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The Schottky-Klein Prime Function on the Schottky Double of Planar Domains

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Abstract. This article provides an overview of the properties and uses of the Schottky-Klein prime function on the Schottky double of multiply connected planar domains. Simple expressions are offered for the conformal mappings from a multiply connected circular domain to canonical multiply connected slit domains. It is argued that these basic functions can be used as "building blocks" for the construction of more complicated functions in a variety of circumstances. As an example, a new geometrical interpretation of a recently-derived formula for multiply connected Schwarz-Christoffel mappings is given.

Keywords. Schwarz-Christoffel, conformal mappings, slit mappings, Schottky-Klein prime function.

2000 MSC. 30C20, 31A15.

1. Background

The fundamental theorem of algebra states that any N-th degree polynomial $P_N(\zeta)$, with $N \geq 1$, can be uniquely factorized into a product of simpler functions of the form

$$P_N(\zeta) = \zeta^N + a_{N-1}\zeta^{N-1} + \cdots + a_1\zeta + a_0 = \prod_{k=1}^{N}(\zeta - \gamma_k) \tag{1}$$

where $\{\gamma_k | k = 1, \ldots, N\}$ are the roots of the polynomial. It is then natural to define the simple monomial function of two variables, $\omega(\zeta, \gamma) \equiv (\zeta - \gamma)$, to be a *prime function* since, in analogy with being able to factorize any integer into a unique product of prime integers, any polynomial can be uniquely factorized

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into a product of such prime functions, as evinced by (1), so that we can write

$$P_N(\zeta) = \prod_{k=1}^{N} \omega(\zeta, \gamma_k).$$

By extension, any rational function $R(\zeta)$ (a function whose only singularities are poles) can be written in the form

$$R(\zeta) = \prod_{k=1}^{N} \omega(\zeta, \alpha_k) \Big/ \prod_{k=1}^{N} \omega(\zeta, \beta_k), \tag{2}$$

where $\{\alpha_k | k = 1, \ldots, N\}$ are the zeros and $\{\beta_k | k = 1, \ldots, N\}$ the poles of the function. The *Schottky-Klein prime function* (henceforth, S-K prime function) is the name given to the function which replaces the simple monomial function $(\zeta - \gamma)$ when the underlying compact Riemann surface has higher genus than the Riemann sphere (i.e., when the Riemann sphere has handles). Any meromorphic function $R(\zeta)$ on such a surface then also has a representation in terms of its zeros and poles very much akin to (2). The prime function for general compact Riemann surfaces was first considered by Schottky [30] and Klein [26]. It is discussed in the paper by Burnside [5] and reported on in a special chapter of the classic monograph by Baker [1] on abelian functions. It has close mathematical connections with the notion of a *prime form* [21] on the Jacobi variety associated with a compact Riemann surface and prime forms have, over the years, found abundant application in, for example, algebraic geometry, mathematical physics and integrable systems. The S-K prime function within the Schottky model of algebraic curves has, by contrast, been used much less often.

Hejhal [25] returns to the S-K prime function in his discussion of the classical kernel functions of planar domains and it is on the particular application of the prime function to planar domains that this article will focus. It is possible to associate with any multiply connected planar domain a compact symmetric Riemann surface called its Schottky double. The S-K prime function on such symmetric Riemann surfaces has certain special properties which will be reviewed here. As a result, a large number of results associated with the function theory of planar domains can be conveniently expressed in terms of the S-K prime function on the Schottky double of the domain.

2. The prime function on a torus

The S-K prime function for the Riemann sphere is simple. The prime function for a sphere with one handle, or torus, is more interesting. A mathematical model of a torus is to consider the two neighbouring annuli $\rho < |\zeta| < 1$ and $1 < |\zeta| < \rho^{-1}$ where $0 < \rho < 1$ is a real parameter. These two annuli already meet at the circle $|\zeta| = 1$ but we also want them to be associated at the two other boundary circles $|\zeta| = \rho$ and $|\zeta| = \rho^{-1}$. A holomorphic identification of these two circles is provided by the Möbius mapping $\zeta \mapsto \rho^2\zeta$. A meromorphic function $F(\zeta)$ on this torus can be defined as a function satisfying the functional relation

$$F(\rho^2\zeta) = F(\zeta) \tag{3}$$

and having only poles in the annulus $\rho \leq |\zeta| < \rho^{-1}$. Such functions have been dubbed *loxodromic functions* [31].

But how do we construct functions satisfying (3)? Consider $P(\zeta)$ defined by the infinite product

$$P(\zeta) \equiv (1-\zeta)\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta)(1-\rho^{2k}\zeta^{-1}). \tag{4}$$

Using standard methods for infinite products [31] the function (4) is known to be convergent for all $\zeta \neq 0$ and for all $0 < \rho < 1$. It is easy to confirm, directly from this definition, that $P(\zeta)$ satisfies the functional relation

$$P(\rho^2\zeta) = -\zeta^{-1}P(\zeta). \tag{5}$$

The function $P(\zeta)$ does not itself satisfy (3), but the ratio of products

$$R(\zeta) \equiv \prod_{k=1}^{N} P(\zeta\alpha_k^{-1}) \Big/ \prod_{k=1}^{N} P(\zeta\beta_k^{-1}) \tag{6}$$

does satisfy (3) provided the parameters in (6) satisfy the single condition

$$\prod_{k=1}^{N}\alpha_k = \prod_{k=1}^{N}\beta_k.$$

This, again, is a simple exercise based on use of (5). $R(\zeta)$ can also be seen to have only poles in the annulus $\rho \leq |\zeta| < \rho^{-1}$ and is therefore meromorphic on the torus. On comparing (6) with (2) it is natural to identify the function $P(\zeta)$ with the prime function for the torus and, up to normalization, this is indeed the case.

It is worth noting that since $P(\zeta)$ is analytic in the annulus $\rho < |\zeta| < 1$ then, in addition to the infinite product expression (4), it also has a convergent Laurent series there. It is given by the rapidly convergent series

$$P(\zeta) = A\sum_{n=-\infty}^{\infty}(-1)^n\rho^{n(n-1)}\zeta^n, \tag{7}$$

where

$$A = \prod_{n=1}^{\infty}(1+\rho^{2n})^2 \Big/ \sum_{n=1}^{\infty}\rho^{n(n-1)}.$$

The Laurent series (7) converges everywhere in the annulus $\rho \le |\zeta| < \rho^{-1}$. These two representations of the same function furnish the identity

$$(8) \qquad (1-\zeta)\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta)(1-\rho^{2k}\zeta^{-1}) = A\sum_{n=-\infty}^{\infty}(-1)^n\rho^{n(n-1)}\zeta^n,$$

which relates an infinite product to an infinite sum. The reader might recognize this as the *Jacobi triple product identity* [33]. We will refer back to this identity later.

3. The Schottky double of a domain

From the theory of conformal mapping [23] it is known that any doubly connected domain is conformally equivalent to some annulus $\rho < |\zeta| < 1$. On the other hand, in the previous section a torus was built by identifying the boundary circles $|\zeta| = \rho$ and $|\zeta| = \rho^{-1}$ of the larger annulus $\rho < |\zeta| < \rho^{-1}$. Note that, given the smaller annulus $\rho < |\zeta| < 1$, the larger annulus $\rho < |\zeta| < \rho^{-1}$ can be constructed by reflection of $\rho < |\zeta| < 1$ in the unit circle.

In the same way, it is known [23] that any M-connected domain (for $M \ge 1$) is conformally equivalent to the unit ζ-disc with M smaller circular discs excised. Let such a domain be denoted D_ζ. One can then reflect the circular boundaries of these discs in the unit circle to produce M additional circles in $|\zeta| > 1$ (by this antiholomorphic reflection, one produces a precise copy, or "backside", of the original multiply connected domain outside the unit ζ-disc). Then, following the example of the torus, we can identify each circle in $|\zeta| < 1$ with its reflection in $|\zeta| > 1$ by means of a holomorphic Möbius transformation. For an $(M+1)$-connected domain there will be M such Möbius transformations. The basic idea of this construction lies at the heart of the Schottky model of algebraic curves and, by forming the union of D_ζ with its backside and identifying the boundary circles as just described, we produce a model of the so-called *Schottky double* of the original multiply connected circular domain. Gustafsson [24] and Varchenko & Etingof [32] have discussed the so-called Hele-Shaw free boundary problem in a multiply connected domain by considering the Schottky double of the domain.

More specifically, consider a multiply connected circular domain D_ζ consisting of the unit ζ-disc with M smaller circular discs excised having centres located at $\{\delta_j | j = 1, \dots, M\}$ and radii $\{q_j | j = 1, \dots, M\}$. The data $\{\delta_j, q_j | j = 1, \dots, M\}$ will be called the *conformal moduli* of D_ζ. Let the unit circle be denoted C_0 and let the M interior circular boundaries be denoted $\{C_k | k = 1, \dots, M\}$. For $k = 1, \dots, M$ let C_k' denote the reflection of C_k in C_0. Figure 1 shows a schematic in the triply connected case $M = 2$. Now, for $k = 0, 1, \dots, M$, we introduce the Möbius transformation $\phi_k(\zeta)$ defined by

$$(9) \qquad \phi_k(\zeta) = \overline{\delta_k} + \frac{{q_k}^2}{\zeta - \delta_k}, \qquad k = 0, 1, \dots, M.$$

It is straightforward to check that for points on the circle C_k

$$\phi_k(\zeta) = \overline{\zeta}.$$

We define the *reflection* of a point ζ in the circle C_k by $\overline{\phi_k(\zeta)}$. Then for $k = 1, \ldots, M$ introduce the Möbius transformation $\theta_k(\zeta)$ defined by

$$\theta_k(\zeta) = \overline{\phi_k(\overline{\zeta}^{-1})}, \qquad k = 1, \ldots, M. \tag{10}$$

It follows from (10) and (9) that

$$\theta_k(\zeta) = \delta_k + \frac{{q_k}^2\zeta}{1 - \overline{\delta_k}\zeta}, \qquad k = 1, \ldots, M. \tag{11}$$

It is straightforward to show that $\theta_k(\zeta)$ maps C_k' onto C_k as illustrated in Figure 1.

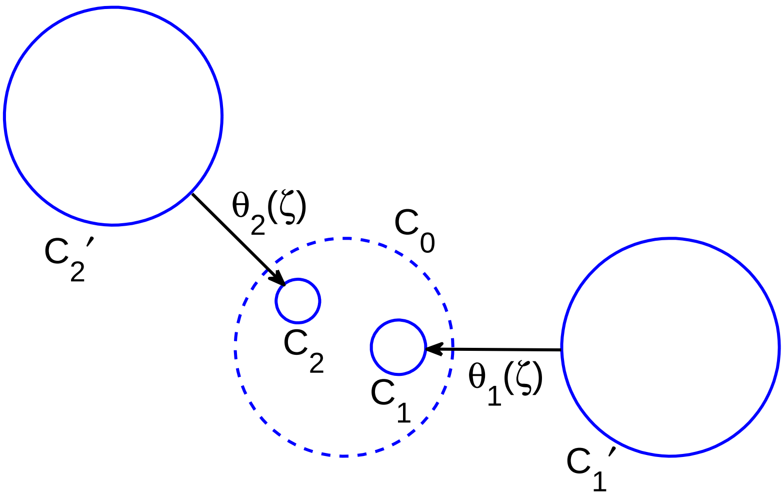

FIGURE 1. The circles $\{C_j, C_j' | j = 1, \ldots, M\}$ and the maps $\{\theta_j(\zeta) | j = 1, \ldots, M\}$ identifying them. C_0 is the unit circle. The case $M = 2$ is shown.

Consider the maps $\{\theta_k(\zeta) | k = 1, \ldots, M\}$. Then the set Θ consisting of all functional compositions of these maps and their inverses is an example of a classical *Schottky group* [1, 2]. We refer to the maps $\{\theta_k(\zeta) | k = 1, \ldots, M\}$ and their inverses as the *generators* of the group Θ. A fundamental region of Θ is a connected region whose images under all maps in Θ tessellate the whole of the plane. Let us define F as the region consisting of D_ζ and its reflection in C_0, i.e., the $2M$-connected region bounded by $\{C_k, C_k' | k = 1, \ldots, M\}$. Then F is a fundamental region for the group Θ.

To return to the genus-1 case, if the previous construction is applied to the case in which D_ζ is the annulus $0 < \rho < |\zeta| < 1$ then the Schottky group is generated by $\theta_1(\zeta) = \rho^2\zeta$ and its inverse. The Schottky group Θ in this case can be written explicitly as

$$\Theta \equiv \left\{\theta_j(\zeta) = \rho^{2j}\zeta | j \in \mathbb{Z}\right\}. \tag{12}$$

A fundamental region F is the annulus $\rho \leq |\zeta| < \rho^{-1}$.

4. The Schottky-Klein prime function

Associated with the Schottky double of the domain D_ζ are M functions known as *integrals of the first kind* which we denote $\{\upsilon_k(\zeta)|k = 1, \ldots, M\}$. These are analytic, but not single-valued, in F. Indeed, for $j, k = 1, \ldots, M$ we have [1]

$$[\upsilon_k(\zeta)]_{C_j} = -[\upsilon_k(\zeta)]_{C'_j} = \delta_{jk}, \tag{13}$$

where $[\upsilon_k(\zeta)]_{C_j}$ and $[\upsilon_k(\zeta)]_{C'_j}$ denote respectively the changes in $\upsilon_k(\zeta)$ on traversing C_j and C'_j with the interior of F on the right. δ_{jk} denotes the Kronecker delta function. Furthermore, for $j, k = 1, \ldots, M$,

$$\upsilon_k(\theta_j(\zeta)) - \upsilon_k(\zeta) = \tau_{jk} \tag{14}$$

for some $\{\tau_{jk}|j, k = 1, \ldots, M\}$ which are constants (independent of ζ). The functions $\{\upsilon_k(\zeta)|k = 1, \ldots, M\}$ are uniquely determined (up to an additive constant) by their periods given by (13) and (14).

It is established in [25] that there exists a unique function $X(\zeta, \alpha)$ defined by the properties:

(i) $X(\zeta, \alpha)$ is single-valued and analytic in F.
(ii) $X(\zeta, \alpha)$ has a second-order zero at each of the points $\theta(\alpha)$, $\theta \in \Theta$.
(iii)
$$\lim_{\zeta \to \alpha} \frac{X(\zeta, \alpha)}{(\zeta - \alpha)^2} = 1.$$
(iv) If $\theta_k(\zeta)$ (for $k = 1, \ldots, M$) is one of the generators of Θ, then

$$X(\theta_k(\zeta), \alpha) = \exp(-2\pi i(2\upsilon_k(\zeta) - 2\upsilon_k(\alpha) + \tau_{kk}))\,\frac{d\theta_k(\zeta)}{d\zeta} X(\zeta, \alpha). \tag{15}$$

The *Schottky-Klein prime function* is then defined as the square root of $X(\zeta, \alpha)$. There is a similar characterization for the prime function itself subject only to an ambiguity in sign (a matter which has recently been discussed by Bogatyrev [4]).

5. Computing the S-K prime function

How can the S-K prime function be evaluated in practice? One possibility is to use a classical infinite product formula for it as recorded, for example, in Baker [1]. It is given by

$$\omega(\zeta, \alpha) = (\zeta - \alpha) \prod_{\theta_k} \frac{(\theta_k(\zeta) - \alpha)(\theta_k(\alpha) - \zeta)}{(\theta_k(\zeta) - \zeta)(\theta_k(\alpha) - \alpha)}, \tag{16}$$

where the product is over all compositions of the maps generating the group Θ excluding the identity and all inverse maps. It has not yet been proven that this product converges for *all* choices of the parameters $\{q_j, \delta_j | j = 1, \ldots, M\}$ and, even if it does, its convergence rate can be so slow as to make use of (16) impractical in many circumstances. The product must, of course, be truncated in some fashion and one way to do this is to truncate at a certain level of composition of the Möbius mappings generating the group. This can quickly become numerically expensive but it can be safely used in some cases, especially of small connectivity.

For a torus, if the group (12) is used in the definition (16), the result is the infinite product (4) which, as already mentioned, is the relevant S-K prime function (to within a normalization factor). It has already been pointed out that the Jacobi triple product identity (8) provides an alternative method of computing the S-K prime function in that case: instead of truncating the infinite product (4) one can alternatively truncate the much more rapidly convergent infinite sum (7). This idea of trading of a slowly convergent infinite product for a more rapidly convergent infinite sum is the motivation behind a recently-devised numerical algorithm for computing the S-K prime function (on the Schottky double of planar domains) presented by Crowdy & Marshall [13]. They devised a novel numerical algorithm for computing the S-K prime function based on Laurent series representations of the function. It can be used to evaluate $\omega(\zeta, \alpha)$, with great speed and accuracy, for broad classes of domains. The algorithm works by writing

$$X(\zeta, \alpha) = (\zeta - \alpha)^2 \hat{X}(\zeta, \alpha),$$

where $X(\zeta, \alpha) = \omega^2(\zeta, \alpha)$ and then computing the coefficients in a truncation of the following Laurent expansion of $\hat{X}(\zeta, \alpha)$:

$$\hat{X}(\zeta, \alpha) = A \left(1 + \sum_{k=1}^{M} \sum_{m=1}^{\infty} \frac{c_m{}^{(k)} q_k{}^m}{(\zeta - \delta_k)^m} + \sum_{k=1}^{M} \sum_{m=1}^{\infty} \frac{d_m{}^{(k)} Q_k{}^m}{(\zeta - \delta_k')^m} \right). \tag{17}$$

where Q_k and δ_k' are, respectively, the radius and centre of C_k'. The coefficients $\{c_m^{(k)}, d_m^{(k)}\}$ are determined numerically using the transformation properties (15) [13]. It is important to note that the algorithm in [13] does not depend on a sum or product over a Schottky group. This feature renders it a much faster practical method of evaluating the prime function than methods based on use of (16). Freely downloadable MATLAB codes have been prepared, based on the algorithm described in [13], and they are available for general use at the website [10].

6. The Schwarz conjugate function

The symmetry of the Schottky double (the fact that it comprises two copies of the planar domain – the domain itself together with a backside) furnishes another functional relation involving the S-K prime function that proves to be invaluable in applications. Indeed the Schwarz conjugate function to the S-K prime function can, in this case, be usefully related to the S-K prime function itself. Defining the Schwarz conjugate of the prime function by

$$\overline{\omega}(\zeta,\gamma) = \overline{\omega(\overline{\zeta},\overline{\gamma})},$$

it can be shown that the following functional relation holds:

$$\overline{\omega}(\zeta^{-1},\gamma^{-1}) = -\zeta^{-1}\gamma^{-1}\omega(\zeta,\gamma).$$

A proof of this result, based on the infinite product definition of the prime function, is given in an appendix to Crowdy & Marshall [11]. It can alternatively be established by simply making use of the definition of the prime function and the symmetry of the underlying Schottky group.

7. Slit mappings as building blocks

We now give evidence that conformal slit mappings can provide useful "building block" functions from which more complicated function theoretic objects can be constructed. The theoretical importance of conformal slit maps is well-known: Schiffer [29], for example, has elucidated a number of useful connections between conformal slit mappings and the fundamental objects of potential theory (Green's functions, modified Green's functions, harmonic measures). It is a remarkable (and useful) fact that conformal mappings of multiply connected circular domains to all the canonical slit domains [28, 3] can be expressed explicitly, by means of compact formulae, in terms of the S-K prime function. Full details can be found in Crowdy & Marshall [12] and those relevant to the remainder of this article will now be reviewed.

7.1. Circular slit domains. Pick a point α in the interior of some multiply connected circular domain D_ζ as shown in Figure 2 for the case $M = 2$. Consider the function

$$\eta_1(\zeta) = \frac{\omega(\zeta,\alpha)}{|\alpha|\omega(\zeta,\overline{\alpha}^{-1})}. \tag{18}$$

As a conformal mapping this takes C_0 to a unit circle L_0 and the point $\zeta = \alpha$ to the centre of L_0. Meanwhile the circles C_1 and C_2 are mapped to circular arc slits L_1 and L_2 concentric with the circle L_0. The image is therefore a circular slit domain that is recognized as being one of the canonical multiply connected slit domains [28, 3].

7.2. Half-space slit domains. Consider the function defined by

$$\eta_2(\zeta) = \frac{\omega(\zeta,\alpha_1)}{\omega(\zeta,\alpha_2)},$$

where the two parameters α_1 and α_2 are both taken to be on the same boundary circle of a multiply connected circular domain D_ζ. Figure 3 shows the case with

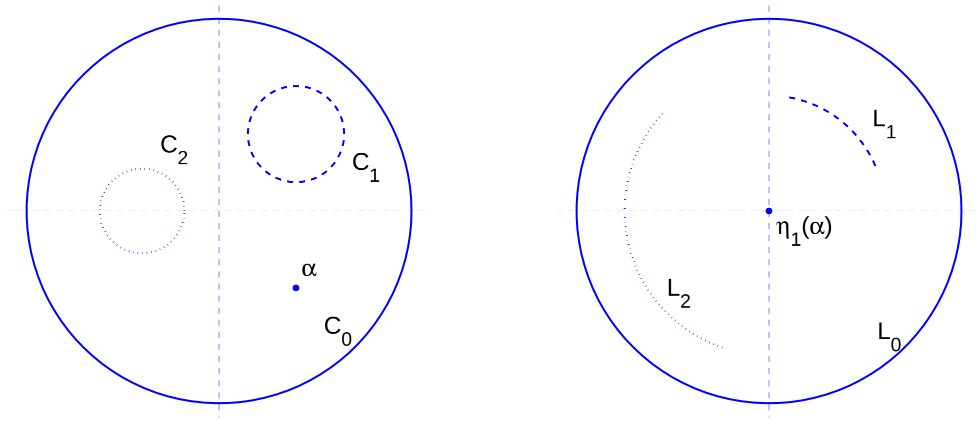

FIGURE 2. Circular slit domain. C_0 maps to a unit circle L_0 while C_1 and C_2 map to concentric circular arc slits inside L_0. α maps to the centre of L_0.

$M = 2$ and where α_1 and α_2 are both chosen to be on C_0. α_1 maps to the origin and α_2 maps to infinity in such a way that C_0 maps to a infinite straight line through the origin. C_1 and C_2 each map to radial slits of finite length. The image will be referred to as a half-space slit domain. Actually, such slit domains are not explicitly discussed in Crowdy & Marshall [12] but can be viewed as special cases of the radial slit mappings to be presented next.

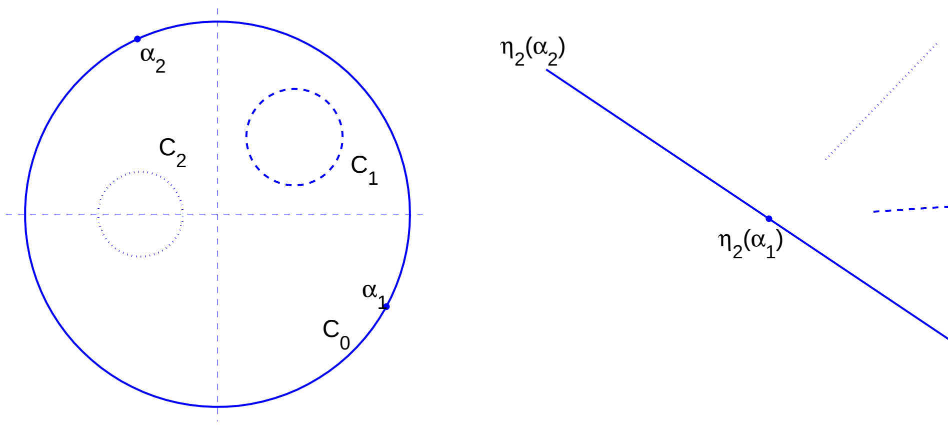

FIGURE 3. Half-space slit domain. C_0 maps to a infinite straight line through the origin and infinity. C_1 and C_2 map to radial slits of finite length.

7.3. Radial slit domains. Pick two points α and β inside a multiply connected circular domain D_ζ and consider the function

$$\eta_3(\zeta) = \frac{\omega(\zeta,\alpha)\omega(\zeta,\overline{\alpha}^{-1})}{\omega(\zeta,\beta)\omega(\zeta,\overline{\beta}^{-1})}.$$

As a conformal mapping it takes $\zeta = \alpha$ to the origin and $\zeta = \beta$ to the point at infinity. It also maps all the circles $\{C_j | j = 0, 1, \ldots, M\}$ to radial slits.

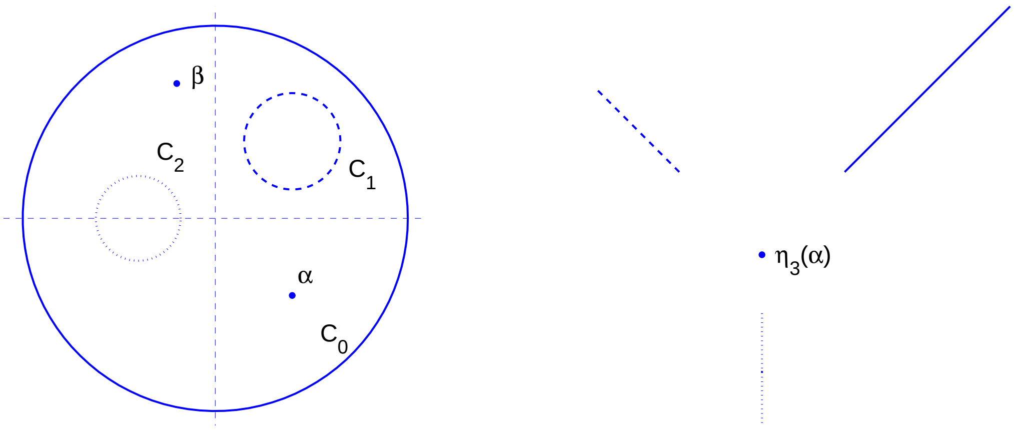

FIGURE 4. Radial slit domain. C_0, C_1 and C_2 all map to radial slits of finite length. α maps to the origin, β maps to infinity.

There are similarly convenient expressions (in terms of the S-K prime function) for the so-called parallel slit domain and the domain consisting of a concentric annulus with enclosed concentric circular arc slits (details are omitted but can be found in Crowdy & Marshall [12]).

8. Schwarz-Christoffel formula

It is interesting that all three of the conformal slit mappings in Sections 7.1–7.3 can be written so compactly in terms of the S-K prime function. All three will now be used to give a geometrical interpretation of the key steps in the construction of a multiply connected Schwarz-Christoffel formula derived by Crowdy [7]. This geometrical interpretation is implicit in the derivation of [7] but here we render it more explicit by emphasizing that the so-called "building-block functions" of that paper are none other than the conformal slit maps discussed in Section 7. The monograph by Driscoll & Trefethen [20] gives a presentation of the theory of Schwarz-Christoffel mappings (in the simply and doubly connected situations) from a geometrical viewpoint (see also DeLillo, Elcrat & Pfaltzgraff [18] for a discussion of the doubly connected case). The conformal slit maps are themselves, of course, examples of multiply connected Schwarz-Christoffel maps but the polygons involved, with boundaries consisting just of finite length slits, are degenerate. The construction that follows shows how to take these mappings to degenerate polygons and build mappings to more elaborate ones. Indeed, the final result will be a formula for the most general mapping to a bounded polygon of any (finite) connectivity and with any (finite) number of sides.

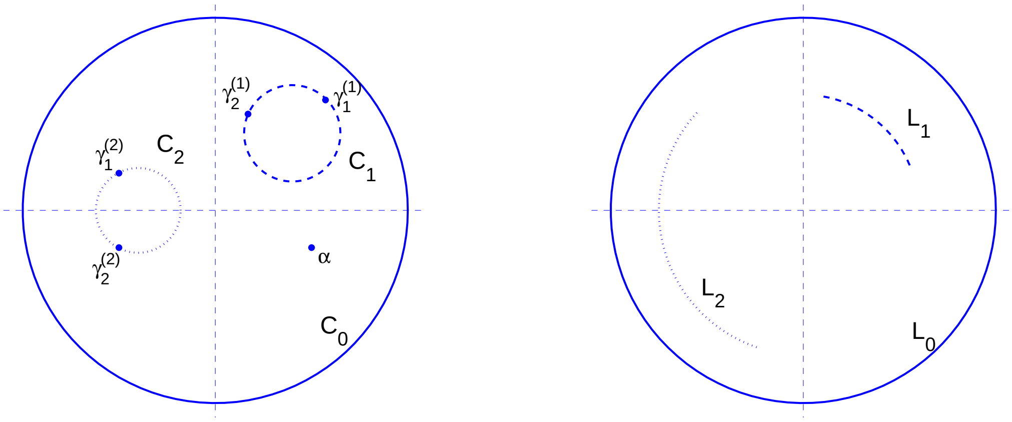

FIGURE 5. Conformal mapping from the circular domain D_ζ to the circular-slit domain D_η. The case shown is a triply connected domain. The points $\gamma_1^{(1)}$ and $\gamma_2^{(1)}$ on C_1 map to the two ends of the circular arc L_1; $\gamma_1^{(2)}$ and $\gamma_2^{(2)}$ on C_2 map to the two ends of the circular arc L_2.

The objective is to find the functional form, up to a finite set of accessory parameters, for the conformal map $z(\zeta)$ from a multiply connected circular domain D_ζ to a given bounded $(M+1)$-connected polygonal region P. Let the outer boundary be denoted P_0, let it have turning angles [20] $\{\pi\beta_j^{(0)}|j=1,\ldots,n_0\}$ and let the prevertices on C_0 be at $\{a_j^{(0)}|j=1,\ldots,n_0\}$ (a prevertex is the name given to the preimage point that maps to a given vertex [20]). A necessary condition that the polygon is closed is that

$$\sum_{j=1}^{n_0}\beta_j^{(0)}=-2. \tag{19}$$

The M internal polygonal boundaries are denoted $\{P_k|k=1,\ldots,M\}$. Suppose that P_k has n_k edges with turning angles $\{\pi\beta_j^{(k)}|j=1,\ldots,n_k\}$ then a necessary condition is that

$$\sum_{j=1}^{n_k}\beta_j^{(k)}=2. \tag{20}$$

Let the vertices on P_k be at $\{z_j^{(k)}|j=1,\ldots,n_k\}$ and let the prevertices on C_k be at $\{a_j^{(k)}|j=1,\ldots,n_k\}$. Then, following the usual arguments for Schwarz-Christoffel mappings [20], in order that $z(\zeta)$ has the correct branch point behaviour it is necessary that, for ζ near $a_j^{(k)}$,

$$\frac{dz}{d\zeta} = \left(\zeta - a_j^{(k)}\right)^{\beta_j^{(k)}} h_j^{(k)}(\zeta),$$

where $h_j^{(k)}(\zeta)$ is some function that is analytic at $a_j^{(k)}$.

For clarity, we present details of the geometrical construction in the triply connected case $M = 2$. It will be clear how the ideas are extendible to any connectivity. The role of the three slit maps discussed in Sections 7.1–7.3 will be made clear.

8.1. Role of circular slit mapping. First, it is expedient to make use of the circular slit mapping given in Section 7.1. Suppose such a mapping takes D_ζ to a circular slit domain in a complex η-plane and denote the mapping by $\eta(\zeta)$ with inverse function $\zeta(\eta)$. Some chosen point α in D_ζ will map to $\eta = 0$. Let the image of C_k be labelled L_k for $k = 0, 1, \ldots, M$. There will be two points on each circle C_k (for $k = 1, \ldots, M$) mapping to the endpoints of the corresponding circular arc slit L_k. Let these be denoted $\gamma_1^{(k)}$ and $\gamma_2^{(k)}$ (for $k = 1, \ldots, M$). Figure 5 shows an example for $M = 2$ with the preimage points clearly indicated.

Now introduce the pullback of the conformal mapping $z(\zeta)$ to the η-plane given by

$$\mathcal{Z}(\eta) = z(\zeta(\eta)).$$

As a conformal mapping, $\mathcal{Z}(\eta)$ takes the circular-slits $\{L_k | k = 0, 1, \ldots, M\}$ in the η-plane to the boundaries $\{P_k | k = 0, 1, \ldots, M\}$ of the polygon in the z-plane. Furthermore, because the chain rule implies

$$\frac{d\mathcal{Z}}{d\eta} = \frac{dz/d\zeta}{d\eta/d\zeta}, \tag{21}$$

$d\mathcal{Z}/d\eta$ inherits branch points at the images under the mapping $\eta(\zeta)$ of all the prevertices on the boundary of D_ζ. Note, however, that it *also* has two first-order poles at $\gamma_1^{(k)}$ and $\gamma_2^{(k)}$ on C_k for each $k = 1, \ldots, M$ (we assume throughout that none of these poles coincides with any of the branch points — this can always be arranged by a different choice of α).

Now the complex tangent on the k-th boundary P_k of the polygon is

$$\frac{dz}{ds} = \pm i \frac{\eta \mathcal{Z}'(\eta)}{r_k |\mathcal{Z}'(\eta)|},$$

where $\mathcal{Z}'(\eta) = d\mathcal{Z}/d\eta$, r_k denotes the radius of L_k and s denotes arclength. Clearly, since the argument of the tangent vector is piecewise constant on the boundaries of the polygon we require the argument of the function

$$\eta \mathcal{Z}'(\eta) \tag{22}$$

to be piecewise constant on all the boundaries of the polygon P.

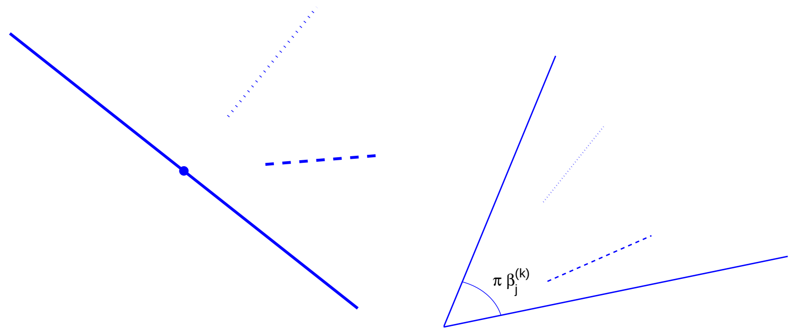

FIGURE 6. Geometrical effect of the functions (23) and (24). To the left is the image of D_ζ as shown in Figure 5 under the mapping (23). To the right is its image under the mapping (24) and shows that a corner with the required turning angle has been introduced.

8.2. Role of half-plane slit mapping. In order to construct the multiply connected Schwarz-Christoffel formula we need to be able to add "corners" to the polygon — and hence branch points of $dz/d\zeta$ — while ensuring that the quantity (22) continues to have piecewise constant argument on all boundaries. To do this we pick one of the prevertices $a_j^{(k)}$ and consider the half-plane slit map (described in Section 7.2) given by

$$\frac{\omega(\zeta, a_j^{(k)})}{\omega(\zeta, \gamma^{(k)})}, \tag{23}$$

where $\gamma^{(k)}$ is an arbitrarily chosen point on C_k. This maps D_ζ to the region shown in the left-most diagram in Figure 6 and, in particular, maps the prevertex $a_j^{(k)}$ to the origin in this plane. Since *all* boundary circles map to radial lines the function

$$\left(\frac{\omega(\zeta, a_j^{(k)})}{\omega(\zeta, \gamma^{(k)})}\right)^{\beta_j^{(k)}} \tag{24}$$

introduces precisely the required branch point behaviour at the prevertex $a_j^{(k)}$ but retains the property that its argument on all boundaries of D_ζ is constant. The image of D_ζ under the mapping (24) is shown in the right-most diagram in Figure 6 and shows that a corner of the appropriate turning angle has been introduced at the origin. It is therefore natural to consider the product

$$\prod_{k=0}^{M}\prod_{j=1}^{n_k}\left(\frac{\omega(\zeta,a_j^{(k)})}{\omega(\zeta,\gamma^{(k)})}\right)^{\beta_j^{(k)}}. \tag{25}$$

This function, by construction, has *all* the required branch points together with the property that its argument is piecewise constant on the boundaries of D_ζ.

Since the boundaries of D_ζ correspond to the boundaries of D_η we can contemplate equating some multiple of the specially constructed product (25) with the required function (22). This cannot be done directly because, owing to condition (20), the product (25) has M unwanted second order poles at the points $\{\gamma^{(k)}|k=1,\dots,M\}$. We also know that $\eta d\mathcal{Z}/d\eta$ has simple poles at $\gamma_1^{(k)}$ and $\gamma_2^{(k)}$ on C_k for each $k=1,\dots,M$ in contrast to the product (25). We can adjust the positions of the poles of (25) by multiplying it by the product of half-plane slit mappings (from Section 7.2) given by

$$\prod_{k=1}^{M}\left(\frac{\omega(\zeta,\gamma^{(k)})}{\omega(\zeta,\gamma_1^{(k)})}\right)\left(\frac{\omega(\zeta,\gamma^{(k)})}{\omega(\zeta,\gamma_2^{(k)})}\right).$$

This multiplicative operation does not affect the piecewise constancy of the product (25) on any of the boundaries of D_ζ. It removes the unwanted second order poles of (25) at $\{\gamma^{(k)}|k=1,\dots,M\}$ and replaces them with simple poles at $\{\gamma_1^{(k)},\gamma_2^{(k)}|k=1,\dots,M\}$. The product is now given by

$$\left(\frac{\omega(\zeta,\gamma^{(0)})^2}{\prod_{k=1}^{M}\omega(\zeta,\gamma_1^{(k)})\omega(\zeta,\gamma_2^{(k)})}\right)\prod_{k=0}^{M}\prod_{j=1}^{n_k}\omega(\zeta,a_j^{(k)})^{\beta_j^{(k)}}. \tag{26}$$

8.3. Role of radial slit mapping. Since the circular slit map $\eta(\zeta)$ vanishes at $\zeta=\alpha$, $\eta d\mathcal{Z}/d\eta$ also has a simple zero at the point $\zeta=\alpha$ in contrast to the product (26). Moreover, owing to the condition (20), the product (26) has an unwanted second order zero at the arbitrarily chosen point $\gamma^{(0)}$. These remaining problems can be fixed by multiplying (26) by the radial slit mapping (as described in Section 7.3)

$$\frac{\omega(\zeta,\alpha)\ \omega(\zeta,\overline{\alpha}^{-1})}{\omega(\zeta,\gamma^{(0)})\omega(\zeta,\gamma^{(0)})}.$$

This multiplicative operation does not affect the piecewise constancy of the product (26) on any of the boundaries of D_ζ. It also removes the unwanted second-order zero at $\gamma^{(0)}$ and replaces it with simple zeros at α and $1/\overline{\alpha}$.

8.4. The final formula. Putting all these together gives us a candidate function having all the properties required of the function $\eta\mathcal{Z}'(\eta)$. Some additional arguments (based on Liouville's theorem or using Riemann-Hilbert methods) can be used to show that $\eta\mathcal{Z}'(\eta)$ is, in fact, proportional to this candidate function.

Finally, use of (21) together with the explicit expression (18) for the circular slit map $\eta(\zeta)$ can be combined to deduce that the Schwarz-Christoffel mapping from a bounded $(M+1)$-connected circular region D_ζ to a bounded $(M+1)$-connected polygonal region is

$$z(\zeta) = A + B \int^{\zeta} S_M(\zeta') \prod_{j=1}^{n_0} \left[\omega(\zeta', a_j^{(0)})\right]^{\beta_j^{(0)}} \prod_{k=1}^{M} \prod_{j=1}^{n_k} \left[\omega(\zeta', a_j^{(k)})\right]^{\beta_j^{(k)}} d\zeta' \tag{27}$$

where $S_0(\zeta) = 1$ (the simply connected case), $S_1(\zeta) = 1/\zeta^2$ (the doubly connected case) and, for all $M \geq 2$,

$$S_M(\zeta) = \frac{\omega_\zeta(\zeta, \alpha)\omega(\zeta, \overline{\alpha}^{-1}) - \omega_\zeta(\zeta, \overline{\alpha}^{-1})\omega(\zeta, \alpha)}{\prod_{k=1}^{M} \omega(\zeta, \gamma_1^{(k)})\omega(\zeta, \gamma_2^{(k)})} \tag{28}$$

where $\omega_\zeta(\cdot, \cdot)$ denotes the derivative of the prime function with respect to its first argument. Until recently, only the formulae for $M = 0$ and $M = 1$ were known [20]. (27) incorporates these as special cases. From its representation (28) the function $S_M(\zeta)$ appears to depend on the choice of the (arbitrary) point α. However a demonstration that it is, in fact, independent of α is given in Crowdy [9].

The construction just described is readily extendible to the case of mappings to unbounded polygonal domains [8]. In this case, DeLillo *et al.* [17] have given an alternative formula for the Schwarz-Christoffel mapping (not expressed in terms of the S-K prime function) using reflection arguments based on consideration of a pre-Schwarzian function. DeLillo [15] has shown how to relate the latter approach to the function theoretic approach of Crowdy [7] based on use of the S-K prime function. Other ideas on how to compute the accessory parameters appearing in these formulae have been discussed by DeLillo *et al.* [16] but much further work on the numerical implementation of the formula remains to be done.

9. Discussion

The Schottky-Klein prime function on the Schottky double of planar domains provides an important linchpin in the function theory associated with planar multiply connected domains. It has many applications in this area and it is only recently that many of these have been identified. With new numerical techniques for its evaluation [13] (and downloadable MATLAB files available at [10]) it is hoped that future workers will incorporate it as an important theoretical and computational tool in their investigations of problems in multiply connected domains.

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