-vertex. We first consider the following two cases where v is adjacent to exactly one 2-vertex:

- Suppose first that v is weary (see Fig 8.b), i.e. v is adjacent to a 2-vertex and to a light vertex. Note that the exclusion of C4 assures that v is adjacent to at most one light vertex. By rule R1, v gives $\frac{2}{5}$ to its adjacent 2-vertex whereas by rule R3, v receives $\frac{1}{5}$ from its non-light neighbour. Hence we have $\omega^*(v) = 3 - \frac{2}{5} + \frac{1}{5} = \frac{14}{5}$.
- Suppose now that v is light but not weary (see Fig. 8.a): v is adjacent to a 2-vertex and to two non light 3^+-vertices. In that case, by rule R1, v gives $\frac{2}{5}$ to its adjacent 2-vertex, and by rule R2, v receives $\frac{1}{10}$ from each adjacent 3^+-vertices; so $\omega^*(v) = 3 - \frac{2}{5} + 2 \cdot \frac{1}{10} = \frac{14}{5}$.

Assume now that v is not adjacent to a 2-vertex. By exclusion of C5, v is adjacent to at most two light vertices. Moreover, by exclusion of C7, v is adjacent to at most one weary vertex. We consider the following cases:

- Suppose v is exhausted (see Fig 8.c), i.e. v is adjacent to a (non-weary) light vertex and to a weary vertex. Its remaining neighbour is a 3^+-vertex that is not light (thus not weary) by exclusion of C5, nor exhausted by exclusion of C6. Hence, applying rules R2 and R3, v gives respectively $\frac{1}{10}$ and $\frac{1}{5}$ to its light and weary neighbours, but by R4, v receives $\frac{1}{10}$ from its third neighbour. We thus have $\omega^*(v) = 3 - \frac{1}{10} - \frac{1}{5} + \frac{1}{10} = \frac{14}{5}$.

- Suppose now that v is adjacent to two (non-weary) light vertices. Its last neighbour is a 3^+-vertex that can be neither light by exclusion of C5, nor exhausted by exclusion of C6. So we only apply rule R2 twice and we get $\omega^*(v) = 3 - 2 \cdot \frac{1}{10} = \frac{14}{5}$.

- Suppose that v is adjacent to exactly one light vertex. If this vertex is weary, then by exclusion of C8, v cannot be adjacent to an exhausted vertex, and applying rule R3 we get $\omega^*(v) = 3 - \frac{1}{5} = \frac{14}{5}$. Otherwise, by exclusion of C9, at most one of v's non-light neighbours is exhausted. It follows that $\omega^*(v) \geq 3 - \frac{1}{10} - \frac{1}{10} = \frac{14}{5}$ by rules R2 and possibly R4.

- Finally, assume that v is not adjacent to any light vertices. By exclusion of C9, v is adjacent to at most one exhausted vertex. Hence, possibly applying R4, we get $\omega^*(v) \geq 3 - \frac{1}{10} > \frac{14}{5}$.

Case $d(v) = 4$. Initially, $\omega(v) = 4$. By exclusion of C10, v is adjacent to at most three 2-vertices. If v is adjacent to three 2-vertices, then its last neighbour cannot be light (by exclusion of C11) nor exhausted (by exclusion of C12); hence, applying rule R1, $\omega^*(v) = 4 - 3 \cdot \frac{2}{5} = \frac{14}{5}$. Otherwise, we apply rule R1 at most twice and possibly two rules from R2 to R4, and we get $\omega^*(v) \geq 4 - 2 \cdot \frac{2}{5} - 2 \cdot \frac{1}{5} = \frac{14}{5}$.

Case $d(v) = k \geq 5$. Applying rules R1 to R4, v may give at most k times $\frac{2}{5}$, and we get $\omega^*(v) \geq k - k \cdot \frac{2}{5} = \frac{3k}{5} \geq 3$ since $k \geq 5$.

In all cases, we got $\omega^*(v) \geq \frac{14}{5}$ as claimed, and this concludes the proof. □

Before proving Theorem 5, we show some useful properties of graphs containing one of the configurations C1-C12.

Note that configurations C1-C12 may exist in a graph G with a different embedding than the ones depicted in Fig. 7, containing short cycles. In [3], the authors reduced the possible number of such embeddings by considering only graphs with girth at least 7. Here, we use a different technique to consider any possible embedding of a configuration without enumerating them. The principle of this technique is to verify that no matter how a configuration is embedded, we still have enough leeway to extend a partition of the rest of the graph to the configuration. This leeway is carried by a few

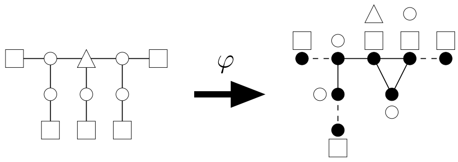

Figure 9: An example of a different embedding of T4.

vertices in the configurations, namely the vertices of the set later denoted $V_\triangle$. Observation 8 to Proposition 12 are simple structural properties of the configurations and their embeddings. One may easily convince himself these hold by looking at the Configurations C1-C12. Then we describe a simple process, in two stages, to extend a partition of the rest of the graph to the configuration. Most of the time, the first stage (steps 1 and 2) should almost suffice to extend the partition. In some cases where vertices of $V_\triangle$ have many common neighbours, fewer vertices are colored, and more leeway makes things harder to describe, but there is no real difficulty in extending the partition.

We use the following definitions. For any $1 \leq i \leq 12$, let Ti be the smallest tree containing configuration Ci ; namely the trees depicted in Fig. 7.

Given a tree Ti, we partition its vertices into three sets, $V_\circ, V_\triangle$ and $V_\square$. The set $V_\square$ contains all the vertices whose degree was not fixed by the configuration. Except for T1, this corresponds to all the leaves. The set $V_\circ$ contains all the vertices adjacent to a vertex in $V_\square$. Finally, $V_\triangle$ contains the remaining vertices, adjacent to no vertices of $V_\square$. This partition is displayed in Fig. 7 with the shape of the vertices.

By construction of $V_\circ, V_\triangle, V_\square$, we observe the following property on any tree in T1-T12 :

Observation 8 *Any vertex in $V_\circ$ has exactly one neighbour in $V_\square$, and any vertex in $V_\triangle$ has no neighbours in $V_\square$.*

$$\begin{array}{rcl} \forall v \in V_\circ, \quad |N(v) \cap V_\square| &=& 1 \\ \forall v \in V_\triangle, \quad |N(v) \cap V_\square| &=& 0 \end{array}$$

Let G be a graph containing an occurrence of configuration Ci. We define the homomorphism $\varphi \colon V(\mathrm{T}i) \longrightarrow V(G)$ that maps each vertex of Ti to the vertex of G having the same role in this occurrence of configuration Ci. Some vertices of G may be the image of more than one vertex of Ti. See for example Fig. 9. Abusing the notation, we will simply denote by $\varphi^{-1}(v)$ the set $\{u \in \mathrm{T}i \mid \varphi(u) = v\}$.

Note that for any edge uv in $E(\mathrm{T}i)$, $\varphi(u)\varphi(v)$ is an edge in G. Also note that, for $\varphi(\mathrm{T}i)$ to be an occurrence of configuration Ci, the restriction of φ from $N[u]$ to $N[\varphi(u)]$ has to be bijective for any $u \in V_\circ \cup V_\triangle$. The mapping φ thus preserves degree, lightness, weariness, and exhaustedness of vertices in $V_\circ \cup V_\triangle$. We later refer to this property saying that φ is locally bijective.

In G, we denote by $E_\diamondsuit$ the set of edges defined by

$$E_\diamondsuit = \{e \in E(G) \mid \varphi^{-1}(e) \subseteq E(V_\circ, V_\square)\}$$

Such edges are dashed in Fig. 9.

The following propositions hold.

Proposition 9 *If $v \in V(G)$ is a vertex of $\varphi(V_\circ)$, then at most one edge incident to v is in $E_\diamondsuit$.*

PROOF. Let $v = \varphi(v')$, $v' \in V_\circ$. Suppose there exists $u \in N(v)$ such that $uv \in E_\diamondsuit$. Let $w \in N(v) \setminus \{u\}$. By local bijectivity of φ, there exist a unique $u' \in \varphi^{-1}(u) \cap N_{\mathrm{T}i}(v')$ and $w' \in \varphi^{-1}(w) \cap N_{\mathrm{T}i}(v')$, and $u' \neq w'$. From Observation 8, $w' \notin V_\Box$ so $vw \notin E_\diamondsuit$. □

Proposition 10 *Suppose a configuration $C \neq \mathrm{C2}$ appears in G. Let uv be an edge of G such that $u, v \in \varphi(V_\circ) \setminus \varphi(V_\triangle)$. If $uv \notin E_\diamondsuit$, then $d(u) = 2$ and $d(v) = 3$, or conversely.*

PROOF. Let uv satisfy the condition of the proposition. Since $uv \notin E_\diamondsuit$, there exist u', v' adjacent vertices in C such that $\phi(u') = u, \phi(v') = v$, and none of u', v' are in $V_\Box$. By hypothesis, u' and v' are also not in $V_\triangle$, thus they must both be in $V_\circ$. It is then easy to check on all configurations that any edge between two vertices in $V_\circ$ has an extremity of degree 2 and the other of degree 3. □

Proposition 11 *On C1 to C12, the homomorphism φ restricted on $V_\triangle \to \varphi(V_\triangle)$ is bijective.*

PROOF. We only need to check injectivity to reach the conclusion. For configurations C1 to C5, C10 and C11, since $|V_\triangle| \leq 1$, this is clearly true. For configurations C6, C7 and C12, the proposition is a corollary of bijectivity of φ on $N[a]$.

Let us now deal first with C8. By bijectivity of φ respectively on $N[b]$ and on $N[c]$, we know that $\varphi(a) \neq \varphi(c)$ and $\varphi(c) \neq \varphi(d)$. Moreover, if $\varphi(a) = \varphi(d)$, then necessarily, $\varphi(b) = \varphi(c)$, since other neighbours of a and of d are either 2-vertices or light-vertices. But this contradicts local bijectivity of φ. Therefore $\varphi(a) \neq \varphi(d)$ and φ is injective on $V_\triangle$.

Finally, we deal similarly with C9. By local bijectivity of φ, we know that $\varphi(a) \neq \varphi(b)$, $\varphi(b) \neq \varphi(d)$, and $\varphi(d) \neq \varphi(e)$. Moreover, since a and e are light vertices but not b and d, $\varphi(a) \neq \varphi(d)$ and $\varphi(b) \neq \varphi(e)$. Finally, if $\varphi(a) = \varphi(e)$, then $\varphi(b) = \varphi(d)$, b and d being the only neither light nor degree 2 neighbours of a and e, and that contradicts bijectivity of φ on $N[c]$. Therefore, φ is also injective on $V_\triangle$ in C9, and the proposition is proven. □

Proposition 12 *Let $\mathrm{T}i$ be a configuration occuring in G. No vertices in $\varphi(\mathrm{T}i)$ may be adjacent to three or more vertices in $\varphi(V_\triangle)$.*

PROOF. The only configurations where this could occur are C8, C9 and C12. In C8, only the vertex $\varphi(b)$ may have three neighbours in $\varphi(V_\triangle)$, if $\varphi(d)$ was a neighbour of $\varphi(b)$. But neighbours of d different from c are 2-vertices or light-vertices, which b is not. In C9, only $\varphi(c)$ could have three neighours in $\varphi(V_\triangle)$, yet for the same reason, $\varphi(c)$ can not be a neighbour of $\varphi(a)$ or of $\varphi(e)$. Finally, in C12, for having three neighbours in $\varphi(V_\triangle)$, a vertex should be adjacent to the 4-vertex, and to $\varphi(a)$, but vertices adjacent to the 4-vertex different from a are of degree 2. Therefore, no vertices may be adjacent to three vertices in $\varphi(V_\triangle)$. $\square$

We now prove Theorem 5, that we recall here:

Theorem 5 *Every graph G with $\mathrm{mad(G)} < \frac{14}{5}$ is 3-star 2-partitionable.*

PROOF. We prove the theorem by contradiction. Suppose it is false and G is a counterexample with minimum order. By Lemma 7, G contains one of the configurations C1 to C12, let C be the configuration of smallest label appearing in G.

Let ν be a valid partition of $G[V \setminus \varphi(V_\circ \cup V_\triangle)]$. *We see this partition as a colouring of the vertices $\nu : V \longrightarrow \{0, 1\}$, where one part is $\{v \in V \mid \nu(v) = 0\}$ and the other is* $\{v \in V \mid \nu(v) = 1\}$.

We extend ν to $G[V \setminus \varphi(V_\triangle)]$ with the following procedure :

Step 1 Colour any edge $uv \in E_\diamond$ properly, that is choose $\nu(u) \neq \nu(v)$. We know this can be done by Proposition 9.

Step 2 For any remaining vertex v in $V \setminus \varphi(V_\triangle)$, with at least one coloured neighbour, choose for $\nu(v)$ a least represented colour in $N(v)$. We reiterate step 2 as long as possible. If at some point, there are some vertices left in $V \setminus \varphi(V_\triangle)$ with no coloured neighbours, pick one, colour it randomly, and continue step 2.

Remark that are coloured during Step 2 precisely all vertices $v \in \varphi(V_\circ) \setminus \varphi(V_\triangle)$ that are incident to no edges of $E_\diamond$. The following observation also holds:

Observation 13 *Any vertex coloured during step 1 or 2 has either no coloured neighbours, or at least one neighbour of the opposite colour.*

Proposition 14 *After application of Steps 1 and 2, either G contains Ci for some $i \in \{1, 2, 3\}$ or ν is such that $G\big[\{v \in \varphi(V_\circ) \mid \nu(v) = i\}\big]$ contains no paths on three vertices for $i = 0, 1$, that is, there is no monochromatic P_3.*

PROOF. Suppose G contains neither of C1, C2 or C3, yet that $u, v, w \in \varphi(V_\circ)$ form a P_3 with edges uv and vw, such that $\nu(u) = \nu(v) = \nu(w)$. Clearly, none of uv and vw is in $E_\diamond$, or the two ends of the edge would have a different label. We can thus apply Proposition 10 for both edges, and conclude that either v is of degree 2, or both u and w are and v is of degree 3. The later case corresponds to the presence of configuration C3 in G. Suppose now that v is of degree 2, whereas u and w are of degree at least 3 to avoid presence of C1 or C2. Since v have no incident edges in $E_\diamond$, it was assigned a colour ν during Step 2. It then must have at least one neighbour coloured differently, (either u or v), a contradiction. $\square$

Proposition 15 *After application of Steps 1 and 2, ν is a proper 3-star 2-partition of* $G\big[V \setminus \varphi(V_\triangle)\big]$.

PROOF. Suppose on the contrary that there exists a component of order at least 4 or a triangle in the subgraph of G induced by a colour class. Since ν is a valid partition of $G\big[V \setminus \varphi(V_\circ \cup V_\triangle)\big]$, this component does not contain only vertices of $V \setminus \varphi(V_\circ \cup V_\triangle)$. Remark that any edge joining a vertex from $\varphi(V_\circ \cup V_\triangle)$ to a vertex of $V \setminus \varphi(V_\circ \cup V_\triangle)$ is necessarily in $E_\diamondsuit$, so is properly coloured. Therefore, the component must be contained in $\varphi(V_\circ \cup V_\triangle)$.

Configurations C1 and C2 are such that $\big|\varphi(V_\circ \cup V_\triangle)\big| \leq 2$, so no component of size at least three may be included in $\varphi(V_\circ \cup V_\triangle)$. In C3, $|V_\circ \cup V_\triangle| = 3$, so we may only find a triangle. However, if the vertex of degree 3 with its two neighbours of degree 2 form a triangle, then the edge joining the vertices of degree 2 is in $E_\diamondsuit$, and is properly coloured, a contradiction. Finally, if none of C1, C2 and C3 appears in G but another configuration appears, then the result holds by Proposition 14. □

From this last proposition, we only have to extend the colour assignment to vertices in $\varphi(V_\triangle)$. We assign colours, trying to preserve the following properties :

P1: The graph induced by $\varphi(V_\circ \cup V_\triangle)$ contains no monochromatic P_3 adjacent to an uncoloured vertex of $\varphi(V_\triangle)$.

P2: The graph induced by $\varphi(V_\circ \cup V_\triangle)$ contains no monochromatic P_2 (path on two vertices) adjacent to 2 uncoloured vertices of $\varphi(V_\triangle)$.

After Steps 1 and 2, P1 is a consequence of Proposition 14. Suppose P2 does not hold. Let a, b form a monochromatic P_2 adjacent to two vertices of $\varphi(V_\triangle)$. The edge ab is not in $E_\diamondsuit$, so applying Proposition 10, we know that a and b are of degree 2 and 3, say b is of degree 2. If b is adjacent to a vertex in $\varphi(V_\triangle)$, then a is its only coloured neighbour, and by Observation 13, it is coloured differently, a contradiction. If a is adjacent to the two vertices of $\varphi(V_\triangle)$, then b is the only coloured neighbour of a, and Observation 13 again leads to a contradiction.

We use the following strategies (in this order) to colour yet uncoloured 3-vertices of $\varphi(V_\triangle)$. We also show that for each strategy, the colouring still corresponds to a 3-star 2-partition and satisfies properties P1 and P2 (note that we leave the colouring of 4-vertices to the very end).

- *If $v \in \varphi(V_\triangle)$ has two uncoloured neighbours*, assign to v the opposite colour of its third neighbour.

In that case, it is clear that we still have a 3-star 2-partition and that P1 and P2 are still satisfied.

- *If $v \in \varphi(V_\triangle)$ has two neighbours of the same colour*, assign to v the opposite colour.

Since v is a 3-vertex, it has at most one neighbour of the same colour, thus, we do not form a monochromatic triangle. From property P1, we easily infer that colouring v, we do not form a component of order more than 3. Suppose a monochromatic P_3 is formed, then by property P2, it is not adjacent to an uncoloured vertex of $\varphi(V_\triangle)$ and P1 and P2 hold trivially. If only a monochromatic P_2 is formed, by Proposition 12, P2 holds, and P1 holds trivially.

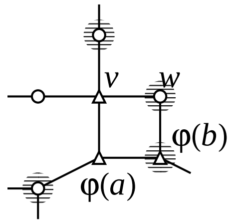

Figure 10: A particular partial embedding of C12.

- *If $u, v \in \varphi(V_\triangle)$ are two adjacent uncolored vertices both adjacent to a vertex of each color*, then properly color the edge uv.

Here again, P1 suffices to conclude that the obtained colouring corresponds to a 3-star 2-partition. Property P2 implies that P1 still holds, and Proposition 12 implies that P2 still holds.

We now need to deal with the remaining 4-vertex v in $\varphi(V_\triangle)$ for configurations C10, C11 and C12. Note that in C12, the first strategy should be applied on a before colouring the other vertices, so that only the 4-vertex remains uncoloured at the end.

If v has at least three neighbours of the same colour, use the opposite colour for v. By property P1, the colouring obtained corresponds to a 3-star 2-partition of the graph.

Suppose now that v has two neighbours of each colour. Among them, three are 2-vertices, and two of them have necessarily the same colour, say 0. If the second neighbour of each of these two vertices are of colour 1, then we choose for v colour 0, forming a monochromatic P_3, and this extends the 3-star 2-partition. The only situation when this is not true is in C12, when one of them, say w, has both its neighbours in $\varphi(V_\triangle)$; otherwise, Observation 13 applies. The second neighbour is then necessarily $\varphi(b)$.

Suppose we are in this situation, which is depicted in Fig. 10. We know that w is of same colour than $\varphi(b)$ and than some other degree 2 neighbour of v. Also, $\varphi(a)$ is of different colour than w and $\varphi(b)$. By the strategy applied, $\varphi(a)$ got a different colour than its only neighbour not in $\varphi(V_\triangle)$. Thus, we can choose for v the same colour has $\varphi(a)$. This forms no more than a monochromatic P_3, and extends the 3-star 2-partition to G.

Finally, in each situation, we proved that G was not a counterexample, reaching a contradiction. This concludes the proof. □

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