Semi-Intuitionistic Logic with Strong Negation

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Abstract

Motivated by the definition of semi-Nelson algebras, a propositional calculus called semi-intuitionistic logic with strong negation is introduced and proved to be complete with respect to that class of algebras. An axiomatic extension is proved to have as algebraic semantics the class of Nelson algebras.

Nelson and semi-Nelson algebras

We present the algebraic background that motivated our definition of semi-intuitionistic logic with strong negation.

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Definition

[2] Nelson algebras are algebras $\mathbf{A} = \langle A; \wedge, \vee, \to, \sim, \top \rangle$ *that satisfy the conditions:*

(N1) $x \wedge (x \vee y) \approx x$,

(N2) $x \wedge (y \vee z) \approx (z \wedge x) \vee (y \wedge x)$,

(N3) $\sim\sim x \approx x$,

(N4) $\sim (x \wedge y) \approx \sim x \vee \sim y$,

(N5) $x \wedge \sim x \approx (x \wedge \sim x) \wedge (y \vee \sim y)$,

(N6) $x \to (y \to z) \approx (x \wedge y) \to z$,

(N7) $x \wedge (x \to y) \approx x \wedge (\sim x \vee y)$.

(N8) $x \to x \approx \top$,

[2] António Monteiro and Luiz Monteiro. *Axiomes indépendants pour les algèbres de Nelson, de łukasiewicz trivalentes, de De Morgan et de Kleene*. In Unpublished papers, I, volume 40 of *Notas Lógica Mat.*, page 13. UNS, 1996.

Nelson and semi-Nelson algebras

Semi-Heyting algebras are a generalization of Heyting algebras introduced by H. Sankappanavar in [4], and they share with Heyting algebras the properties of being pseudocomplemented, distributive and having their congruences determined by filters.

[4] HANAMANTAGOUDA P. SANKAPPANAVAR. *Semi-Heyting algebras: an abstraction from Heyting algebras*. In *Proceedings of the 9th "Dr. Antonio A. R. Monteiro" Congress (Spanish)*, Actas Congr. "Dr. Antonio A. R. Monteiro", pages 33–66, Bahía Blanca, 2008. Univ. Nac. del Sur.

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Definition

An algebra $\mathbf{A} = \langle A; \wedge, \vee, \rightarrow, \bot, \top \rangle$ *is said to be a semi-Heyting algebra if* $\langle A; \wedge, \vee, \bot, \top \rangle$ *is a bounded lattice, and it satisfies the identities:* $x \wedge (x \rightarrow y) \approx x \wedge y$, $x \wedge (y \rightarrow z) \approx x \wedge ((x \wedge y) \rightarrow (x \wedge z))$, *and* $x \rightarrow x \approx \top$.

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[4] Hanamantagouda P. Sankappanavar. *Semi-Heyting algebras: an abstraction from Heyting algebras*. In *Proceedings of the 9th "Dr. Antonio A. R. Monteiro" Congress (Spanish)*, Actas Congr. "Dr. Antonio A. R. Monteiro", pages 33–66, Bahía Blanca, 2008. Univ. Nac. del Sur.

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To define Heyting algebras, all we need to do is to replace the last identity by $(x \wedge y) \rightarrow y \approx \top$.

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[4] Hanamantagouda P. Sankappanavar. *Semi-Heyting algebras: an abstraction from Heyting algebras*. In *Proceedings of the 9th "Dr. Antonio A. R. Monteiro" Congress (Spanish)*, Actas Congr. "Dr. Antonio A. R. Monteiro", pages 33–66, Bahía Blanca, 2008. Univ. Nac. del Sur.

Nelson and semi-Nelson algebras

In [5], D. Vakarelov presented a construction of Nelson algebras from Heyting algebras.

[5] D. Vakarelov. *Notes on $\mathcal{N}$-lattices and constructive logic with strong negation*. Studia Logica, 36(1–2):109–125, 1977.

Nelson and semi-Nelson algebras

In [5], D. Vakarelov presented a construction of Nelson algebras from Heyting algebras. This construction has proven fruitful for the study of Nelson algebras, and in [1] we applied it to semi-Heyting algebras, which motivated the definition of semi-Nelson algebras.

[1] Juan M. Cornejo and Ignacio D. Viglizzo. *Semi-Nelson algebras*. Order, 2016.

[5] D. Vakarelov. *Notes on $\mathcal{N}$-lattices and constructive logic with strong negation*. Studia Logica, 36(1–2):109–125, 1977.

Nelson and semi-Nelson algebras

Definition

An algebra $\mathbf{A} = \langle A; \wedge, \vee, \to, \sim, \top \rangle$ *of type* $(2, 2, 2, 1, 0)$ *is a semi-Nelson algebra if the following conditions are satisfied:*

(SN1) $x \wedge (x \vee y) \approx x,$

(SN2) $x \wedge (y \vee z) \approx (z \wedge x) \vee (y \wedge x),$

(SN3) $\sim\sim x \approx x,$

(SN4) $\sim (x \wedge y) \approx \sim x \vee \sim y,$

(SN5) $x \wedge \sim x \approx (x \wedge \sim x) \wedge (y \vee \sim y),$

(SN6) $x \wedge (x \to_N y) \approx x \wedge (\sim x \vee y),$

(SN7) $x \to_N (y \to_N z) \approx (x \wedge y) \to_N z,$

(SN8) $(x \to_N y) \to_N [(y \to_N x) \to_N [(x \to z) \to_N (y \to z)]] \approx \top,$

(SN9) $(x \to_N y) \to_N [(y \to_N x) \to_N [(z \to x) \to_N (z \to y)]] \approx \top,$

(SN10) $(\sim (x \to y)) \to_N (x \wedge \sim y) \approx \top,$

(SN11) $(x \wedge \sim y) \to_N (\sim (x \to y)) \approx \top,$

where $x \to_N y$ *stands for the term* $x \to (x \wedge y)$.

Nelson and semi-Nelson algebras

The term $x \to_N y = x \to (x \wedge y)$ is quite important for the study of semi-Nelson algebras.

Nelson and semi-Nelson algebras

The term $x \rightarrow_N y = x \rightarrow (x \wedge y)$ is quite important for the study of semi-Nelson algebras. There can be many semi-Nelson implication operations defined over the same underlying distributive lattice, but all of them will yield the same implication when the term defining $\rightarrow_N$ is calculated. Furthermore, the algebra with this implication turns out to be the only Nelson algebra definable over the lattice.

Nelson and semi-Nelson algebras

The term $x \to_N y = x \to (x \wedge y)$ is quite important for the study of semi-Nelson algebras. There can be many semi-Nelson implication operations defined over the same underlying distributive lattice, but all of them will yield the same implication when the term defining $\to_N$ is calculated. Furthermore, the algebra with this implication turns out to be the only Nelson algebra definable over the lattice. In fact, we have the following theorem:

Theorem

[1, Theorem 2.8] The variety **N** *of Nelson algebras is the subvariety of* **SN** *defined by the identity* $x \to y \approx x \to_N y$.

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[1] Juan M. Cornejo and Ignacio D. Viglizzo. *Semi-Nelson algebras*. Order, 2016.

Semi-intuitionistic logic with strong negation

We define *semi Intuitionistic logic with strong negation* $\mathcal{SN}$ over the language $\mathbf{L} = \{\top, \sim, \wedge, \vee, \rightarrow\}$ in terms of the following set of axiom schemata, in which we use the following definitions:

- $\alpha \rightarrow_N \beta := \alpha \rightarrow (\alpha \wedge \beta)$,
- $\alpha \Rightarrow \beta := (\alpha \rightarrow_N \beta) \wedge (\sim \beta \rightarrow_N \sim \alpha)$.

Semi-intuitionistic logic with strong negation

(A1) $(\alpha \to_N \beta) \to_N ((\beta \to_N \gamma) \to_N (\alpha \to_N \gamma))$,
(A2) $(\alpha \to_N \beta) \to_N ((\alpha \to_N \gamma) \to_N (\alpha \to_N (\beta \wedge \gamma)))$,
(A3) $(\alpha \wedge \beta) \to_N \alpha$,
(A4) $(\alpha \wedge \beta) \to_N \beta$,
(A5) $\alpha \to_N (\alpha \vee \beta)$,
(A6) $\beta \to_N (\alpha \vee \beta)$,
(A7) $\sim (\alpha \vee \beta) \to_N \sim \alpha$,
(A8) $\sim (\alpha \vee \beta) \to_N \sim \beta$,
(A9) $(\alpha \to_N \gamma) \to_N ((\beta \to_N \gamma) \to_N ((\alpha \vee \beta) \to_N \gamma))$,
(A10) $(\sim \alpha \to_N \sim \beta) \to_N ((\sim \alpha \to_N \sim \gamma) \to_N (\sim \alpha \to_N \sim (\beta \vee \gamma)))$,
(A11) $\alpha \Rightarrow (\sim\sim \alpha)$,
(A12) $(\sim\sim \alpha) \Rightarrow \alpha$,
(A13) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\alpha \to \gamma) \to_N (\beta \to \gamma)]]$,
(A14) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\gamma \to \alpha) \to_N (\gamma \to \beta)]]$,
(A15) $[(\alpha \wedge \beta) \to_N \gamma] \Rightarrow [\alpha \to_N (\beta \to_N \gamma)]$,
(A16) $(\sim (\alpha \wedge \beta)) \Rightarrow (\sim \alpha \vee \sim \beta)$,
(A17) $(\sim \alpha \vee \sim \beta) \Rightarrow (\sim (\alpha \wedge \beta))$,
(A18) $(\alpha \wedge (\sim \alpha \vee \beta)) \Rightarrow (\alpha \wedge (\alpha \to_N \beta))$,
(A19) $(\alpha \to_N (\beta \to_N \gamma)) \Rightarrow ((\alpha \wedge \beta) \to_N \gamma)$,
(A20) $(\sim (\alpha \to \beta)) \to_N (\alpha \wedge \sim \beta)$,
(A21) $(\alpha \wedge \sim \beta) \to_N (\sim (\alpha \to \beta))$,
(A22) $[\sim (\alpha \wedge ((\gamma \wedge \alpha) \vee (\beta \wedge \alpha)))] \to_N [\sim (\alpha \wedge (\beta \vee \gamma))]$,
(A23) $\top$.

Semi-intuitionistic logic with strong negation

The only inference rule is Modus Ponens for the implication $\to_N$, which we denominate $\mathcal{N}$*-Modus Ponens* ($\mathcal{N}$-MP): $\Gamma \vdash_{\mathcal{SN}} \phi$ and $\Gamma \vdash_{\mathcal{SN}} \phi \to_N \gamma$ yield $\Gamma \vdash_{\mathcal{SN}} \gamma$.

Why these axioms?

(A1) $(\alpha \to_N \beta) \to_N ((\beta \to_N \gamma) \to_N (\alpha \to_N \gamma))$,

(A2) $(\alpha \to_N \beta) \to_N ((\alpha \to_N \gamma) \to_N (\alpha \to_N (\beta \wedge \gamma)))$,

(A3) $(\alpha \wedge \beta) \to_N \alpha$,

(A4) $(\alpha \wedge \beta) \to_N \beta$,

(A5) $\alpha \to_N (\alpha \vee \beta)$,

(A6) $\beta \to_N (\alpha \vee \beta)$,

(A7) $\sim (\alpha \vee \beta) \to_N \sim \alpha$,

(A8) $\sim (\alpha \vee \beta) \to_N \sim \beta$,

(A9) $(\alpha \to_N \gamma) \to_N ((\beta \to_N \gamma) \to_N ((\alpha \vee \beta) \to_N \gamma))$,

(A10) $(\sim \alpha \to_N \sim \beta) \to_N ((\sim \alpha \to_N \sim \gamma) \to_N (\sim \alpha \to_N \sim (\beta \vee \gamma)))$,

(A11) $\alpha \Rightarrow (\sim\sim \alpha)$,

(A12) $(\sim\sim \alpha) \Rightarrow \alpha$,

(A13) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\alpha \to \gamma) \to_N (\beta \to \gamma)]]$,

(A14) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\gamma \to \alpha) \to_N (\gamma \to \beta)]]$,

(A15) $[(\alpha \wedge \beta) \to_N \gamma] \Rightarrow [\alpha \to_N (\beta \to_N \gamma)]$,

(A16) $(\sim (\alpha \wedge \beta)) \Rightarrow (\sim \alpha \vee \sim \beta)$,

(A17) $(\sim \alpha \vee \sim \beta) \Rightarrow (\sim (\alpha \wedge \beta))$,

(A18) $(\alpha \wedge (\sim \alpha \vee \beta)) \Rightarrow (\alpha \wedge (\alpha \to_N \beta))$,

(A19) $(\alpha \to_N (\beta \to_N \gamma)) \Rightarrow ((\alpha \wedge \beta) \to_N \gamma)$,

(A20) $(\sim (\alpha \to \beta)) \to_N (\alpha \wedge \sim \beta)$,

(A21) $(\alpha \wedge \sim \beta) \to_N (\sim (\alpha \to \beta))$,

(A22) $[\sim (\alpha \wedge ((\gamma \wedge \alpha) \vee (\beta \wedge \alpha)))] \to_N [\sim (\alpha \wedge (\beta \vee \gamma))]$,

(A23) $\top$.

These axioms are used to prove that **SN** is an implicative logic in the sense of [3] (definition forthcoming).

[3] HELENA RASIOWA. *An algebraic approach to non-classical logics*.

Why these axioms?

(A1) $(\alpha \to_N \beta) \to_N ((\beta \to_N \gamma) \to_N (\alpha \to_N \gamma))$,

(A2) $(\alpha \to_N \beta) \to_N ((\alpha \to_N \gamma) \to_N (\alpha \to_N (\beta \wedge \gamma)))$,

(A3) $(\alpha \wedge \beta) \to_N \alpha$,

(A4) $(\alpha \wedge \beta) \to_N \beta$,

(A5) $\alpha \to_N (\alpha \vee \beta)$,

(A6) $\beta \to_N (\alpha \vee \beta)$,

(A7) $\sim (\alpha \vee \beta) \to_N \sim \alpha$,

(A8) $\sim (\alpha \vee \beta) \to_N \sim \beta$,

(A9) $(\alpha \to_N \gamma) \to_N ((\beta \to_N \gamma) \to_N ((\alpha \vee \beta) \to_N \gamma))$,

(A10) $(\sim \alpha \to_N \sim \beta) \to_N ((\sim \alpha \to_N \sim \gamma) \to_N (\sim \alpha \to_N \sim (\beta \vee \gamma)))$,

(A11) $\alpha \Rightarrow (\sim\sim \alpha)$,

(A12) $(\sim\sim \alpha) \Rightarrow \alpha$,

(A13) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\alpha \to \gamma) \to_N (\beta \to \gamma)]]$,

(A14) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\gamma \to \alpha) \to_N (\gamma \to \beta)]]$,

(A15) $[(\alpha \wedge \beta) \to_N \gamma] \Rightarrow [\alpha \to_N (\beta \to_N \gamma)]$,

(A16) $(\sim (\alpha \wedge \beta)) \Rightarrow (\sim \alpha \vee \sim \beta)$,

(A17) $(\sim \alpha \vee \sim \beta) \Rightarrow (\sim (\alpha \wedge \beta))$,

(A18) $(\alpha \wedge (\sim \alpha \vee \beta)) \Rightarrow (\alpha \wedge (\alpha \to_N \beta))$,

(A19) $(\alpha \to_N (\beta \to_N \gamma)) \Rightarrow ((\alpha \wedge \beta) \to_N \gamma)$,

(A20) $(\sim (\alpha \to \beta)) \to_N (\alpha \wedge \sim \beta)$,

(A21) $(\alpha \wedge \sim \beta) \to_N (\sim (\alpha \to \beta))$,

(A22) $[\sim (\alpha \wedge ((\gamma \wedge \alpha) \vee (\beta \wedge \alpha)))] \to_N [\sim (\alpha \wedge (\beta \vee \gamma))]$,

(A23) $\top$.

To prove the completeness of **SN** with respect to the variety of semi-Nelson algebras we use these axioms to verify:
(SN1) and (SN2), this is, that the class is conformed by bounded distributive lattices.

Why these axioms?

(A1) $(\alpha \to_N \beta) \to_N ((\beta \to_N \gamma) \to_N (\alpha \to_N \gamma))$,
(A2) $(\alpha \to_N \beta) \to_N ((\alpha \to_N \gamma) \to_N (\alpha \to_N (\beta \wedge \gamma)))$,
(A3) $(\alpha \wedge \beta) \to_N \alpha$,
(A4) $(\alpha \wedge \beta) \to_N \beta$,
(A5) $\alpha \to_N (\alpha \vee \beta)$,
(A6) $\beta \to_N (\alpha \vee \beta)$,
(A7) $\sim (\alpha \vee \beta) \to_N \sim \alpha$,
(A8) $\sim (\alpha \vee \beta) \to_N \sim \beta$,
(A9) $(\alpha \to_N \gamma) \to_N ((\beta \to_N \gamma) \to_N ((\alpha \vee \beta) \to_N \gamma))$,
(A10) $(\sim \alpha \to_N \sim \beta) \to_N ((\sim \alpha \to_N \sim \gamma) \to_N (\sim \alpha \to_N \sim (\beta \vee \gamma)))$,
(A11) $\alpha \Rightarrow (\sim\sim \alpha)$,
(A12) $(\sim\sim \alpha) \Rightarrow \alpha$,
(A13) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\alpha \to \gamma) \to_N (\beta \to \gamma)]]$,
(A14) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\gamma \to \alpha) \to_N (\gamma \to \beta)]]$,
(A15) $[(\alpha \wedge \beta) \to_N \gamma] \Rightarrow [\alpha \to_N (\beta \to_N \gamma)]$,
(A16) $(\sim (\alpha \wedge \beta)) \Rightarrow (\sim \alpha \vee \sim \beta)$,
(A17) $(\sim \alpha \vee \sim \beta) \Rightarrow (\sim (\alpha \wedge \beta))$,
(A18) $(\alpha \wedge (\sim \alpha \vee \beta)) \Rightarrow (\alpha \wedge (\alpha \to_N \beta))$,
(A19) $(\alpha \to_N (\beta \to_N \gamma)) \Rightarrow ((\alpha \wedge \beta) \to_N \gamma)$,
(A20) $(\sim (\alpha \to \beta)) \to_N (\alpha \wedge \sim \beta)$,
(A21) $(\alpha \wedge \sim \beta) \to_N (\sim (\alpha \to \beta))$,
(A22) $[\sim (\alpha \wedge ((\gamma \wedge \alpha) \vee (\beta \wedge \alpha)))] \to_N [\sim (\alpha \wedge (\beta \vee \gamma))]$,
(A23) $\top$.

To prove the completeness of **SN** with respect to the variety of semi-Nelson algebras we use these axioms to verify (SN3).

Why these axioms?

(A1) $(\alpha \to_N \beta) \to_N ((\beta \to_N \gamma) \to_N (\alpha \to_N \gamma))$,
(A2) $(\alpha \to_N \beta) \to_N ((\alpha \to_N \gamma) \to_N (\alpha \to_N (\beta \wedge \gamma)))$,
(A3) $(\alpha \wedge \beta) \to_N \alpha$,
(A4) $(\alpha \wedge \beta) \to_N \beta$,
(A5) $\alpha \to_N (\alpha \vee \beta)$,
(A6) $\beta \to_N (\alpha \vee \beta)$,
(A7) $\sim (\alpha \vee \beta) \to_N \sim \alpha$,
(A8) $\sim (\alpha \vee \beta) \to_N \sim \beta$,
(A9) $(\alpha \to_N \gamma) \to_N ((\beta \to_N \gamma) \to_N ((\alpha \vee \beta) \to_N \gamma))$,
(A10) $(\sim \alpha \to_N \sim \beta) \to_N ((\sim \alpha \to_N \sim \gamma) \to_N (\sim \alpha \to_N \sim (\beta \vee \gamma)))$,
(A11) $\alpha \Rightarrow (\sim\sim \alpha)$,
(A12) $(\sim\sim \alpha) \Rightarrow \alpha$,
(A13) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\alpha \to \gamma) \to_N (\beta \to \gamma)]]$,
(A14) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\gamma \to \alpha) \to_N (\gamma \to \beta)]]$,
(A15) $[(\alpha \wedge \beta) \to_N \gamma] \Rightarrow [\alpha \to_N (\beta \to_N \gamma)]$,
(A16) $(\sim (\alpha \wedge \beta)) \Rightarrow (\sim \alpha \vee \sim \beta)$,
(A17) $(\sim \alpha \vee \sim \beta) \Rightarrow (\sim (\alpha \wedge \beta))$,
(A18) $(\alpha \wedge (\sim \alpha \vee \beta)) \Rightarrow (\alpha \wedge (\alpha \to_N \beta))$,
(A19) $(\alpha \to_N (\beta \to_N \gamma)) \Rightarrow ((\alpha \wedge \beta) \to_N \gamma)$,
(A20) $(\sim (\alpha \to \beta)) \to_N (\alpha \wedge \sim \beta)$,
(A21) $(\alpha \wedge \sim \beta) \to_N (\sim (\alpha \to \beta))$,
(A22) $[\sim (\alpha \wedge ((\gamma \wedge \alpha) \vee (\beta \wedge \alpha)))] \to_N [\sim (\alpha \wedge (\beta \vee \gamma))]$,
(A23) $\top$.

To prove the completeness of **SN** with respect to the variety of semi-Nelson algebras we use these axioms to verify (SN4).

Why these axioms?

(A1) $(\alpha \to_N \beta) \to_N ((\beta \to_N \gamma) \to_N (\alpha \to_N \gamma))$,

(A2) $(\alpha \to_N \beta) \to_N ((\alpha \to_N \gamma) \to_N (\alpha \to_N (\beta \wedge \gamma)))$,

(A3) $(\alpha \wedge \beta) \to_N \alpha$,

(A4) $(\alpha \wedge \beta) \to_N \beta$,

(A5) $\alpha \to_N (\alpha \vee \beta)$,

(A6) $\beta \to_N (\alpha \vee \beta)$,

(A7) $\sim (\alpha \vee \beta) \to_N \sim \alpha$,

(A8) $\sim (\alpha \vee \beta) \to_N \sim \beta$,

(A9) $(\alpha \to_N \gamma) \to_N ((\beta \to_N \gamma) \to_N ((\alpha \vee \beta) \to_N \gamma))$,

(A10) $(\sim \alpha \to_N \sim \beta) \to_N ((\sim \alpha \to_N \sim \gamma) \to_N (\sim \alpha \to_N \sim (\beta \vee \gamma)))$,

(A11) $\alpha \Rightarrow (\sim\sim \alpha)$,

(A12) $(\sim\sim \alpha) \Rightarrow \alpha$,

(A13) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\alpha \to \gamma) \to_N (\beta \to \gamma)]]$,

(A14) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\gamma \to \alpha) \to_N (\gamma \to \beta)]]$,

(A15) $[(\alpha \wedge \beta) \to_N \gamma] \Rightarrow [\alpha \to_N (\beta \to_N \gamma)]$,

(A16) $(\sim (\alpha \wedge \beta)) \Rightarrow (\sim \alpha \vee \sim \beta)$,

(A17) $(\sim \alpha \vee \sim \beta) \Rightarrow (\sim (\alpha \wedge \beta))$,

(A18) $(\alpha \wedge (\sim \alpha \vee \beta)) \Rightarrow (\alpha \wedge (\alpha \to_N \beta))$,

(A19) $(\alpha \to_N (\beta \to_N \gamma)) \Rightarrow ((\alpha \wedge \beta) \to_N \gamma)$,

(A20) $(\sim (\alpha \to \beta)) \to_N (\alpha \wedge \sim \beta)$,

(A21) $(\alpha \wedge \sim \beta) \to_N (\sim (\alpha \to \beta))$,

(A22) $[\sim (\alpha \wedge ((\gamma \wedge \alpha) \vee (\beta \wedge \alpha)))] \to_N [\sim (\alpha \wedge (\beta \vee \gamma))]$,

(A23) $\top$.

To prove the completeness of **SN** with respect to the variety of semi-Nelson algebras we use these axioms to verify (SN5).

Why these axioms?

(A1) $(\alpha \to_N \beta) \to_N ((\beta \to_N \gamma) \to_N (\alpha \to_N \gamma))$,
(A2) $(\alpha \to_N \beta) \to_N ((\alpha \to_N \gamma) \to_N (\alpha \to_N (\beta \wedge \gamma)))$,
(A3) $(\alpha \wedge \beta) \to_N \alpha$,
(A4) $(\alpha \wedge \beta) \to_N \beta$,
(A5) $\alpha \to_N (\alpha \vee \beta)$,
(A6) $\beta \to_N (\alpha \vee \beta)$,
(A7) $\sim (\alpha \vee \beta) \to_N \sim \alpha$,
(A8) $\sim (\alpha \vee \beta) \to_N \sim \beta$,
(A9) $(\alpha \to_N \gamma) \to_N ((\beta \to_N \gamma) \to_N ((\alpha \vee \beta) \to_N \gamma))$,
(A10) $(\sim \alpha \to_N \sim \beta) \to_N ((\sim \alpha \to_N \sim \gamma) \to_N (\sim \alpha \to_N \sim (\beta \vee \gamma)))$,
(A11) $\alpha \Rightarrow (\sim\sim \alpha)$,
(A12) $(\sim\sim \alpha) \Rightarrow \alpha$,
(A13) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\alpha \to \gamma) \to_N (\beta \to \gamma)]]$,
(A14) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\gamma \to \alpha) \to_N (\gamma \to \beta)]]$,
(A15) $[(\alpha \wedge \beta) \to_N \gamma] \Rightarrow [\alpha \to_N (\beta \to_N \gamma)]$,
(A16) $(\sim (\alpha \wedge \beta)) \Rightarrow (\sim \alpha \vee \sim \beta)$,
(A17) $(\sim \alpha \vee \sim \beta) \Rightarrow (\sim (\alpha \wedge \beta))$,
(A18) $(\alpha \wedge (\sim \alpha \vee \beta)) \Rightarrow (\alpha \wedge (\alpha \to_N \beta))$,
(A19) $(\alpha \to_N (\beta \to_N \gamma)) \Rightarrow ((\alpha \wedge \beta) \to_N \gamma)$,
(A20) $(\sim (\alpha \to \beta)) \to_N (\alpha \wedge \sim \beta)$,
(A21) $(\alpha \wedge \sim \beta) \to_N (\sim (\alpha \to \beta))$,
(A22) $[\sim (\alpha \wedge ((\gamma \wedge \alpha) \vee (\beta \wedge \alpha)))] \to_N [\sim (\alpha \wedge (\beta \vee \gamma))]$,
(A23) $\top$.

To prove the completeness of **SN** with respect to the variety of semi-Nelson algebras we use these axioms to verify:
(SN6).

Why these axioms?

(A1) $(\alpha \to_N \beta) \to_N ((\beta \to_N \gamma) \to_N (\alpha \to_N \gamma))$,

(A2) $(\alpha \to_N \beta) \to_N ((\alpha \to_N \gamma) \to_N (\alpha \to_N (\beta \wedge \gamma)))$,

(A3) $(\alpha \wedge \beta) \to_N \alpha$,

(A4) $(\alpha \wedge \beta) \to_N \beta$,

(A5) $\alpha \to_N (\alpha \vee \beta)$,

(A6) $\beta \to_N (\alpha \vee \beta)$,

(A7) $\sim (\alpha \vee \beta) \to_N \sim \alpha$,

(A8) $\sim (\alpha \vee \beta) \to_N \sim \beta$,

(A9) $(\alpha \to_N \gamma) \to_N ((\beta \to_N \gamma) \to_N ((\alpha \vee \beta) \to_N \gamma))$,

(A10) $(\sim \alpha \to_N \sim \beta) \to_N ((\sim \alpha \to_N \sim \gamma) \to_N (\sim \alpha \to_N \sim (\beta \vee \gamma)))$,

(A11) $\alpha \Rightarrow (\sim\sim \alpha)$,

(A12) $(\sim\sim \alpha) \Rightarrow \alpha$,

(A13) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\alpha \to \gamma) \to_N (\beta \to \gamma)]]$,

(A14) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\gamma \to \alpha) \to_N (\gamma \to \beta)]]$,

(A15) $[(\alpha \wedge \beta) \to_N \gamma] \Rightarrow [\alpha \to_N (\beta \to_N \gamma)]$,

(A16) $(\sim (\alpha \wedge \beta)) \Rightarrow (\sim \alpha \vee \sim \beta)$,

(A17) $(\sim \alpha \vee \sim \beta) \Rightarrow (\sim (\alpha \wedge \beta))$,

(A18) $(\alpha \wedge (\sim \alpha \vee \beta)) \Rightarrow (\alpha \wedge (\alpha \to_N \beta))$,

(A19) $(\alpha \to_N (\beta \to_N \gamma)) \Rightarrow ((\alpha \wedge \beta) \to_N \gamma)$,

(A20) $(\sim (\alpha \to \beta)) \to_N (\alpha \wedge \sim \beta)$,

(A21) $(\alpha \wedge \sim \beta) \to_N (\sim (\alpha \to \beta))$,

(A22) $[\sim (\alpha \wedge ((\gamma \wedge \alpha) \vee (\beta \wedge \alpha)))] \to_N [\sim (\alpha \wedge (\beta \vee \gamma))]$,

(A23) $\top$.

To prove the completeness of **SN** with respect to the variety of semi-Nelson algebras we use these axioms to verify:
(SN7).

Why these axioms?

(A1) $(\alpha \to_N \beta) \to_N ((\beta \to_N \gamma) \to_N (\alpha \to_N \gamma))$,
(A2) $(\alpha \to_N \beta) \to_N ((\alpha \to_N \gamma) \to_N (\alpha \to_N (\beta \wedge \gamma)))$,
(A3) $(\alpha \wedge \beta) \to_N \alpha$,
(A4) $(\alpha \wedge \beta) \to_N \beta$,
(A5) $\alpha \to_N (\alpha \vee \beta)$,
(A6) $\beta \to_N (\alpha \vee \beta)$,
(A7) $\sim (\alpha \vee \beta) \to_N \sim \alpha$,
(A8) $\sim (\alpha \vee \beta) \to_N \sim \beta$,
(A9) $(\alpha \to_N \gamma) \to_N ((\beta \to_N \gamma) \to_N ((\alpha \vee \beta) \to_N \gamma))$,
(A10) $(\sim \alpha \to_N \sim \beta) \to_N ((\sim \alpha \to_N \sim \gamma) \to_N (\sim \alpha \to_N \sim (\beta \vee \gamma)))$,
(A11) $\alpha \Rightarrow (\sim\sim \alpha)$,
(A12) $(\sim\sim \alpha) \Rightarrow \alpha$,
(A13) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\alpha \to \gamma) \to_N (\beta \to \gamma)]]$,
(A14) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\gamma \to \alpha) \to_N (\gamma \to \beta)]]$,
(A15) $[(\alpha \wedge \beta) \to_N \gamma] \Rightarrow [\alpha \to_N (\beta \to_N \gamma)]$,
(A16) $(\sim (\alpha \wedge \beta)) \Rightarrow (\sim \alpha \vee \sim \beta)$,
(A17) $(\sim \alpha \vee \sim \beta) \Rightarrow (\sim (\alpha \wedge \beta))$,
(A18) $(\alpha \wedge (\sim \alpha \vee \beta)) \Rightarrow (\alpha \wedge (\alpha \to_N \beta))$,
(A19) $(\alpha \to_N (\beta \to_N \gamma)) \Rightarrow ((\alpha \wedge \beta) \to_N \gamma)$,
(A20) $(\sim (\alpha \to \beta)) \to_N (\alpha \wedge \sim \beta)$,
(A21) $(\alpha \wedge \sim \beta) \to_N (\sim (\alpha \to \beta))$,
(A22) $[\sim (\alpha \wedge ((\gamma \wedge \alpha) \vee (\beta \wedge \alpha)))] \to_N [\sim (\alpha \wedge (\beta \vee \gamma))]$,
(A23) $\top$.

To prove the completeness of **SN** with respect to the variety of semi-Nelson algebras we use these axioms to verify:
(SN8) and (SN9), respectively.

Why these axioms?

(A1) $(\alpha \to_N \beta) \to_N ((\beta \to_N \gamma) \to_N (\alpha \to_N \gamma))$,
(A2) $(\alpha \to_N \beta) \to_N ((\alpha \to_N \gamma) \to_N (\alpha \to_N (\beta \wedge \gamma)))$,
(A3) $(\alpha \wedge \beta) \to_N \alpha$,
(A4) $(\alpha \wedge \beta) \to_N \beta$,
(A5) $\alpha \to_N (\alpha \vee \beta)$,
(A6) $\beta \to_N (\alpha \vee \beta)$,
(A7) $\sim (\alpha \vee \beta) \to_N \sim \alpha$,
(A8) $\sim (\alpha \vee \beta) \to_N \sim \beta$,
(A9) $(\alpha \to_N \gamma) \to_N ((\beta \to_N \gamma) \to_N ((\alpha \vee \beta) \to_N \gamma))$,
(A10) $(\sim \alpha \to_N \sim \beta) \to_N ((\sim \alpha \to_N \sim \gamma) \to_N (\sim \alpha \to_N \sim (\beta \vee \gamma)))$,
(A11) $\alpha \Rightarrow (\sim\sim \alpha)$,
(A12) $(\sim\sim \alpha) \Rightarrow \alpha$,
(A13) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\alpha \to \gamma) \to_N (\beta \to \gamma)]]$,
(A14) $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\gamma \to \alpha) \to_N (\gamma \to \beta)]]$,
(A15) $[(\alpha \wedge \beta) \to_N \gamma] \Rightarrow [\alpha \to_N (\beta \to_N \gamma)]$,
(A16) $(\sim (\alpha \wedge \beta)) \Rightarrow (\sim \alpha \vee \sim \beta)$,
(A17) $(\sim \alpha \vee \sim \beta) \Rightarrow (\sim (\alpha \wedge \beta))$,
(A18) $(\alpha \wedge (\sim \alpha \vee \beta)) \Rightarrow (\alpha \wedge (\alpha \to_N \beta))$,
(A19) $(\alpha \to_N (\beta \to_N \gamma)) \Rightarrow ((\alpha \wedge \beta) \to_N \gamma)$,
(A20) $(\sim (\alpha \to \beta)) \to_N (\alpha \wedge \sim \beta)$,
(A21) $(\alpha \wedge \sim \beta) \to_N (\sim (\alpha \to \beta))$,
(A22) $[\sim (\alpha \wedge ((\gamma \wedge \alpha) \vee (\beta \wedge \alpha)))] \to_N [\sim (\alpha \wedge (\beta \vee \gamma))]$,
(A23) $\top$.

To prove the completeness of **SN** with respect to the variety of semi-Nelson algebras we use these axioms to verify: (SN10) and (SN11), respectively.

Semi-intuitionistic logic with strong negation

Theorem

(Deduction Theorem) *Let* $\Gamma \cup \{\alpha, \beta\} \subseteq FmL$*. Then*

$$\Gamma \vdash \alpha \to_N \beta \text{ if and only if } \Gamma, \alpha \vdash \beta$$

Completeness

Definition

[3] Let $\mathcal{L}$ be a logic in a language with a binary connective $\to$, either primitive or defined by a term in exactly two variables. Then $\mathcal{L}$ is called an implicative logic with respect to the binary connective $\to$ if the following conditions are satisfied:

(IL1) $\vdash_{\mathcal{L}} \alpha \to \alpha$.

(IL2) $\alpha \to \beta, \beta \to \gamma \vdash_{\mathcal{L}} \alpha \to \gamma$.

(IL3) *For each connective f in the language of arity $n > 0$,*
$$\left\{ \begin{array}{l} \alpha_1 \to \beta_1, \ldots, \alpha_n \to \beta_n \\ \beta_1 \to \alpha_1, \ldots, \beta_n \to \alpha_n \end{array} \right\} \vdash_{\mathcal{L}} f(\alpha_1, \ldots, \alpha_n) \to f(\beta_1, \ldots, \beta_n).$$

(IL4) $\alpha, \alpha \to \beta \vdash_{\mathcal{L}} \beta$.

(IL5) $\alpha \vdash_{\mathcal{L}} \beta \to \alpha$.

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[3] Helena Rasiowa. *An algebraic approach to non-classical logics.* North-Holland Publishing Co., Amsterdam, 1974. Studies in Logic and the Foundations of Mathematics, Vol. 78.

Completeness

Theorem

$\mathcal{SN}$ *is implicative with respect to the connective* $\Rightarrow$.

Completeness

Theorem

$\mathcal{SN}$ is implicative with respect to the connective $\Rightarrow$.

Definition

[3, Definition 6, page 181] Let $\mathcal{L}$ be an implicative logic on the language **L**. *An $\mathcal{L}$-algebra is an algebra* **A** *of similarity type* **L** *that has an element $\top$ with the following properties:*

(LALG1) *For all $\Gamma \cup \{\phi\} \subseteq Fm\mathbf{L}$ and all $h \in Hom(Fm\mathbf{L}, \mathbf{A})$, if $\Gamma \vdash_{\mathcal{L}} \phi$ and $h\Gamma \subseteq \{\top\}$ then $h\phi = \top$, where $h\Gamma = \{h\gamma : \gamma \in \Gamma\}$.*

(LALG2) *For all $a, b \in A$, if $a \to b = \top$ and $b \to a = \top$ then $a = b$.*

The class of $\mathcal{L}$-algebras is denoted by $\mathrm{Alg}^*\mathcal{L}$.

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[3] Helena Rasiowa. *An algebraic approach to non-classical logics.* North-Holland Publishing Co., Amsterdam, 1974. Studies in Logic and the Foundations of Mathematics, Vol. 78.

Completeness

Since $\mathcal{SN}$ is an implicative logic with respect to the binary connective $\Rightarrow$, we have the next result using [3, Theorem 7.1, pag 222].

Theorem

The logic $\mathcal{SN}$ is complete with respect to the class $\mathrm{Alg}^*\mathcal{SN}$. *In other words, for all* $\Gamma \cup \{\phi\} \subseteq Fm\mathbf{L}$,

$$\Gamma \vdash_{\mathcal{SN}} \phi \text{ if and only if } h\Gamma \subseteq \{\top\} \text{ implies } h\phi = \top,$$

for all $h \in Hom(Fm\mathbf{L}, \mathbf{A})$ *and all* $\mathbf{A} \in \mathrm{Alg}^*\mathcal{SN}$.

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[3] Helena Rasiowa. *An algebraic approach to non-classical logics.* North-Holland Publishing Co., Amsterdam, 1974. Studies in Logic and the Foundations of Mathematics, Vol. 78.

Completeness

Theorem

$\mathrm{Alg}^*\mathcal{SN} = \mathbf{SN}$ *where* **SN** *is the variety of semi Nelson algebras.*

Corollary

The logic $\mathcal{SN}$ *is complete with respect to the class* **SN**.

Completeness

Since adding the identity $x \to y = x \to_N y$ to the definition of semi-Nelson algebras yields a characterization of Nelson algebras, we can carry this result to the logic $\mathcal{SN}$.

Completeness

Since adding the identity $x \rightarrow y = x \rightarrow_N y$ to the definition of semi-Nelson algebras yields a characterization of Nelson algebras, we can carry this result to the logic $\mathcal{SN}$.

Theorem

The logic $\mathcal{N}$, which is $\mathcal{SN}$ together with the axioms:

(A24) $(\alpha \rightarrow_N \beta) \rightarrow_N (\alpha \rightarrow \beta)$,

(A25) $\sim (\alpha \rightarrow_N \beta) \rightarrow_N \sim (\alpha \rightarrow \beta)$,

has the variety of Nelson algebras as its algebraic semantis.

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Thank you for your attention!