# The Vertices of the Platonic Solids are Tight Frames

Richard Vale, Shayne Waldron

**Abstract.** Here we give a simple proof that many highly symmetric configurations of vectors, such as the $n^{th}$ roots of unity in $\mathbb{R}^2$ and the vertices of the platonic solids in $\mathbb{R}^3$, are tight frames.

## §1. Introduction

Let $V = \mathbb{R}^d, \mathbb{C}^d$ be $d$–dimensional real or complex Euclidean space. Then

**Definition 1.1.** *A finite subset $\Phi = \{\phi_j\}_{j=1}^n$ of $V$ is a* tight frame *for $V$ if there is a $c > 0$ with*

$$\|f\|^2 = c \sum_{j=1}^n |\langle f, \phi_j \rangle|^2, \qquad \forall f \in V. \tag{1}$$

This is equivalent (by the polarisation identity) to a representation of the form

$$f = c \sum_{j=1}^n \langle f, \phi_j \rangle \phi_j, \qquad \forall f \in V. \tag{2}$$

In particular, $\Phi$ must be a spanning set for $V$.

By computing (1), it has been shown that many highly symmetric configurations of vectors in $\mathbb{R}^d$ such as the $n^{\text{th}}$ roots of unity in $\mathbb{R}^2$, the vertices of the platonic solids in $\mathbb{R}^3$, the vertices of the regular simplex, and vertices of the 'soccer ball' (cf [1]) are tight frames. Here we use group theory to give a unified way to show that all of these are tight frames. This approach does not require (1) to be computed (in some coordinate system), and explains precisely why such configurations are tight frames. There is a short discussion of the history of this result.

Advances in Constructive Approximation
XXX (eds.), pp. 1–7.


ISBN 0-9728482-x-x

## §2. The general theorem

The group $G$ generated by rotation through $\frac{2\pi}{3}$ in the plane maps any nonzero vector in $\mathbb{R}^2$ to 3 equally spaced vectors. For these to be tight frame they must span $\mathbb{R}^2$ (which clearly they do). This is in fact all that is required.

Let $\mathcal{U}(V)$ be the group of unitary transformations of $V$, i.e., the $d\times d$ real orthogonal matrices for $V$ real, and the $d \times d$ unitary matrices for $V$ complex. The group $G$ above generalises to an irreducible group.

**Definition 2.1.** *A finite subgroup $G \subset \mathcal{U}(V)$ is* irreducible *if*

$$G\phi := \{g\phi : g \in G\}$$

*spans $V$ for every nonzero $\phi \in V$. The set $G\phi$ is called the $G$–*orbit *of $\phi$.*

**Theorem 2.2.** *If $G \subset \mathcal{U}(V)$ is a finite group of unitary transformations which is irreducible, then every nonzero $G$–orbit is a tight frame for $V$.*

**Proof:** Let $\phi \in V$, $\phi \neq 0$, and define a linear map $S_\phi : V \to V$ by

$$S_\phi f := \sum_{g\in G} \langle f, g\phi\rangle g\phi, \qquad \forall f \in V.$$

We will show that $S_\phi$ is a positive scalar multiple of the identity map $\mathrm{id}_V : V \to V$, thereby obtaining (2) for the $G$–orbit $\Phi = G\phi$. This follows as each element of $G\phi$ occurs the same number of times in the list $(g\phi)_{g\in G}$.

Since $\langle S_\phi f, f\rangle > 0$ for all nonzero $f \in V$, $S_\phi$ is positive definite, and so has an eigenvalue $\lambda > 0$. Let $v \in V$ be a corresponding eigenvector. Then for $h \in G$, $hv$ is also a $\lambda$–eigenvector, since

$$S_\phi(hv) = \sum_{g\in G} \langle hv, g\phi\rangle g\phi = \sum_{g\in G} \langle v, h^{-1}g\phi\rangle hh^{-1}g\phi = hS_\phi(v) = \lambda\, hv.$$

Hence $S_\phi f = \lambda f$ for all $f \in \mathrm{span}\{hv : h \in G\}$. But since $G$ is irreducible, $\mathrm{span}\{hv : h \in G\} = V$, and so $S_\phi = \lambda\,\mathrm{id}_V$ as supposed. This argument is essentially 'Schurs Lemma' for irreducible modules. □

Clearly the converse holds: if every nonzero $G$–orbit is a tight frame, then $G$ is irreducible.

This result shows that the $n^{\mathrm{th}}$ roots of unity (vertices of an $n$-sided regular polygon) in $\mathbb{R}^2$ form a tight frame because they are an orbit of the cyclic group of order $n$ (generated by rotation through $\frac{2\pi}{n}$) which is irreducible. By way of comparison, to calculate (1) directly in this case would require the identities

$$\sum_{j=1}^{n} \left(\cos\frac{2\pi j}{n}\right)^2 = \sum_{j=1}^{n} \left(\sin\frac{2\pi j}{n}\right)^2 = \frac{n}{2}, \qquad \sum_{j=1}^{n} \cos\frac{2\pi j}{n} \sin\frac{2\pi j}{n} = 0,$$

which can now be viewed as a consequence of Theorem 2.2.

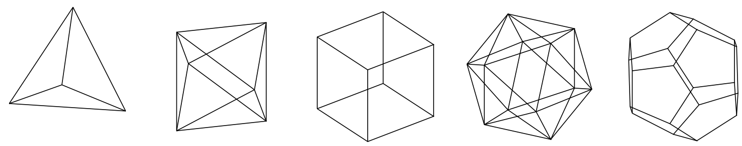

**Fig. 1.** The platonic solids (courtesy of Matthew Fickus).

**Corollary 2.3.** *The vertices of each platonic solid is a tight frame for* $\mathbb{R}^3$.

**Proof:** Let $X$ be a platonic solid (tetrahedron, octahedron, dodecahedron, icosahedron, cube) with centre of gravity at the origin. Then the set of vertices of $X$ is an orbit of the symmetry group $G$ of $X$, which is irreducible. (If the symmetry group was not irreducible, then there would be a plane $P \subset \mathbb{R}^3$ that was $G$–invariant. The restriction of the action of $G$ to $P$ would then be a faithful representation of $G$, so $G$ would be isomorphic to a finite subgroup of $O(2)$. But this is impossible, since every finite subgroup of $O(2)$ is cyclic or dihedral. See [2]). □

Similarly, the vertices of the truncated icosahedron (aka the 'soccer ball', 'bucky ball') form a tight frame for $\mathbb{R}^3$, since they are an orbit of its symmetry group, which is irreducible.

## §3. History of the result

Let $P_W$ denote the orthogonal projection onto a subspace $W$ of $V$. If $V_\phi$ is the 1-dimensional subspace spanned by a nonzero $\phi \in V$ then the conclusion that $G\phi$ is a tight frame can be expressed as

$$f = \frac{d}{|G|} \sum_{g \in G} P_{gV_\phi} f, \qquad \forall f \in V. \tag{3}$$

Now suppose $W = W_1 \oplus \cdots \oplus W_m \subset V$, $m \le d$ is an orthogonal direct sum of 1-dimensional subspaces, so that $P_W = P_{W_1} + \cdots + P_{W_m}$, and average the above to obtain

$$f = \frac{d}{m} \frac{1}{|G|} \sum_{g \in G} P_{gW} f, \qquad \forall f \in V, \tag{4}$$

for all $m$-dimensional subspaces $W$ of $V$.

Results of this type go back at least to Schönhardt who showed that if a vector $v$ in the plane is projected onto the sides of a regular $n$-gon then the arithmetic mean of these $n$ projections is $v/2$. This and similar results led Brauer and Coxeter [3] to prove that if $G$ is an irreducible finite group of real orthogonal transformations then (4) holds. Our Theorem 2.2 (obtained independently) shows that $G$ being irreducible implies (3), from which (4) follows. It also applies in the case where $V$ is complex, which was not considered in [3].

**References**

1. Benedetto, J. and M. Fickus. Finite Normalized Tight Frames. Adv. Comput. Math. **18** (2003) no. 2–4, 357–385.
2. Benson, C.T. and L.C. Grove. *Finite Reflection Groups*. Bogden and Quigley, Tarrytown-on-Hudson, N.Y., 1971.
3. Brauer, R., and H. S. M. Coxeter, A Generalization of Theorems of Schönhardt and Mehmke on Polytopes. Trans. Roy. Soc. Canada. Sect. III (3) **34**, (1940). 29–34.

Richard Vale
Dept. of Mathematics
University of Auckland
Private Bag 92019, Auckland
New Zealand
vale@math.auckland.ac.nz

Shayne Waldron
Dept. of Mathematics
University of Auckland
Private Bag 92019, Auckland
New Zealand
waldron@math.auckland.ac.nz
http://www.math.auckland.ac.nz/∼waldron/