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Better-reply dynamics and global convergence to Nash equilibrium in aggregative games

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Abstract

We consider n-person games with quasi-concave payoffs that depend on a player's own action and the sum of all players' actions. We show that a discrete-time, stochastic process in which players move towards better replies—the *better-reply dynamics*—converges globally to a Nash equilibrium if actions are either strategic substitutes or strategic complements for all players around each Nash equilibrium that is asymptotically stable under a deterministic, adjusted best-reply dynamics. We present an example of a 2-person game with a unique equilibrium where the derivatives of the best-reply functions have different signs and the better-reply dynamics does not converge.

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1. Introduction

In his study of duopoly, Cournot (1960) introduced the noncooperative equilibrium later generalized by Nash (1950; 1951), and investigated its stability under a version

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of the best-reply dynamics in which firms alternate changing output from its current level to a best reply to the opponent's level. The more recent literature has studied several versions of the best-reply dynamics, both in the framework of oligopoly models and in the more general setup of a noncooperative game, with the focus on finding conditions that guarantee either the global or the local asymptotic stability of a Nash equilibrium. The general message is that these conditions are very strong, especially when global stability of a particular Nash equilibrium is required, since in this case they must imply uniqueness of the equilibrium. (See Al-Nowaihi and Levine, 1985; Dastidar, 2000; and Vives, 1990 for results on convergence of the continuous-time version of the dynamics, and Gabay and Moulin, 1980 and Moulin, 1984 for results on convergence of the discrete-time dynamics.)

In this paper we are interested in studying global convergence to a Nash equilibrium, but we do not require that the equilibrium be unique. Instead, we study conditions under which the system eventually settles in an equilibrium, without imposing that all possible paths converge to the same equilibrium. This is only the first of several differences between our approach and the rest of the literature on the best-reply dynamics. A more fundamental difference is that we look at stochastic, rather than deterministic, adjustment processes. In our model players have status quo actions and are randomly selected, one at the time, to sample new actions. When a player is selected to sample, she randomly draws one of her available actions and only changes her status quo to the sampled action if this improves her payoff (i.e., if the move constitutes a *single-player improvement*). We call the stochastic process so generated the *better-reply dynamics*, because players move from their current actions to a better reply, not necessarily a best reply; even though players move in the direction of their best replies, they can overshoot or undershoot them.

Our better-reply dynamics can be viewed as a simple stimulus–response model of the behavior of boundedly rational players. It can be seen as a formalization of results from experimental research in economics and psychology showing that players' behavior gravitates towards actions that have been successful (see Roth and Erev, 1995). Our dynamics is related to the recent literature on learning in games (see Fudenberg and Levine, 1998 for a survey). However, the focus of this literature is on how players may learn to play a mixed strategy Nash equilibrium in a finite game (often with only two players), while we focus on convergence to a pure strategy Nash equilibrium in a game with continuous action spaces and several players. A distinguishing feature of our model is that the bound on players' rationality and knowledge is more severe than in most of the learning literature. The better-reply dynamics is consistent with a player not having precise knowledge, or memory, of her own and her opponents' payoff functions and past actions.

A standard criticism levied against the deterministic, best-reply dynamics first studied by Cournot is that when a player moves to a best reply to her opponents' current actions, she acts as if her opponents never changed their actions, in spite of collecting repeated evidence that actions do change. This criticism is less pertinent to our model, because our players need not know the actions of their opponents or their own best-reply functions. Our players simply experiment new actions and make definite changes after experiencing an increase in payoff.

A version of the better-reply dynamics studied in this paper was first introduced by Friedman and Mezzetti (2000). They noticed that in finite games having the *weak finite* *improvement property* (weak FIP), the better-reply dynamics globally converges to a Nash equilibrium. The weak FIP requires that starting from any action profile there exists a finite path of single-player improvements that leads to a Nash equilibrium of the game. Friedman and Mezzetti (2000) showed that finite supermodular games and generic, continuous, two-person, quasi-concave games have the weak FIP.

The focus of our analysis in this paper is on *aggregative*, n-person, noncooperative games. In an aggregative game the payoff of each player is a function of the player's own action and of the sum of the actions (or, equivalently, the mean action) of all players. We take the players' action spaces to be closed intervals on the real line and assume that a player's payoff is a quasi-concave function of her own action. We restrict attention to games in which, for each player, the slope of the best-reply function is bounded below by -1.[1] The class of aggregative games contains many interesting games from economics and political science. A wide class of oligopoly games, including Cournot's original model, models of the private provision of a public good, models of the joint exploitation of a common resource, collective actions models, and macroeconomic models with catching-up-with-the-Joneses are all examples of aggregative games.

Two recent papers that also study aggregative games are Kukushkin (2004) and Dubey et al. (2004). Kukushkin (2004) established that every deterministic best response path leads to a Nash equilibrium in finite aggregative games satisfying one of three possible versions of a single crossing condition. Dubey et al. (2004) proved existence of a pure strategy Nash equilibrium in a fairly general class of aggregative games (including games with non-convex strategy sets), when actions are either strategic complements, or strategic substitutes.

After describing the model in the next section, in Section 3 we study the stochastic process generated by the better-reply dynamics. We provide a sufficient condition for the better-reply dynamics to globally converge to a Nash equilibrium of almost all aggregative games. This condition is that actions be either strategic substitutes or strategic complements for all players (i.e., the derivatives of the best-reply functions have the same sign) at all Nash equilibria that are asymptotically stable under a deterministic, continuous-time, adjusted best-reply dynamics. This sufficient condition is a local condition, actions need to be either strategic complements or strategic substitutes only at the Nash equilibria. Furthermore, actions are allowed to be strategic substitutes at some equilibria and strategic complements at other equilibria; we only need to rule out equilibria where actions are strategic complements for some players and strategic substitutes for other players.

In Section 4 we provide an example of a 2-person game with a unique Nash equilibrium at which the derivatives of the two best-reply functions have different signs; we show that in such a game the stochastic process generated by the better-reply dynamics does not converge to the equilibrium. This demonstrates that our condition on the derivatives of the best reply functions at the Nash equilibria cannot be easily relaxed; without it the better reply dynamics may fail to converge.

[1] In most aggregative games such an assumption is not very restrictive; for example, in the Cournot model it requires that the difference between price and marginal cost is a decreasing function of the firm's output.

In Section 5 we prove that aggregative games have the weak finite improvement property. This implies that any discretization of a game with a continuous action space also has the weak FIP and that in such a discretized game the better-reply dynamics converges to an action profile that is close to a Nash equilibrium of the original game. At first, it may seem puzzling that with a discrete state space global convergence requires less stringent conditions (actions need not be strategic complements or substitutes at a Nash equilibrium). The puzzle is easily resolved by noting that although a discretized version of the non-convergent example of a 2-person game described in Section 4 would converge to the Nash equilibrium, the average time that it takes to converge goes to infinity as the discretized version converges to the continuous game.

Section 6 contains some concluding remarks. There we argue that our convergence results for the stochastic better-reply dynamics are considerably stronger than existing results on the convergence of the deterministic best-reply dynamics. The proofs of our results, except for Theorems 1 and 6, are in Appendix A.

2. The model

We study n-person games $g = \langle N, \{A_i\}_{i\in N}, \{U_i\}_{i\in N}\rangle$ where each player $i \in N = \{1,\dots,n\}$ has a one dimensional, compact, convex strategy set $A_i \subset \mathbb{R}$ and a payoff function $U_i : A \to \mathbb{R}$, with $A \equiv \bigtimes_{i=1,\dots,n} A_i$, that only depends on player i's own action and the sum of the actions of all players. That is, there exists a function $\phi_i : A_i \times A_\Sigma \to \mathbb{R}$ such that, for all $a \in A$ it is $U_i(a) = \phi_i(a_i, \Sigma)$, where $\Sigma = a_1 + \cdots + a_n$ and $A_\Sigma = \{a_1 + \cdots + a_n\colon a_i \in A_i \text{ for all } i \in N\}$ is the set of admissible sums. (Note that this condition always hold for 2-person games.) We assume that U_i is twice continuously differentiable in its arguments and strictly quasi-concave with respect to $a_i \in A_i$; that is, $U_i(a_i', a_{-i}) \geqslant U_i(a_i'', a_{-i})$, with $a_i' \neq a_i''$, implies $U_i(\lambda a_i' + (1-\lambda)a_i'', a_{-i}) > U_i(a_i'', a_{-i})$ for all $\lambda \in (0,1)$. The partial derivative of U_i with respect to a_i, $\partial U_i/\partial a_i$ must have the same form as U_i.[2] It is thus possible to define a function D_i that depends on a_i and Σ as follows:

$$D_i(a_i, \Sigma) = \frac{\partial U_i(a)}{\partial a_i} = \frac{\partial \phi_i}{\partial a_i}(a_i, \Sigma) + \frac{\partial \phi_i}{\partial \Sigma}(a_i, \Sigma). \tag{1}$$

At an interior solution, player i's best response function $b_i(a_{-i})$ only depends on the sum of the opponent's actions $\Sigma_{-i} = \Sigma - a_i$; that is, there is a function $B_i(\Sigma_{-i})$ such that $b_i(a_{-i}) = B_i(\Sigma_{-i})$, for all $a_{-i} \in A_{-i} = \bigtimes_{j\neq i} A_j$. The function $B_i(\Sigma_{-i})$ is an implicit solution to the equation

$$D_i\big(B_i(\Sigma_{-i}), B_i(\Sigma_{-i}) + \Sigma_{-i}\big) = 0. \tag{2}$$

If for some Σ_{-i} we have $D_i(a_i, a_i + \Sigma_{-i}) \neq 0$ for all $a_i \in A_i$, then B_i is simply the right endpoint of A_i if $D_i > 0$, and the left one if $D_i < 0$. Strict quasi-concavity of U_i with respect to $a_i \in A_i$ implies that:

[2] Since U_i is twice continuously differentiable and A_i is compact, the partial and cross partial derivatives of U_i are bounded and obtain a maximum and a minimum.

(i) B_i is continuous and single-valued (i.e., the implicit equation (2) has, at most, a unique solution for all Σ_{-i}; at this solution U_i attains its maximum) and
(ii) player i's payoff declines as a_i moves away from $B_i(\Sigma_{-i})$.

This latter property and differentiability of U_i imply that for all $a_i \in A_i$ and all $\Sigma \in A_\Sigma$:

$$D_i(a_i, \Sigma) \geqslant 0 \quad \text{if } a_i < B_i(\Sigma_{-i}); \qquad D_i(a_i, \Sigma) \leqslant 0 \quad \text{if } a_i > B_i(\Sigma_{-i}) \tag{3}$$

and

$$\frac{\mathrm{d}D_i(a_i, \Sigma)}{\mathrm{d}a_i} = \frac{\partial D_i(a_i, \Sigma)}{\partial a_i} + \frac{\partial D_i(a_i, \Sigma)}{\partial \Sigma} \leqslant 0 \quad \text{if } a_i = B_i(\Sigma_{-i}) \in A_i^0 \tag{4}$$

where A_i^0 is the interior of A. We also make the additional assumption that the partial derivative of D_i with respect to a_i is negative everywhere; that is, for all $a_i \in A_i$ and all $\Sigma \in A_\Sigma$:

$$\frac{\partial D_i(a_i, \Sigma)}{\partial a_i} = \frac{\partial_i^2 \phi(a_i, \Sigma)}{\partial a_i^2} + \frac{\partial_i^2 \phi(a_i, \Sigma)}{\partial \Sigma \partial a_i} < 0. \tag{5}$$

This condition does not imply, and is not implied by, the concavity of U_i with respect to a_i. For example, if $U_i(a) = \beta a_i^2 + \gamma \Sigma^2$, then this assumption requires $\beta < 0$, while concavity requires $\beta + \gamma < 0$. In many games condition (5) is not very restrictive; together with condition (4), it implies that the slope of the best-reply function of each player is bounded below by -1:

$$\frac{\mathrm{d}B_i(\Sigma_{-i})}{\mathrm{d}\Sigma_{-i}} = -\frac{\partial D_i(a_i, \Sigma)/\partial \Sigma}{dD_i(a_i, \Sigma)/da_i} = \frac{\partial D_i(a_i, \Sigma)/\partial a_i}{dD_i(a_i, \Sigma)/da_i} - 1 > -1 \quad \text{for } a_i = B_i(\Sigma_{-i}).$$

Since $\partial D_i(a_i, \Sigma)/\partial a_i$ is a continuous function (U_i is twice continuously differentiable) on a compact domain, it attains a maximum; since the function must be negative, it follows that the value at the maximum must be negative. Thus, $\partial D_i(a_i, \Sigma)/\partial a_i$ must be bounded away from zero.

Definition 1. The game $g = \langle N, \{A_i\}_{i \in N}, \{U_i\}_{i \in N} \rangle$ is an *aggregative game* if for all $i \in N$:

(a) $A_i \subset \mathbb{R}$ is compact and convex;
(b) $U_i(a) = \phi_i(a_i, \Sigma)$ is twice continuously differentiable in all its arguments and strictly quasi-concave with respect to $a_i \in A_i$;
(c) U_i satisfies condition (5).

Examples of aggregative games include oligopoly, and many collective action and search models.

Example 1. In a homogeneous product, Cournot oligopoly with n firms, let a_i be the output level of firm i and Σ be total output. $P(\Sigma)$ is the inverse demand function, $C_i(a_i)$ is the cost function of firm i and $U_i(a) = P(\Sigma)a_i - C_i(a_i)$. Condition (5) requires that the difference between price and marginal cost be a decreasing function of a firm's output, $P'(\Sigma) - C_i''(a_i) < 0$.

Example 2. In a collective action problem, each of n players privately chooses a_i at a cost $C_i(a_i)$ (a_i could be i's private provision of a public good, or her private use of a common resource). The sum of individual choices determines the benefit $V_i(\Sigma)$ to the player and $U_i(a) = V_i(\Sigma) - C_i(a_i)$. Condition (5) is satisfied provided cost is a convex function.

3. The better-reply dynamics: convergence results

We are interested in studying the convergence properties of a stochastic, discrete time, adjustment process, called the *better-reply dynamics*, in which, at each point in time a player is randomly selected to sample among her available actions. The selected player only changes her status quo to the sampled action if this improves her payoff. We assume that the probability that a randomly selected player i samples a strategy belonging to any subset E of A_i is positive if E has positive Lebesgue measure. Formally, we associate with the strategy space A_i of player i a probability measure P_i defined on the Borel subsets of A_i. For any Borel set $E \subset A_i$ the number $P_i(E)$ expresses the likelihood that player i samples a strategy that belongs to E. The only condition we impose on P_i is that for any open interval $I \subset A_i$ we have $P_i(I) > 0$. This does not exclude singular measures; that is, the measure P_i can have one or more points x where $P_i(\{x\}) > 0$.

Let $a\backslash x_i$ denote the n-tuple $(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_n) \in A$. The strategy profile $a\backslash x_i \in A$ is a *single-player improvement* over $a \in A$ if and only if the payoff to player i is higher under $a\backslash x_i$ than under a: $U_i(a\backslash x_i) > U_i(a)$.

Definition 2 (*The better-reply dynamics*). Consider a continuous game g. Let P_i be a probability measure on the Borel subsets of A_i such that for any open interval $I \subset A_i$, $P_i(I) > 0$. At each discrete time period t there is a status quo action profile a^t. A single player $i \in N$ is randomly selected, with all players having positive selection probability. Player i randomly samples action $x_i \in A_i$ according to the probability measure P_i. If $a^t\backslash x_i$ is a single-player improvement over a^t, then it becomes the new status quo, $a^{t+1} = a^t\backslash x_i$. If $U_i(a^t\backslash x_i) \leqslant U_i(a^t)$ then the status quo does not change, $a^{t+1} = a^t$.

The process described in Definition 2 is essentially the same as the one defined by Friedman and Mezzetti (2000), except that they had finite strategy spaces and required all players to have the same probability of being selected to sample a new strategy and all strategies to have the same probability of being sampled. Note that the experimentation of a new strategy on the part of the player sampling at time t has no effect on the other players. In particular, it does not affect the payoff that other players associate with their status quo action. The simplest way to justify this assumption is to think of time as a continuous variable, with players experimenting new actions at (possibly random) discrete points in time. When a player is sampling a new strategy at time t, she has experienced the same payoff for the time interval $(t-1, t)$ and views it as the status quo payoff.

We will derive results that hold for almost all transversal, aggregative games. Let a^* be a Nash equilibrium of a game $g = \langle N, A, U\rangle$ and $B_i'(\Sigma_{-i}^*)$ be the derivative of player i's best reply function at a^*. Player i and j's best reply functions at a^* are said to be *transversal* if they are not tangent; that is, if $B_i'(\Sigma_{-i}^*) B_j'(\Sigma_{-j}^*) \neq 1$. The game g is called a *transversal* *game* if at all Nash equilibria a^*, $B_i'(\Sigma_{-i}^*)B_j'(\Sigma_{-j}^*) \neq 1$ for all i and $j \neq i$ (i.e., if the best-reply functions of all pairs of players are transversal). Transversal games have a finite number of equilibria (equilibria are isolated). Let $S(n,q)$ be the set of n-person, transversal, aggregative games with q equilibria, $a(1),\ldots,a(q)$ (we can unambiguously order equilibria by putting $a_1(h-1) \leqslant a_1(h)$ for all h, $a_2(h-1) \leqslant a_2(h)$ if $a_1(h-1) = a_1(h),\ldots$). Consider the map $\xi_q^n : S(n,q) \to \mathbb{R}^{nq}$ that associates to each game $g \in S(n,q)$ the vector s_q^n of the slopes of the best-reply functions at the Nash equilibria of g:

$$s_q^n = \big(B_1'\big(\Sigma_{-1}(1)\big),\ldots,B_n'\big(\Sigma_{-n}(1)\big),\ldots,B_1'\big(\Sigma_{-1}(q)\big),\ldots,B_n'\big(\Sigma_{-n}(q)\big)\big).$$

Let $I(\xi_q^n)$ be the range of ξ_q^n. We will say that a property $\mathcal{P}$ holds for almost all games in $S(n,q)$ if there exists a subset I^* of $I(\xi_q^n)$ such that (i) $I(\xi_q^n) \setminus I^*$ has zero Lebesgue measure in $\mathbb{R}^{nq}$, and (ii) the property $\mathcal{P}$ holds for all games g with $\xi_q^n(g) \in I^*$. A property holds for *almost all transversal, n-person, aggregative games* if it holds for almost all games in $S(n,q)$, for all q.

We begin with some preliminary results. If Eq. (5) holds, then for any given $\Sigma \in A_\Sigma$ there is (at most) a unique solution $M_i(\Sigma)$ to the implicit equation:

$$D_i\big(M_i(\Sigma),\Sigma\big) = 0. \tag{6}$$

If Eq. (6) does not have a solution in A_i (i.e., $D_i(a_i,\Sigma) \neq 0$ for all $a_i \in A_i$), then let $M_i(\Sigma)$ be the right endpoint of the interval A_i if $D_i(a_i,\Sigma) > 0$ and the left endpoint otherwise. The function $M_i(\Sigma)$ is a piece-wise C^1 function and at points where it is not differentiable it has a right and a left derivative. We will use the convention that at such points $\mathrm{d}M_i/\mathrm{d}\Sigma$ is the left derivative of $M_i(\Sigma)$.

Consider the following system of differential equations:

$$\dot{a}_i = M_i(\Sigma) - a_i, \quad i = 1,2,\ldots,n. \tag{7}$$

If g is an aggregative game, then one can show that $M_i(\Sigma) - a_i$ and $B_i(\Sigma_{-i}) - a_i$ have always the same sign. For this reason, we will call (7) the *continuous-time, adjusted best-reply* (*CAB*) *dynamics*.

Lemma 1. *Let $g = \langle N, \{A_i\}_{i\in N}, \{U_i\}_{i\in N}\rangle$ be an aggregative game. For all $a \in A$, we have*:

(a) $(M_i(\Sigma) - a_i)(B_i(\Sigma_{-i}) - a_i) \geqslant 0$, *and*
(b) $B_i(\Sigma_{-i}) = a_i$, *if and only if* $M_i(\Sigma) = a_i$.

Recall that, given a system of ordinary differential equations $\dot{a} = f(a)$, with $f(a)$ a Lipschitz function, we can think of the unique solution a^t, with initial condition a^0, as the trajectory of the deterministic dynamical system starting at a^0. Any point $a \in A$ with the property that there exists a sequence $t_1, t_2, \ldots$ such that $\lim_{m\to\infty} a^{t_m} = a$ is called an *ω-limit point* of the trajectory a^t; the set of all such points is called the *ω-limit set* of the trajectory a^t. If the ω-limit set of the trajectory a^t contains a single element $a^* \in A$, then a^* is a stationary point (i.e., $f(a^*) = 0$) and if the system starts at a^0 then it will converge to a^*; $\lim_{t\to\infty} a^t = a^*$. The system of ordinary differential equations $\dot{a} = f(a)$ *globally converges*, if for all $a^0 \in A$ the ω-limit set of the system with initial condition a^0 is a singleton.

If it is globally convergent, then from any given initial state a^0 the system converges to an equilibrium a^*; $\lim_{t\to\infty} a^t = a^*$. Note that global convergence of a system does not imply that the system has a unique equilibrium or stationary point. Rather, it means that starting from any initial position one of the equilibria is eventually reached. Cycling or chaotic dynamics are ruled out. Lemma 2 shows that the CAB dynamics defined by (7) globally converges to some Nash equilibrium of the game g.

Lemma 2. *Let g be an n-person, aggregative game. Then the CAB dynamics defined by Eq.* (7) *is globally convergent.*

The intuition behind the global convergence of the CAB dynamics defined by Eq. (7) is simple. By adding up the n differential equations in (7) one obtains a single ordinary differential equation in Σ. Such an equation cannot exhibit any cyclic or chaotic behavior.

Lemma 2 is needed to establish the next lemma, which shows that for almost all transversal, aggregative games there is a finite sequence of single-player improvements that ends arbitrarily close to a Nash equilibrium.

Lemma 3. *For almost all transversal, n-person, aggregative games g, given any $r > 0$ and any strategy profile a^0, there is a finite sequence of single-player improvements that starts at a^0 and ends inside a ball of radius r around an isolated Nash equilibrium a^* of g.*

Lemma 3 implies that with probability one the better-reply dynamics ends up arbitrarily close to a Nash equilibrium of an aggregative game. This is because, with probability one, the better-reply dynamics will eventually follow a path arbitrarily close to a trajectory of the deterministic system (7). The next lemma provides sufficient conditions for convergence to a Nash equilibrium when the system is already close to the equilibrium.

Lemma 4. *Let g be an n-person, aggregative game. Consider a Nash equilibrium a^* of g. Let B_i' be the first derivative of the best-reply function $B_i(\Sigma_{-i})$ of player i evaluated at a^*. Assume that either* (a) *$B_i' > 0$ for all i and $\sum_{i=1}^n B_i'/(1+B_i') < 1$, or* (b) *$0 > B_i'$ for all i. Then there exists a neighborhood V of the Nash equilibrium a^* such that almost every path $a^0, a^1, a^2, \ldots$ generated by the stochastic process described in Definition* 2 *that starts in V stays in V and, moreover,* $\lim_{t\to\infty} a^t = a^*$.

Two sets of conditions guarantee that when starting close to a Nash equilibrium a^* almost all paths generated by the better-reply dynamics converge to a^*. The first condition is that all the derivatives B_i' of the best-reply functions evaluated at a^* have the same sign; that is, actions are either locally strategic substitutes or locally strategic complements. The second condition is $\sum_{i=1}^n B_i'/(1+B_i') < 1$. Note that if $B_i' < 0$ for all $i \in N$ this condition is automatically satisfied, because (4) and (5) guarantee that $B_i' > -1$ for all $i \in N$. Let C_i be the derivative of the M_i functions defined in Eq. (6), evaluated at a^*. Equation (6) implies that $C_i = B_i'/(1+B_i')$, hence $B_i' > -1$ is equivalent to $C_i < 1$. Furthermore, $\sum_{i=1}^n B_i'/(1+B_i') < 1$ is equivalent to $\sum_{i=1}^n C_i < 1$. This condition is sufficient for a^* to be locally asymptotically stable under the CAB dynamics defined by Eq. (7).[3] Note also that if $0 > B_i' > -1$ for all i, or $B_i' > 0$ for all i and $\sum_{i=1}^n B_i'/(1+B_i') < 1$, then at the equilibrium a^* the best reply functions of all players are transversal, since it must be $B_i' B_j' < 1$ for all i, j.

The next theorem shows that the stochastic better-reply dynamics globally converges if actions are either locally strategic substitutes or locally strategic complements for all players at all Nash equilibria that are asymptotically stable under the CAB dynamics. To use Lemma 4 in the proof, we also need to add the technical assumption that $\sum_{i=1}^n B_i'/(1+B_i') \neq 1$.[4] Note that actions are allowed to be strategic substitutes at a Nash equilibrium and strategic complements at another equilibrium, as we only need to rule out equilibria where actions are strategic complements for some players and strategic substitutes for other players.

Theorem 1. *Let g be a transversal, n-person, aggregative game. Suppose that at each Nash equilibrium a^* that is asymptotically stable under the CAB dynamics defined by Eq.* (7):

(a) $\sum_{i=1}^n B_i'(\Sigma_{-i}^*)/[1+B_i'(\Sigma_{-i}^*)] \neq 1$, *and*
(b) *the derivatives of all the best-reply functions have the same sign.*

Then, regardless of the initial position, for almost any path $a^0, a^1, a^2, \ldots$ generated by the stochastic process described in Definition 2 (*the better-reply dynamics*) *we have*: $\lim_{t\to\infty} a^t = a^*$, *where a^* is a Nash equilibrium of the game g.*

Proof. First we must argue as in Lemma 3 that, starting from any non-equilibrium point, the path $a^0, a^1, a^2, \ldots$ will eventually end up in some neighborhood V of a locally asymptotically stable equilibrium a^* of the system (7). Once there we can apply Lemma 4 to conclude the proof. (Recall that at an asymptotically stable equilibrium of (7) it must be $\sum_{i=1}^n B_i'/(1+B_i') \leqslant 1$; hence $\sum_{i=1}^n B_i'/(1+B_i') \neq 1$ at such an equilibrium implies $\sum_{i=1}^n B_i'/(1+B_i') < 1$.) □

Theorem 1 is the main result of the paper. Why is global convergence obtained and why do we need actions to be either strategic substitutes or strategic complements around a Nash equilibrium? An intuitive explanation consists of two parts. First, in aggregative games the stochastic process generated by the better-reply dynamics will eventually get within a small ball around a Nash equilibrium. Second, if the derivatives of the best reply functions have the same sign at a Nash equilibrium, players always move in the same direction and when close to equilibrium the stochastic process will not exit from a small ball around it. If the derivatives of the best reply functions have different signs at a Nash equilibrium, there is no tendency for players to move in the same direction. In the next section we will present an example that shows that in this case the better-reply dynamics of Definition 2 need not globally converge, because the stochastic process may leave any small ball around a Nash equilibrium.

[3] If $\sum_{i=1}^n C_i > 1$ then a^* is unstable, while if $\sum_{i=1}^n C_i = 1$ then a^* could be either stable or unstable.
[4] This assumption is satisfied by almost all games.

Consider again the examples introduced in Section 2.

Example 1. In the case of a homogeneous product, Cournot oligopoly, a sufficient condition for the slopes of the best-reply functions to have the same (negative) sign is $P''(\Sigma^*) \leqslant 0$ (i.e., at the total output level corresponding to a Nash equilibrium the slope of the inverse demand function is decreasing).

Example 2. In a collective action problem, the slopes of the best-reply functions have the same (negative) sign at a Nash equilibrium a^* provided each player's marginal benefit function is decreasing in Σ^*, $V_i''(\Sigma^*) < 0$.

The next theorem shows that for 2-person games condition (5) need not be satisfied (i.e., the slope of the best-reply functions need not be bounded below by -1) for the better-reply dynamics to globally converges to a Nash equilibrium.

Theorem 2. *Let g be a transversal, 2-person game with payoff functions that are quasi-concave in own action. Suppose $B_1'(a_2^*)B_2'(a_1^*) > 0$ (i.e., the derivatives of the best-reply functions have the same sign) at each Nash equilibrium a^* of g such that $B_1'(a_2^*)/(1 + B_1'(a_2^*)) + B_2'(a_1^*)/(1 + B_2'(a_1^*)) < 1$. Then, regardless of the initial position, for almost any path $a^0, a^1, a^2, \ldots$ generated by the stochastic process described in Definition 2 we have: $\lim_{t\to\infty} a^t = a^*$, where a^* is a Nash equilibrium of g.*

4. An example of non-convergence of the better-reply dynamics

In this section we construct an example of a 2-person, quasi-concave game g^E with a unique Nash equilibrium $a^* = (0, 0)$ at which the derivatives of the best-reply functions have different signs and show that for almost all paths $a^0, a^1, \ldots$ the better-reply dynamics does not converge to the equilibrium. First we introduce a needed lemma.

Lemma 5. *Let $\rho^0, \rho^1, \ldots$ be an infinite path of a discrete time Markov process with $r > \rho^0 > 0$.*[5] *Suppose the probability law governing the stochastic process in the interval $(0, r)$ satisfies the following inequalities:*

$$P\left(\frac{\rho^{t+1}}{\rho^t} \geqslant 2^{2^4}\right) > \frac{1}{4}, \tag{8}$$

$$P\left(\frac{\rho^{t+1}}{\rho^t} \leqslant \varepsilon\right) < \varepsilon^2 \quad \textit{for any } \varepsilon > 0. \tag{9}$$

Then for almost all path $\rho^0, \rho^1, \ldots$ there exists T such that $\rho^T > r$.

[5] The stochastic process considered in this lemma may depend on some hidden, time-varying variables, provided that their values do not influence the validity of inequalities (8) and (9).

Now we are ready to construct our example. The strategy set of each player $i = 1, 2$ is $A_i = [-2, 2]$. Let $\beta_1, \beta_2 > 0$ and define the best-reply functions of the players as follows:

$$B_1(a_2) = \begin{cases} -\beta_1 a_2 & \text{for } a_2 \in \left[-\frac{1}{\beta_1}, \frac{1}{\beta_1}\right], \\ -\frac{2\beta_1+\beta_1 a_2-2}{2\beta_1-1} & \text{for } a_2 > \frac{1}{\beta_1}, \\ -\frac{2+\beta_1 a_2-2\beta_1}{2\beta_1-1} & \text{for } a_2 < -\frac{1}{\beta_1}; \end{cases}$$

$$B_2(a_1) = \begin{cases} \beta_2 a_1 & \text{for } a_1 \in \left[-\frac{1}{\beta_2}, \frac{1}{\beta_2}\right], \\ \frac{2\beta_2+\beta_2 a_1-2}{2\beta_2-1} & \text{for } a_1 > \frac{1}{\beta_2}, \\ \frac{2+\beta_2 a_2-2\beta_2}{2\beta_2-1} & \text{for } a_1 < -\frac{1}{\beta_2}. \end{cases}$$

Let the utility function of each player be the square of the Euclidean distance from the best reply. This defines a two-player, continuous game g^E having a unique Nash equilibrium at the point $a^* = (0, 0)$. If $\beta_1 < 1$, then condition (5) is satisfied. In this game the CAB dynamics defined by (7) always converges to a^*. Also, around the equilibrium the game has linear best-reply functions with $B_1'(0)B_2'(0) = -\beta_1\beta_2 < 0$.

Theorem 3 shows that in the game g^E, if the evolution of the action profile a follows the better-reply dynamics, then play will not converge to the Nash equilibrium a^*. For simplicity we will suppose that each player's sampling probability is uniform on $A_i = [-2, 2]$ and that the probability that each player is selected to sample a new strategy is $1/2$.

Theorem 3. *In the game g^E, consider the stochastic process generated by the better-reply dynamics described in Definition* 2. *Assume that each player's sampling probability is uniform on $A_i = [-2, 2]$ and that the probability that each player is selected to sample a new strategy is* $1/2$. *For all $\beta_1 > 0$ there exists $\beta_0 > 0$ such that if $\beta_2 > \beta_0$, then for almost all paths $a^0, a^1, \ldots,$ with $a^0 \neq (0, 0)$, the stochastic process does not converge to the Nash equilibrium a^* of g^E.*

Lemma 3 shows that there is a positive probability that in the game g^E discussed in this section the stochastic process described in Definition 2 enters any small neighborhood U of the equilibrium $(0, 0)$; Theorem 3 shows that it is also the case that almost any path of the process will leave the neighborhood U. In fact, almost all orbits are dense in the square $[-2, 2]^2$; that is, any set of positive measure is visited infinitely many times. This explains why in Theorem 2 we must impose the condition $B_1'(a_2^*)B_2'(a_1^*) > 0$ at a Nash equilibrium a^* to guarantee that the system globally converges.

Theorem 3 is related to a result by Gale and Rosenthal (1999). They studied a model with an experimenter and an imitator. At each point in time, the experimenter samples new actions and moves to a better response, while the imitator adjusts her action towards the current action of the experimenter. The experimenter's (player 1) best-reply function is $B_1 = \gamma a_2$, where γ could be positive or negative. Gale and Rosenthal (1999) showed that if γ is negative (and sufficiently small) then the system leaves any sufficiently small neighborhood of the unique Nash equilibrium $(0, 0)$ with probability one, while if γ is positive then the system globally converges to the equilibrium. To relate this result to our model, note that we can think that the imitator acts as if her best-reply function were $B_2 = a_1$.

Then, applying our better-reply dynamics, we would also obtain global convergence when $\gamma > 0$ and no convergence for a sufficiently small, negative γ.

5. The better-reply dynamics for finite games

So far we have considered games with continuous strategy sets. In this section we show that there are important differences in the convergence properties of the better-reply dynamics with a discrete and with a continuous state space.

Friedman and Mezzetti (2000) introduced the following definition.

Definition 3. The game g has the *weak finite improvement property* (*weak FIP*) if from all action profiles $a \in A$ there exists a finite sequence of single-player improvements that ends in a pure strategy Nash equilibrium.

The weak FIP is an important property in the study of adaptive dynamics in finite games. In a finite game with the weak FIP the better-reply dynamics converges to a Nash equilibrium. For continuous games, Friedman and Mezzetti (2000) proved the following theorem.

Theorem 4. *Any transversal,* 2*-person, quasi-concave game has the weak FIP.*

We will extend Theorem 4 by showing that almost all transversal, n-person, aggregative games also have the weak FIP. We begin with a lemma that deals with 2-person games with quadratic payoffs $U_i(a) = \alpha_i - \gamma_i(a_i^2 - 2\beta_i a_1 a_2)^2$, where $\alpha_i, \beta_i, \gamma_i > 0$ are constants. In such games the best-reply and M_i functions are:

$$\begin{aligned} &B_i(a_j) = \beta_i a_j, \\ &M_i(a_1 + a_2) = C_i(a_1 + a_2), \end{aligned} \tag{10}$$

where β_i is a constant and $C_i = \beta_i/(1 + \beta_i)$, $i = 1, 2$, $j \neq i$.

Lemma 6. *Let $\mathcal{C}$ be the set of all pairs C_1, C_2 with $C_i < 1$ and $C_1 + C_2 < 1$ and let $g = \langle\{1, 2\}, A, U\rangle$ be a transversal,* 2*-person game with best-reply and M_i functions given by* (10) *with $(C_1, C_2) \in \mathcal{C}$.*[6] *For almost all $(C_1, C_2) \in \mathcal{C}$* (*i.e., with the possible exception of a subset of $\mathcal{C}$ having zero Lebesgue measure*) *the following claims hold*:

(a) *Given any action profile $a^0 = (a_1^0, a_2^0)$ and any number $\theta \in (-1, 1)$, there exists a finite sequence of single-player improvements $\{a^0, a^1, \dots, a^T\}$ such that $a_1^T + a_2^T = \theta(a_1^0 + a_2^0)$.*
(b) *If $\theta = 0$, the sequence $\{a^0, a^1, \dots, a^T\}$ can be chosen so that $a^T = (0, 0)$.*

[6] Note that the game g is transversal. For 2-person games the best reply functions at a Nash equilibrium are transversal if and only if $\beta_1\beta_2 \neq 1$, or equivalently $C_1 + C_2 \neq 1$.

Let $\ell(\theta a_0)$ be the line in $\mathbb{R}^2$ with slope -1 which intercepts the segment with endpoints a^0 and $-a^0$ at the interior point θa_0. Part (a) of Lemma 6 says that it is possible to find a single-player improvement path that starts at a^0 and reaches a point a^T on the line $\ell(\theta a_0)$ after T steps. Part (b) says that if the line goes through the origin we can choose $a^T = (0,0)$; that is, there is a single-player improvement path from a^0 to the Nash equilibrium $(0,0)$. Lemma 6 is used in the proof of the following theorem.

Theorem 5. *Almost all transversal, n-person, aggregative games have the weak finite improvement property.*

We will use Theorem 5 to show that in any sufficiently fine, finite discretization of almost all continuous, aggregative games, the better-reply dynamics converges in finite time to a point arbitrarily close to a Nash equilibrium. To do so, we need first to modify the better reply dynamics to fit the case in which the strategy space of each player is finite.

Definition 4 (*The better-reply dynamics for finite games*). Consider a finite game g^F. At each discrete time period t there is a status quo action profile a^t. A single player $i \in N$ is randomly selected, with all players having positive selection probability. Player i randomly samples an action $x_i \in A_i^F$, with all the elements of A_i^F having positive probability of being sampled. If $U_i^F(a^t \backslash x_i) > U_i^F(a^t)$ then $a^{t+1} = a^t \backslash x_i$. If $U_i^F(a^t \backslash x_i) \leqslant U_i^F(a^t)$ then $a^{t+1} = a^t$.

We now discretize the strategy sets of a continuous game. So as not to introduce artificial instability, we assume that all the actions corresponding to a Nash equilibrium in the original game are available to the players in the discretized version.

Definition 5. Let $A_i = [\underline{a}_i, \bar{a}_i]$ be the strategy set of player i in the continuous game $g = \langle N, \{A_i\}_{i\in N}, \{U_i\}_{i\in N}\rangle$. We say that a partition $\underline{a}_i = a_{i,0} < a_{i,1} < a_{i,2} < \cdots < a_{i,H} = \bar{a}_i$ of A_i is ε-fine if for all $h = 1, \ldots, H$: $|a_{i,h} - a_{i,h-1}| < \varepsilon$. We call a finite game $g^F = \langle N, \{A_i^F\}_{i\in N}, \{U_i^F\}_{i\in N}\rangle$, where $A^F = A_1^F \times \cdots \times A_n^F$ an ε-fine discretization of the continuous game g if the following properties hold:

(a) Each set A_i^F is an ε-fine partition of A_i.
(b) If $a^* = (a_1^*, \ldots, a_n^*)$ is a Nash equilibrium of g, then $a_i^* \in A_i^F$ for all $i \in N$.
(c) The payoff functions U_i^F of the game g^F are the restrictions of the payoff functions U_i of the game g to the set A^F.

All transversal, continuous games g have a finite number of Nash equilibria and thus admit at least one ε-fine discretization g^F, for any $\varepsilon > 0$. By choosing ε sufficiently small, the finite game g^F can be made arbitrarily close to the continuous game g. We now show that for a sufficiently fine discretization g^F of an aggregative game g, the better-reply dynamics converges in finite time to a Nash equilibrium of g^F which lies within a small distance from a Nash equilibrium of g.

Theorem 6. *For almost all transversal, n-person, aggregative games g and for all $r > 0$ there exists $\varepsilon_0 > 0$ such that if g^F is an ε-fine discretization of g and $0 < \varepsilon < \varepsilon_0$, then g^F has the weak FIP and the Markov process described in Definition* 4 *converges in finite time to a Nash equilibrium a^F of g^F which is contained in a ball of radius r around a Nash equilibrium a^* of g.*

Proof. Theorem 5 shows that, given any $r > 0$, each trajectory a^t of the dynamical system (7) can be replaced by a finite sequence of single-player improvements leading first inside a ball of radius r around a Nash equilibrium a^* of g and then into a^*. This implies that, given r, if ε is sufficiently small (i.e., $\varepsilon < \varepsilon_0$), then any ε-fine discretization g^F of g has the property that starting from all $a^0 \in A^F$ there is a finite sequence of single-player improvements leading inside a ball of radius r around a Nash equilibrium of g. Observe, however, that inside the ball there may be Nash equilibria of the discretized game g^F that are not Nash equilibria of g. For example, in Fig. 1 the profile N is not a Nash equilibrium of the original game, but it is an equilibrium of an ε-fine discretization. We can nevertheless conclude that g^F has the weak finite improvement property. It follows that, starting from any state $a^0 \in A^F$, almost all paths of the stochastic process described in Definition 4 will reach a Nash equilibrium of g^F in finite time. □

Comparing Theorem 6 with Theorem 1 reveals that global convergence to Nash equilibrium requires less stringent conditions when the state space is discrete than when it is continuous (actions need not be strategic complements or substitutes at any Nash equilibrium). This may seem puzzling. In particular, the convergence result of Theorem 6 for a discrete state space, and the non-convergence of the example in Section 4 with a continuous state space may seem in conflict. However, observe that we can use the proof of Theorem 3

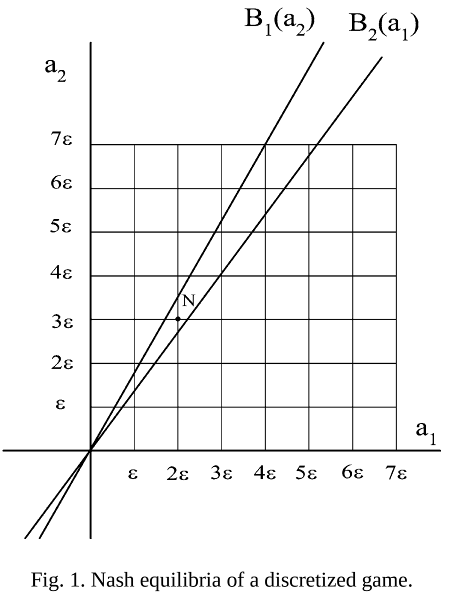

Fig. 1. Nash equilibria of a discretized game.

to show that in an ε-fine discretization of the game g^E described in Section 4, the average time that the better-reply dynamics of Definition 5 takes to converge to the Nash equilibrium goes to infinity as ε goes to zero. Thus, there is really no conflict between Theorems 6 and 3.

6. Conclusions

We have studied the global convergence properties of a stochastic adjustment process, the better-reply dynamics, in which at each discrete point in time a player is randomly selected to sample one of her available actions. The player only changes her current action if the sampled action improves her payoff.

Our convergence results are considerably stronger than existing results on the convergence of the deterministic best-reply dynamics. Gabay and Moulin (1980) and Moulin (1984) (see also Moulin, 1986) showed that the deterministic, discrete-time, best-reply dynamics globally converges to the unique Nash equilibrium if players' payoff functions are strictly concave and an additional condition on the second derivatives of the payoff function is satisfied. This condition requires that the sum of the absolute values of the cross partial derivatives of a player's payoff function with respect to her own action and the other players' actions is less than the absolute value of the second derivative of the player's payoff function with respect to her own action. Al-Nowaihi and Levine (1985) proved global convergence to the unique Nash equilibrium for the continuous-time version of the best-reply dynamics of the homogeneous-product, Cournot model when the difference between price and marginal cost is a decreasing function of the firm's output, the best-reply functions have negative slope everywhere and there are at most 5 firms (Al-Nowaihi and Levine show that the claim made by Hahn, 1962 and Okuguchi, 1964 that this result holds for any number of firms is incorrect). Dastidar (2000) showed that if there is a unique Cournot equilibrium, then the equilibrium is locally stable under fairly general conditions. Vives (1990) observed that a result of Hirsch (1985) implies that the continuous-time, best-reply dynamics globally converges to a Nash equilibrium if the signs of the partial derivatives of the best-reply functions of all players are positive everywhere. Thorlund-Petersen (1990) studied a variant of the deterministic, discrete-time, best-reply dynamics of the Cournot model, in which players best reply to the time average of the total output of their opponents, rather than to their current total output. This dynamics is analogous to the process known as fictitious play in finite games (see Fudenberg and Levine, 1998 for a survey of results on fictitious play in finite games). Thorlund-Petersen (1990) showed that if the difference between price and marginal cost is a decreasing function of the firm's output and the best-reply functions have negative slope everywhere, then his dynamics globally converges to the unique Nash equilibrium, independently of the number of firms.

We have shown that when the action space is continuous, global convergence to a Nash equilibrium in an aggregative game occurs provided that the actions of all players are either locally strategic complements or locally strategic substitutes at all Nash equilibria that are stable under the deterministic, continuous-time, adjusted best-reply dynamics defined by Eq. (7). We used an example to show that if the slopes of the best-reply functions at a Nash equilibrium have different signs, then the better-reply dynamics may not converge to a Nash equilibrium. We also showed that in any discretization of a continuous aggregative game the better-reply dynamics converges to an action profile that is close to a Nash equilibrium of the original game.

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Appendix A

Proof of Lemma 1. If $B_i(\Sigma_{-i}) = a_i$ then (a) holds. Consider any $a \in A$ with $B_i(\Sigma_{-i}) > a_i$. Condition (3) guarantees that for any given $\Sigma_{-i} = \sum_{j \neq i} a_j$, if $B_i(\Sigma_{-i}) > a_i$, then $D_i(a_i, a_i + \Sigma_{-i}) = D_i(a_i, \Sigma) \geqslant 0$. Equation (5) implies that for any given Σ, the function $D_i(a_i, \Sigma)$ is decreasing in a_i. So, if $M_i(\Sigma) = M_i(a_i + \Sigma_{-i})$ were less than a_i, it would follow that $D_i(M_i(\Sigma), \Sigma) > D_i(a_i, \Sigma) \geqslant 0$. This implies that Eq. (6) does not have a solution, and that $M_i(\Sigma)$ is equal to the right endpoint of the interval A_i. Thus, $M_i(\Sigma) \geqslant a_i$, which is a contradiction. It follows that $a_i \leqslant M_i(\Sigma)$. A similar argument can be made for the case $a_i > B_i(\Sigma_{-i})$. This concludes the proof of part (a).

By (2) and (6), if $B_i(\Sigma_{-i}) = a_i$, then it must also be $M_i(\Sigma) = a_i$, since for all a_{-i}:

$$M_i(B_i(\Sigma_{-i}) + \Sigma_{-i}) = B_i(\Sigma_{-i}),$$

while if $M_i(\Sigma) = a_i$, then $B_i(\Sigma_{-i}) = a_i$, since for all Σ

$$B_i\big(\Sigma - M_i(\Sigma)\big) = M_i(\Sigma).$$

This concludes the proof of part (b). □

Proof of Lemma 2. If we sum the system (7) we obtain a differential equation for Σ. Namely:

$$\dot{\Sigma} = \sum_{i=1}^{n} M_i(\Sigma) - \Sigma.$$

This is a single ordinary differential equation satisfying a Lipschitz condition. Because one dimensional autonomous equations cannot exhibit oscillations, it follows that given any initial condition Σ^0 the trajectory Σ^t is monotonic. Since Σ^t is also bounded, it follows that there is $\Sigma^\infty(\Sigma^0) \in A_\Sigma$ such that

$$\Sigma^\infty\big(\Sigma^0\big) = \lim_{t\to\infty} \Sigma^t.$$

This implies that for any $i \in N$: $M_i(\Sigma^t) \to M_i(\Sigma^\infty(\Sigma^0))$ as $t \to \infty$. Hence, for large t the system (7) with initial condition $a^0 \in A$, such that $\sum_{i=1}^n a_i^0 = \Sigma^0$, becomes:

$$\dot{a}_i = M_i\big(\Sigma^\infty\big(\Sigma^0\big)\big) - a_i + h_i\big(t, \Sigma^0\big), \quad i = 1, 2, \ldots, n,$$

where the functions h_i satisfy $h_i(t, \Sigma^0) \to 0$, as $t \to \infty$. This immediately yields that for all $i \in N$, $a_i^t \to M_i(\Sigma^\infty(\Sigma^0))$ as $t \to \infty$; that is, the ω-limit set of the system (7) with initial condition a^0 contains a single element $a^* \in A$ and for all $i \in N$, $a_i^* = M_i(\Sigma^*)$, where $\Sigma^* = \sum_{i=1}^n a_i^*$. Lemma 1 part (b) then implies $a_i^* = B_i(\Sigma_{-i}^*)$ for all $i \in N$; that is, a^* is a Nash equilibrium of g. $\square$

Proof of Lemma 3. Since the game g is transversal, it follows that the set of Nash equilibria is finite. Take $r > 0$ and any $a^0 \in A$. If a^0 is not already contained in a ball of radius r around a Nash equilibrium, consider the dynamics (7) for the game g with initial condition a^0. By Lemma 2, there is a time $T > 0$ such that for all $t > T$, the trajectory a^t lies in some neighborhood V_r contained in a ball of radius r around a Nash equilibrium a^*. The only possible instance in which a^* is not an isolated, asymptotically stable equilibrium of (7) is if (7) has a stable manifold converging to an unstable Nash equilibrium, and a^0 belongs to such a manifold (i.e., it belongs to a trajectory converging to an unstable equilibrium).[7] In such an instance a small deviation (say by player i) from the trajectory a^t leads to another trajectory that converges to an asymptotically stable, isolated equilibrium of (7). Since, as we shall see below, we can always replace the continuous dynamics with a finite sequence of single-player improvements, it is always possible to find a single-player improvement that leads away from a trajectory belonging to a stable manifold of an unstable equilibrium. Hence it is always possible to reach a neighborhood V_r contained in a ball of radius r around an isolated, asymptotically stable equilibrium of (7).

To replace the continuous dynamic with a finite sequence of single-player improvements, we will use a simple Euler scheme to approximate the integral curve a^t. Take an integer $Z \in \mathbb{N}$ and consider the Z-th approximation $\hat{a}^t$ of a^t defined as follows:

$$\begin{aligned} t_z &= \frac{z}{Z}T, \quad z = 0, 1, \ldots, Z, \\ \hat{a}^0 &= a^0, \\ \hat{a}^{t_{z+1}} &= \hat{a}^{t_z} + \frac{T}{Z}\left[M\left(\sum_{i=1}^n \hat{a}_i^{t_z}\right) - \hat{a}^{t_z}\right], \quad z = 0, 1, \ldots, Z-1, \\ \hat{a}^t &= \hat{a}^{t_z}, \quad \text{for } t_z \leqslant t < t_{z+1}. \end{aligned} \tag{A.1}$$

As $Z \to \infty$, $\hat{a}^t$ converges to a^t uniformly on $[0, T]$. In particular there is a Z large enough for which $\hat{a}^T \in V_r$. If we show that for each $z = 0, 1, 2, \ldots Z$ there is a finite single-player improvement path from $\hat{a}^{t_z}$ to $\hat{a}^{t_{z+1}}$ then we are done, because the existence of a path from $\hat{a}^0 = a^0$ to $\hat{a}^{t_Z} \in V_r$ follows.

To show that there is an improvement path from $\hat{a}^{t_z}$ to $\hat{a}^{t_{z+1}}$, take:

$$\begin{aligned} y^0 &= \hat{a}^{t_z}, \\ y^h &= y^{h-1} + e_h \frac{T}{Z}\left[M_h\left(\sum_{i=1}^n \hat{a}_i^{t_z}\right) - \hat{a}^{t_z}\right], \quad h = 1, 2, \ldots, n. \end{aligned}$$

Here e_h is the n-dimensional vector whose hth element is 1 and all other elements are zero. Since M_h is a continuous function, Z can be chosen large enough to make

[7] A stable manifold of an unstable equilibrium a^*, if one exists, has the property that for all points a in the manifold $\sum_{i=1}^n a_i = \sum_{i=1}^n a_i^* = \Sigma^*$.

$M_h(\sum_{i=1}^n \hat{a}_i^{t_z})$ as close as desired to $M_h(\sum_{i=1}^n y_i^h)$ and $\hat{a}^{t_z}$ as close as desired to $\hat{a}^{t_{z+1}}$. This implies that

$$\left[M_h\left(\sum_{i=1}^n \hat{a}_i^{t_z}\right) - \hat{a}_h^{t_z}\right]\left[M_h\left(\sum_{i=1}^n y_i^h\right) - y_h^h\right] > 0.$$

By Lemma 1, we have

$$\left[M_h\left(\sum_{i=1}^n y_i^h\right) - y_h^h\right]\left[B_h\left(\sum_{i\neq h} y_i^h\right) - y_h^h\right] > 0.$$

This implies that the move from y^{h-1} to y^h by player h is in the direction of his best reply; that is, it is a single-player improvement. This completes the proof. □

We now present a useful lemma. Consider the quadratic form γ defined for any $a \in \mathbb{R}^n$ and for $\beta_i \neq 0$, $i = 1, \ldots, n$, as follows:

$$\gamma(a) = \sum_{i=1}^n \frac{1}{\beta_i} a_i^2 - 2\left(\sum_{1\leqslant i<j\leqslant n} a_i a_j\right) = aQa^T \tag{A.2}$$

where Q is the following matrix:

$$Q = \begin{bmatrix} \frac{1}{\beta_1} & -1 & \ldots & -1 \\ -1 & \ddots & \ldots & -1 \\ \vdots & \vdots & \vdots & \vdots \\ -1 & \ldots & -1 & \frac{1}{\beta_n} \end{bmatrix}.$$

Lemma 7. (a) *The function $\gamma : \mathbb{R}^n \to \mathbb{R}$ is bounded from below with greatest lower bound equal to* 0 *if and only if $\beta_i > 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^n \beta_i/(1+\beta_i) \leqslant 1$. If the second inequality is strict, then γ has a unique global minimum at* $(0, 0, \ldots, 0)$.

(b) *If $0 > \beta_i \geqslant -1$, for $i = 1, \ldots, n$, then the function γ is bounded from above with least upper bound equal to* 0. *If the second inequality is strict, then γ has a unique global maximum at* $(0, 0, \ldots, 0)$.

(c) *If $n = 2$, then γ is bounded from above with least upper bound equal to* 0 *if and only if $0 > \beta_i$, $i = 1, 2$ and $\beta_1\beta_2 \leqslant 1$. If the second inequality is strict, then γ has a unique global maximum at* $(0, 0)$.

Proof. Parts (a) and (b) follow from Theorem 2 of Al-Nowaihi and Levine (1985). Part (c) follows from the definition of negative semi-definiteness of a 2×2 matrix. □

We are now ready to prove Lemma 4 and Theorem 2.

Proof of Lemma 4. Consider a Nash equilibrium of g. By changing coordinates we can assume, without loss of generality, that this equilibrium is at the point $(0, 0, \ldots, 0) \in \mathbb{R}^n$. For action profiles a sufficiently close to the equilibrium, we can linearize the best-reply function B_i of each player and write $B_i(\Sigma_{-i}) = \beta_i \Sigma_{-i}$. Thus, a move by player i from

$a = (a_1, a_2, \dots, a_n)$ to $\hat{a} = (a_1, a_2, \dots, a_{i-1}, \hat{a}_i, a_{i+1}, \dots, a_n)$ is payoff improving if and only if

$$|\hat{a}_i - \beta_i \Sigma_{-i}| < |a_i - \beta_i \Sigma_{-i}|.$$

Geometrically, this means that to improve her payoff player i must move to a point on the line segment parallel to the vector $e_i = (0, 0, \dots, 1, 0, \dots, 0)$ (with 1 in the i-th position), with one endpoint at $a = (a_i, a_{-i})$, the middle point at $(\beta_i \sum_{j \neq i} a_j, a_{-i})$ and the other endpoint at $b = (b_i, a_{-i}) = (2\beta_i \sum_{j \neq i} a_j - a_i, a_{-i})$. Consider now the function γ defined in Eq. (A.2), assuming $\beta_i \neq 0$; we claim that $\gamma(a) = \gamma(b)$. We will check this claim for $i = 1$; we have:

$$\begin{aligned}\gamma(b) - \gamma(a) &= \frac{1}{\beta_1}\left[(2\beta_1 \Sigma_{-1} - a_1)^2 - a_1^2\right] - 2\sum_{j=2}^{n}\left[(2\beta_1 \Sigma_{-1} - a_1)a_j - a_1 a_j\right] \\ &= 4\beta_1 \Sigma_{-1}^2 - 4a_1 \Sigma_{-1} - 4\beta_1 \Sigma_{-1}^2 + 4a_1 \Sigma_{-1} = 0.\end{aligned}$$

Now suppose we are in case (a) in the statement of the lemma; that is, $\beta_i > 0$ for all $i = 1, \dots, n$, and $\sum_{i=1}^{n} \beta_i/(1 + \beta_i) < 1$. Then, for any given a_{-i}, the function $a_i \mapsto \gamma(a)$ is quadratic in a_i and goes to $+\infty$ as $|a_i| \to \infty$. Hence, since $\gamma(a) = \gamma(b)$, for all points d on the line segment connecting a and b we have $\gamma(d) \leqslant \gamma(a)$ with strict inequality if d is inside the segment; a single-player improvement by any player i reduces the value of γ.

We now show that there exists a neighborhood V of the Nash equilibrium $(0, \dots, 0)$ such that if $a^0 \in V$ then almost all paths $a^0, a^1, a^2, a^3, \dots$ generated by the stochastic process described in Definition 2 stay all the time in V. Moreover ,

$$\lim_{t \to \infty} a^t = (0, 0, \dots, 0).$$

Let the neighborhood V be given by $\{x\colon\ \gamma(x) < c\}$ for some $c > 0$. It follows that any infinite path $\{a^t\}_{t=0}^{\infty}$ of the stochastic process with $a^0 \in V$ is associated with a nonincreasing sequence of real numbers $\gamma(a^0), \gamma(a^1), \gamma(a^2), \gamma(a^3), \dots$ and thus $a^t \in V$, for all t. Note that in this sequence we have infinitely many times a strict inequality, since if $a^t \neq (0, 0, \dots, 0)$ then there is a positive probability that one of the players samples a strategy that improves his payoff, hence for some $n \geqslant t$, $\gamma(a^{n+1}) < \gamma(a^n)$. By Lemma 7 the sequence $\gamma(a^0), \gamma(a^1), \dots$ is bounded from below, since the function γ reaches its strict, global minimum at $(0, 0)$. Hence it must converge.

To see that $\lim_{t \to \infty} a^t = (0, 0, \dots, 0)$, assume to the contrary that there exists a subsequence $\{a^h\}_{h=0}^{\infty}$ of $\{a^t\}_{t=0}^{\infty}$ with $\lim_{h \to \infty} a^h = a$, with $\gamma(a) = m > 0$. Since $a \neq (0, 0, \dots, 0)$, there is $p > 0$ and $\varepsilon > 0$ such that with probability of at least p the stochastic process moves from a to a point b for which $\gamma(b) < \gamma(a) - \varepsilon$. Because of the continuous nature of the game, it must also be true that with probability of at least p the stochastic process moves from a^h to b, where a^h is any point on the path converging to a that is sufficiently close to a. It follows that the probability that the function γ stays above m along a path of the stochastic process is zero. Since this is true for any $m > 0$, for almost any path γ goes to zero and therefore $\lim_{t \to \infty} a^t = (0, 0, \dots, 0)$.

In case (b) in the statement of the theorem, first note that (4) and (5) imply that $\beta_i > -1$ for all i. Then we can apply Lemma 7 and the proof is similar to the proof of (a), except that we need to use the function $-\gamma$ in place of γ. $\square$

Proof of Theorem 2. Consider the best-reply dynamics:

$$\begin{aligned}\dot{a}_1 &= B_1(a_2) - a_1,\\ \dot{a}_2 &= B_2(a_1) - a_2,\end{aligned} \tag{A.3}$$

with initial condition $(a_1^0, a_2^0) = a^0$. It follows from Liouville's theorem (see Corchon and Mas-Colell, 1996) that the ω-limit set of every solution of this system is a Nash equilibrium a^* of g and that for almost any initial condition (a_1^0, a_2^0) the point in the ω-limit set is a stable equilibrium (a stable manifold of an unstable equilibrium is at most one dimensional).[8,9] By approximating the system (A.3) with an Euler scheme as in the proof of Lemma 3, we can then argue that for any neighborhood V of a^* there is a positive probability $p > 0$ that the stochastic path $a^0, a^1, a^2, \ldots$ generated by the better-reply dynamics will eventually end up and stay in V. If by W we denote the union of a set of small neighborhoods of all stable equilibria of the system (A.3), then there exists an integer k and a number $p > 0$ such that regardless of our initial position the path $a^0, a^1, a^2, \ldots, a^k$ generated by the better-reply dynamics leads to W with probability at least p. Once a path is in W, it stays there indefinitely. On the other hand if $a^k \notin W$ then again with probability at least p the path $a^k, a^{k+1}, a^{k+2}, \ldots, a^{2k}$ leads to W. It follows that eventually almost every path ends in W.

We have shown that the better-reply dynamics leads with probability one to an arbitrarily small neighborhood of a stable equilibrium a^* of the system (A.3). We now show that a^* is a stable equilibrium of (A.3) if and only if $B_1'(a_2^*)/(1 + B_1'(a_2^*)) + B_2'(a_1^*)/(1 + B_2'(a_1^*)) < 1$ (which is equivalent to a^* being asymptotically stable under the adjusted best-reply dynamics (7)). To see this, consider the linearization of (A.3) around a^*:

$$\begin{aligned}\dot{a}_1 &= B_1'\big(a_2^*\big)a_2 - a_1,\\ \dot{a}_2 &= B_2'\big(a_1^*\big)a_1 - a_2.\end{aligned} \tag{A.4}$$

The stability of a^* under (A.3) implies that the real parts of the eigenvalues of the linearized system (A.4) must be non-positive. The characteristic equation of (A.4) is $(1 + \lambda)^2 = B_1'(a_2^*)B_2'(a_1^*)$ and thus the linearized system has a zero eigenvalue, $\lambda = 0$, if and only if $B_1'(a_2^*)B_2'(a_1^*) = 1$, which is ruled out by the assumption that the game g is transversal ($B_1'(a_2^*)B_2'(a_1^*) \neq 1$ at all Nash equilibria). Thus, the equilibrium a^* of (A.3) is stable if and only if $B_1'(a_2^*)B_2'(a_1^*) < 1$. Finally note that $B_1'(a_2^*)B_2'(a_1^*) < 1$ is equivalent to $B_1'(a_2^*)/(1 + B_1'(a_2^*)) + B_2'(a_1^*)/(1 + B_2'(a_1^*)) < 1$.

Now suppose that the inequality $B_1'(a_2^*)B_2'(a_1^*) > 0$ also holds. Then Lemma 4 applies and for almost any path $a^0, a^1, \ldots$ in a small neighborhood of a^* generated by the better-reply dynamics described in Definition 2 we have $\lim_{t\to\infty} a^t = a^*$. This concludes the proof. □

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[8] The equilibrium a^* is stable if for every neighborhood V of a^* there is a neighborhood $V' \subset V$ of a^* such that every trajectory a^t with a^0 in V' is defined and in V for all $t > 0$.

[9] Corchon and Mas-Colell (1996) also showed that with more than two players the best-reply dynamics need not converge; there are games with payoff functions that yield chaotic dynamics (e.g., if the differential equations are Lorenz's equations; see Guckenheimer and Holmes, 1983).

Proof of Lemma 5. To prove this lemma it is sufficient to prove that if the inequalities (8) and (9) are satisfied on the interval $(0,\infty)$, then

$$\lim_{k\to\infty} P\left(\frac{\rho^{t^0+k+1}}{\rho^{t^0}} > 2^{\sqrt{k}}\right) = 1 \quad \text{for all } t^0 \in \mathbb{N} \tag{A.5}$$

and thus $\lim_{t\to\infty} \rho^t = \infty$.

By the central limit theorem for the binomial distribution (e.g., see Billingsley, 1986), if p is the probability of a random event, k is the number of independent draws, and X is the random variable that counts the occurrence of the event, then

$$\frac{X - kp}{\sqrt{kp(1-p)}} \sim N(0,1) \quad \text{as } k \to \infty \tag{A.6}$$

where $N(0,1)$ is the standard normal distribution with distribution function

$$\Phi(Y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{Y} \exp\left\{-\frac{s^2}{2}\right\} \mathrm{d}s. \tag{A.7}$$

It is convenient to take $k = 2^{4K}$ for some integer $K > 0$. Suppose that the random event is $\rho^{t+1}/\rho^t \geqslant 2^{2^4}$. By (8) the probability p of this event is greater than $1/4$; then we have

$$P\left(\left\{\#t\colon \frac{\rho^{t+1}}{\rho^t} \geqslant 2^{2^4}\right\} < 2^{4K-2}\right) = \Phi\left(\frac{2^{4K-2} - 2^{4K}p}{\sqrt{2^{4K}p(1-p)}}\right) \to 0 \quad \text{as } K \to \infty. \tag{A.8}$$

By (9), the probability p of the random event $\rho^{t+1}/\rho^t \in [1/2^{m+1}, 1/2^m]$ is less than $1/2^{2m}$, for $m = 1, \ldots, 3K/2 - 1$, and hence

$$\begin{aligned} &P\left(\left\{\#t\colon \frac{\rho^{t+1}}{\rho^t} \in \left[\frac{1}{2^{m+1}}, \frac{1}{2^m}\right]\right\} \leqslant \frac{2^{4K+1}}{2^{2m}}\right) \\ &\quad = \Phi\left(\frac{2^{4K+1-2m} - 2^{4K}p}{\sqrt{2^{4K}p(1-p)}}\right) \to 1 \quad \text{as } K \to \infty. \end{aligned} \tag{A.9}$$

Similarly, the probability p of the random event $\rho^{t+1}/\rho^t \in [1/2^{5K/2}, 1/2^{3K/2}]$ is

$$p < P\left(\frac{\rho^{t+1}}{\rho^t} \leqslant \frac{1}{2^{3K/2}}\right) < \frac{1}{2^{3K}}$$

and by (A.6) and (A.7) we have

$$\begin{aligned} &P\left(\left\{\#t\colon \frac{\rho^{t+1}}{\rho^t} \in \left[\frac{1}{2^{5K/2}}, \frac{1}{2^{3K/2}}\right]\right\} > 2^{K+1}\right) \\ &\quad = 1 - \Phi\left(\frac{2^{K+1} - 2^{4K}p}{\sqrt{2^{4K}p(1-p)}}\right) \to 0 \quad \text{as } K \to \infty. \end{aligned} \tag{A.10}$$

Finally, the probability p of the random event $\rho^{t+1}/\rho^t \geqslant 1/2^{5K/2}$ is

$$p = P\left(\frac{\rho^{t+1}}{\rho^t} \geqslant \frac{1}{2^{5K/2}}\right) > \frac{2^{5K} - 1}{2^{5K}},$$

and thus

$$P\left(\left\{\#t\colon \frac{\rho^{t+1}}{\rho^t} < \frac{1}{2^{5K/2}}\right\} = 0\right) = P\left(\left\{\#t\colon \frac{\rho^{t+1}}{\rho^t} \geqslant \frac{1}{2^{5K/2}}\right\} > 2^{4K} - 1\right)$$
$$= 1 - \Phi\left(\frac{2^{4K}(1-p) - 1}{\sqrt{2^{4K}p(1-p)}}\right) \to 1 \quad \text{as } K \to \infty. \tag{A.11}$$

Equation (A.8) says that as $K \to \infty$, for almost all paths $\rho^0, \rho^1, \ldots, \rho^{2^{4K}}$ the number of indices $t \in \{1, 2, \ldots, 2^{4K}\}$ for which $\rho^{t+1}/\rho^t \geqslant 2^{2^4}$ is at least 2^{4K-2}. Clearly, since $k = 2^{4K}$, the number of indices t for which $\rho^{t+1}/\rho^t \in [1/2, 1]$ is at most 2^{4K}. As $K \to \infty$, by (A.9) the number of indices t for which $\rho^{t+1}/\rho^t \in [1/2^{m+1}, 1/2^m]$, $m = 1, 2, \ldots, 3K/2-1$, is at most $2^{4K+1-2m}$, and by Eq. (A.10) the number of indices t for which $\rho^{t+1}/\rho^t \in [1/2^{5K/2}, 1/2^{3K/2}]$ is at most 2^{K+1}. Finally, by Eq. (A.11), as $K \to \infty$ the number of indices t for which $\rho^{t+1}/\rho^t < 1/2^{5K/2}$ is zero. Then we can estimate that for a sufficiently large K:

$$\frac{\rho^{t^0+2^{4K}+1}}{\rho^{t^0}} \simeq \prod_{t=t^0}^{t^0+2^{4K}} \frac{\rho^{t+1}}{\rho^t}$$
$$\geqslant \left(2^{2^4}\right)^{2^{4K-2}} 2^{-2^{4K}} \prod_{m=1}^{3K/2-1} \left(\frac{1}{2^{m+1}}\right)^{2^{4K+1-2m}} \left(\frac{1}{2^{5K/2}}\right)^{2^{K+1}}. \tag{A.12}$$

To evaluate the product on the right-hand side of (A.12) we first use logarithms to change the product into a sum:

$$\ln\left(\prod_{m=1}^{3K/2-1} \left(\frac{1}{2^{m+1}}\right)^{2^{4K+1-2m}}\right)$$
$$= 2^{4K+1}\left(\sum_{m=1}^{3K/2-1} 2^{-2m}(-m-1)\right)\ln 2$$
$$> -2^{4K+1}\left(\sum_{m=1}^{\infty} \frac{m+1}{2^{2m}}\right)\ln 2 \geqslant -2^{4K+1}\ln 2 \tag{A.13}$$

where the second inequality follows from

$$\sum_{m=1}^{\infty} \frac{m+1}{2^{2m}} < \sum_{m=1}^{\infty} \frac{2^m}{2^{2m}} = \sum_{m=0}^{\infty} \frac{1}{2^m} - 1 = 1.$$

Using inequality (A.13) in Eq. (A.12) gives:

$$\frac{\rho^{t^0+2^{4K}+1}}{\rho^{t^0}} \geqslant \left(2^{2^4}\right)^{2^{4K-2}} 2^{-2^{4K}} 2^{-2^{4K+1}} \left(2^{-5K/2}\right)^{2^{K+1}} = 2^{2^{4K+2}} 2^{-(3)2^{4K}} 2^{-(5K)2^K}$$
$$= 2^{-(5K)2^K} 2^{2^{4K}} = 2^{2^{2K}(2^{2K} - 5K2^{-K})} > 2^{2^{2K}} = 2^{\sqrt{2^{4K}}}.$$

This is exactly what we claimed in (A.5). Hence the lemma follows. □

Proof of Theorem 3. Denote with $a^{d(t)}$, with $d(t) > t$, the first action profile different from a^t in a path of the stochastic process described in Definition 2, and with $P(a^{d(t)} \in S \subset A \mid a^t = a)$ the probability that $a^{d(t)}$ belongs to the set S given that $a = a^t$ (i.e., a is the status quo at time t). Define the distance $\rho^*(a)$ of a point $a = (a_1, a_2) \in [-2, 2] \times [-2, 2]$ from the origin $(0, 0)$ by

$$\rho^*(a) = \max\{|a_1|, |a_2|\}. \tag{A.14}$$

By letting $\rho^t = \rho^*(a^t)$ and $\rho^{t+1} = \rho^*(a^{d(t)})$, we can define a new process that keeps track of the evolution of the distance ρ^*. Let $A(r) = \{a \in A\colon\ \rho^*(a) = r\}$, $A^-(r) = \{a \in A\colon\ \rho^*(a) \leqslant r\}$ and $A^+(r) = \{a \in A\colon\ \rho^*(a) \geqslant r\}$. Define $\lambda^{A(r)}$ as the Lebesgue (or uniform) probability measure over $A^{(r)}$; then we have

$$P\left(\rho^{t+1} \leqslant s \mid \rho^t = r\right) = \int\limits_{A(r)} P\left(a^{d(t)} \in A^-(s) \mid a^t = a\right) \mathrm{d}\lambda^{A(r)},$$

$$P\left(\rho^{t+1} \geqslant s \mid \rho^t = r\right) = \int\limits_{A(r)} P\left(a^{d(t)} \in A^+(s) \mid a^t = a\right) \mathrm{d}\lambda^{A(r)}.$$

Let $V = \{a\colon\ \rho^*(a) < \varepsilon^V\}$ be a small neighborhood of the point $(0, 0)$. To prove our claim about nonconvergence we need to show that almost all paths starting in V leave V; that is, there is T for which $a^T \notin V$. To establish this we will show that the probability law of the process governing the evolution of the distance ρ^* satisfies the inequalities in Lemma 5.

First note that we can take $\varepsilon^V > 0$ small enough so that in the neighborhood V around $(0, 0)$ the game has linear best-reply functions. Second, recall that if a is the prevailing strategy profile, then the strategies that improve player 1's payoff are the strategies belonging to the interval $I_1(a)$ with endpoints a_1 and $-2\beta_1 a_2 - a_1$, while the strategies improving player 2's payoff are the ones in the interval $I_2(a)$ with endpoints a_2 and $2\beta_2 a_1 - a_2$.

We will begin by showing that in V we have $P(\rho^{t+1} \geqslant 2^{2^4} r \mid \rho^t = r) > 1/4$ for all r with $\varepsilon^V \geqslant r > 0$ and hence Eq. (8) holds. Suppose $\rho^t = r$, or equivalently $a^t \in A(r)$. Let $A_V(r) = \{a \in A(r)\colon\ |a_1| = r\}$ be the vertical sides of the square $A(r)$ in $\mathbb{R}^2$ (see Fig. 2) and $\lambda^{A_V(r)}$ be the Lebesgue (or uniform) probability measure over $A_V^{(r)}$. Observe that

$$P\left(\rho^{t+1} \geqslant 2^{2^4} r \mid \rho^t = r\right) \geqslant \frac{1}{2} \int\limits_{A_V(r)} P\left(a^{d(t)} \in A^+\left(2^{2^4} r\right) \,\middle|\, a^t = a\right) \mathrm{d}\lambda^{A_V(r)}.$$

We will show that for $\beta_2 > 2^{2^4} + 1 + \beta_1$ it is $P(a^{d(t)} \in A^+(2^{2^4} r) \mid a^t \in A_V(r)) > 1/2$ and hence $P(\rho^{t+1} \geqslant 2^{2^4} r \mid \rho^t = r) > 1/4$. For $\beta_2 > 2^{2^4} + 1 + \beta_1$ we have

$$\begin{aligned} &P\left(a^{d(t)} \in A^+\left(2^{2^4} r\right) \,\middle|\, a^t \in A_V(r)\right) \\ &\quad > \frac{2\beta_2 r - a_2^t - 2^{2^4} r}{2\beta_2 r - 2a_2^t} \cdot \frac{2\beta_2 r - 2a_2^t}{2\beta_2 r - 2a_2^t + |2\beta_1 a_2^t + 2r|} \end{aligned}$$

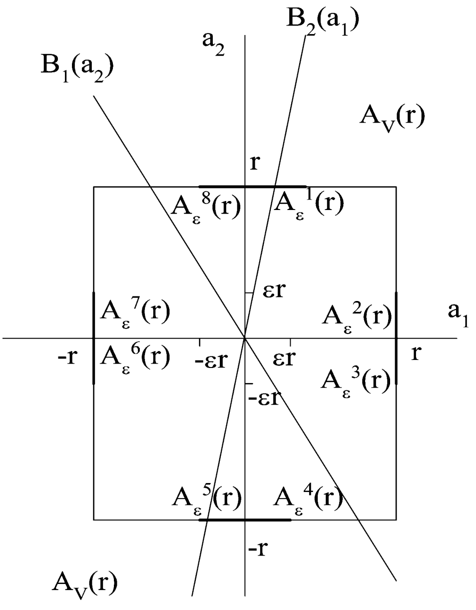

Fig. 2. The set $A(r) = \{a \in A\colon\ \rho(a) = r\}$.

where the first term is the probability that $a_2^{d(t)} > 2^{2^4} r$ given that player 2 is the first to move, and the second term is the probability that player 2 is the first to move. It follows that $P(a^{d(t)} \in A^+(2^{2^4} r) \mid a^t \in A_V(r)) > 1/2$ provided that

$$2\big(2\beta_2 r - a_2^t - 2^{2^4} r\big) > 2\beta_2 r - 2a_2^t + \big|2\beta_1 a_2^t + 2r\big|, \quad \text{or}$$
$$\beta_2 r > 2^{2^4} r + \big|\beta_1 a_2^t + r\big|$$

which holds if $\beta_2 > 2^{2^4} + 1 + \beta_1$. This completes the proof that in V Eq. (8) holds.

It remains to show that Eq. (9) also holds in V. Take $a^t \in A(r)$, so that $\rho^t = r$. We need to show that $P(\rho^{t+1} \leqslant \varepsilon r \mid \rho^t = r) < \varepsilon^2$ for all $\varepsilon^V \geqslant r > 0$ and all $1 \geqslant \varepsilon > 0$. The only way the distance from the origin can decrease rapidly, that is $\rho^{t+1} \leqslant \varepsilon r$, is if $|a_i^t| \leqslant \varepsilon r$ for some $i = 1, 2$, and player $j \neq i$ moves to $a_j^{d(t)}$ with $|a_j^{d(t)}| \leqslant |a_i^t|$. Let (see Fig. 2)

$$A_\varepsilon^1(r) = \big\{a \in A(r)\colon\ 0 \leqslant a_1^t \leqslant \varepsilon r \text{ and } a_2^t = r\big\},$$
$$A_\varepsilon^2(r) = \big\{a \in A(r)\colon\ a_1^t = r \text{ and } 0 \leqslant a_2^t \leqslant \varepsilon r\big\},$$
$$A_\varepsilon^3(r) = \big\{a \in A(r)\colon\ a_1^t = r \text{ and } -\varepsilon r \leqslant a_2^t \leqslant 0\big\},$$
$$A_\varepsilon^4(r) = \big\{a \in A(r)\colon\ 0 \leqslant a_1^t \leqslant \varepsilon r \text{ and } a_2^t = -r\big\},$$
$$A_\varepsilon^5(r) = \big\{a \in A(r)\colon\ -\varepsilon r \leqslant a_1^t \leqslant 0 \text{ and } a_2^t = -r\big\},$$
$$A_\varepsilon^6(r) = \big\{a \in A(r)\colon\ a_1^t = -r \text{ and } -\varepsilon r \leqslant a_2^t \leqslant 0\big\},$$
$$A_\varepsilon^7(r) = \big\{a \in A(r)\colon\ a_1^t = -r \text{ and } 0 \leqslant a_2^t \leqslant \varepsilon r\big\},$$
$$A_\varepsilon^8(r) = \big\{a \in A(r)\colon\ -\varepsilon r \leqslant a_1^t \leqslant 0 \text{ and } a_2^t = r\big\}.$$

Let $\lambda^{A^i_\varepsilon(r)}$ be the uniform probability measure over $A^i_\varepsilon(r)$; and observe that

$$P\big(\rho^{t+1} \leqslant \varepsilon r \mid \rho^t = r\big) = \frac{\varepsilon}{8}\sum_{i=1}^{8}\int_{A^i_\varepsilon(r)} P\big(a^{d(t)} \in A^-(\varepsilon r) \mid a^t = a\big)\,\mathrm{d}\lambda^{A^i_\varepsilon(r)}. \tag{A.15}$$

There are four different cases. (1) If either $a \in A^1_\varepsilon(r)$ or $a \in A^5_\varepsilon(r)$, then we have

$$P\big(a^{d(t)} \in A^-(\varepsilon r) \mid a^t = a\big) \leqslant \frac{2\varepsilon r}{(r+\varepsilon r) + (2|a^t_1| + 2\beta_1 r)} < \frac{2\varepsilon}{1+2\beta_1}; \tag{A.16}$$

(2) If either $a \in A^2_\varepsilon(r)$ or $a \in A^6_\varepsilon(r)$, then we have

$$P\big(a^{d(t)} \in A^-(\varepsilon r) \mid a^t = a\big) \leqslant \frac{2\varepsilon r}{|2\beta_2 r - 2|a^t_2|| + (2r + 2\beta_1|a^t_2|)} < \varepsilon; \tag{A.17}$$

(3) If either $a \in A^3_\varepsilon(r)$ or $a \in A^7_\varepsilon(r)$, then we have

$$P\big(a^{d(t)} \in A^-(\varepsilon r) \mid a^t = a\big) \leqslant \frac{2\varepsilon r}{(2\beta_2 r + 2|a^t_2|) + r + \varepsilon r} < \frac{2\varepsilon}{1+2\beta_2}; \tag{A.18}$$

Finally, (4) if either $a \in A^4_\varepsilon(r)$ or $a \in A^8_\varepsilon(r)$, then we have

$$P\big(a^{d(t)} \in A^-(\varepsilon r) \mid a^t = a\big) \leqslant \frac{2\varepsilon r}{(2\beta_2|a^t_1| + 2r) + 2|\beta_1 r - |a^t_1||} < \varepsilon. \tag{A.19}$$

Adding up the left-hand sides of Eqs. (A.16) and (A.18) we obtain

$$\frac{2\varepsilon}{1+2\beta_1} + \frac{2\varepsilon}{1+2\beta_2} < 2\varepsilon \quad \text{for } \beta_2 > \frac{1}{4\beta_1}.$$

Thus, for a sufficiently large β_2 Eq. (A.15) implies that $P(\rho^{t+1} \leqslant \varepsilon r \mid \rho^t = r) < \varepsilon^2$. This completes the proof that Eq. (8) holds in V.

Applying Lemma 5 to the stochastic process governing the evolution of the distance of the state of the system from the origin, we see that for any $(0,0) \neq a^t \in V$ and for a sufficiently large k, $a^{n+k} \notin V$. Hence we cannot have $(0,0) = a^* = \lim_{t\to\infty} a^t$. This concludes the proof. □

Proof of Lemma 6. Without loss of generality we can assume that $C_1 \leqslant C_2$. Also, since we only need to prove that (a) and (b) in the lemma hold for almost all $(C_1, C_2) \in \mathcal{C}$, we can assume that $C_1 \neq C_2$ and $C_1, C_2 \neq 0$. Therefore, there are three distinct cases we have to consider. All cases must satisfy $C_1 + C_2 < 1$.

(1) $0 < C_1 < C_2 < 1$, or, equivalently $0 < \beta_1 < \beta_2 < \infty$.
(2) $C_1 < 0 < C_2 < 1$, or, equivalently $-\infty < \beta_1 < 0 < \beta_2 < \infty$.
(3) $C_1 < C_2 < 0$, or, equivalently $-\infty < \beta_1 < \beta_2 < 0$.

Recall that $1/\beta_1$ is the slope of the best-reply function of the first player and β_2 is the slope of the best-reply function of the second player in the a_1–a_2 plane. The lemma says that, starting from any point a^0, and for any given $\theta \in (-1, 1)$, it is possible to construct a sequence of single-player improvements that reaches some point on the straight line $x_1 + x_2 = \theta(a_1^0 + a_2^0)$.

Case 1. When $0 < C_1 < C_2$, condition $C_1 + C_2 < 1$ is equivalent to $1/\beta_1 > \beta_2$; that is, the slope in the $a_1 - a_2$ plane of player 1's best-reply function is greater than the slope of the best-reply function of player 2. Consider the function γ defined by

$$\gamma(a) = \frac{1}{\beta_1}a_1^2 + \frac{1}{\beta_2}a_2^2 - 2a_1a_2.$$

For any $c > 0$, the set $E = \{a \in \mathbb{R}^2\colon \gamma(a) = c\}$ is an ellipse centered at the origin. Denote by E_+ and E_- the intersections of E with the region that lies between the best-reply functions in the first and third quadrant, respectively. Clearly, E_+ and E_- are symmetric with respect to the origin. Moreover, the slope of the ellipse in the $a_1 - a_2$ plane is zero for the two points on the best-reply function of player 1, and it is infinity for the two points on the best-reply function of player 2 (see Fig. 3). The symmetry of the ellipse, the slope of player 1's best-reply function being greater than the slope of player 2's best-reply function, and the slope of the ellipse being zero or infinity at the intersections with the best-reply functions imply that starting from any $a \in E_+$ we can define two finite sequences $\{a^0, a^1, a^2, \ldots, a^{T_2}\}$ and $\{b^0, b^1, b^2, \ldots, b^{T_1}\}$ of points on the ellipse E with the following properties:

(1) $a^0 = b^0 = (a_1, a_2) \in E_+$, $a^{T_2} \in E_-$, and $b^{T_1} \in E_-$;
(2) $\{a^1, a^2, \ldots, a^{T_2}\} = \{(2\beta_1 a_2 - a_1, a_2),\ (2\beta_1 a_2 - a_1, 2\beta_2(2\beta_1 a_2 - a_1) - a_2),\ \ldots,\ a^{T_2}\}$ and $\{b^1, b^2, \ldots, b^{T_1}\} = \{(a_1, 2\beta_2 a_1 - a_2),\ (2\beta_1(2\beta_2 a_1 - a_2) - a_1, 2\beta_2 a_1 - a_2),\ \ldots,\ b^{T_1}\}$;
(3) $a^h \notin E_+ \cup E_-$ for $h = 1, \ldots, T_2 - 1$ and $b^h \notin E_+ \cup E_-$ for $h = 1, \ldots, T_1 - 1$.

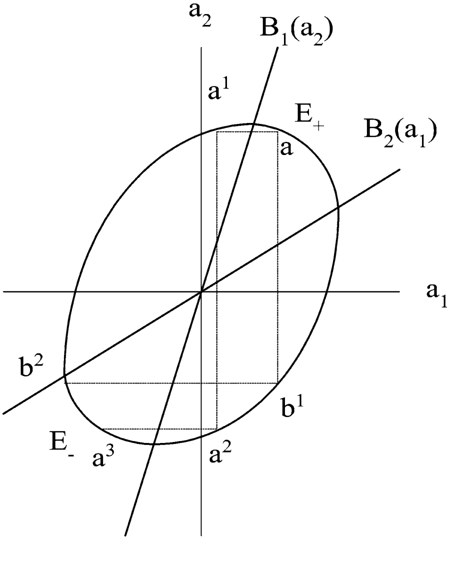

Fig. 3. The ellipse $E = \{a \in \mathbb{R}^2\colon \gamma(a) = c\}$.

Each step in the sequences consists of a payoff neutral change by one of the two players. Players take turns changing action; in the sequence $\{a^0, a^1, a^2, \dots, a^{T_2}\}$ player 1 is the first to change action, in the other sequence the first to change action is player 2.

By letting $\varphi_1(a) = a^{T_1}$ and $\varphi_2(a) = a^{T_2}$ we can define two continuous maps from E_+ into E_-. Consider a smooth parameterization $f_+ : [0,1] \to E_+$ of the arc E_+ of the ellipse E and note that by letting $f_- = -f_+$ we obtain a smooth parametrization of the arc E_-, $f_- : [0,1] \to E_-$. We are now ready to define two continuous maps $\Phi_h : [0,1] \times \{1,2\} \to [0,1] \times \{1,2\}$, $h = 1, 2$ with

$$\Phi_h(t,i) = \bigl(f_-^{-1}\bigl(\varphi_i\bigl(f_+(t)\bigr)\bigr), j(i,t,h)\bigr) \quad \text{for any } t \in [0,1],\ i = 1,2 \text{ and } h = 1,2,$$

where, for all $t \in [0,1]$, $j(i,t,h)$ is defined as follows:

$$j(i,t,h) = \begin{cases} h & \text{if } \varphi_1(-\varphi_i(f_+(t))) = -f_+(t), \\ 3-h & \text{if } \varphi_1(-\varphi_i(f_+(t))) \neq -f_+(t) \text{ and } \varphi_2(-\varphi_i(f_+(t))) = -f_+(t). \end{cases} \tag{A.20}$$

Note that the symmetry of the ellipse (see Fig. 3), implies that one of the two conditions in Eq. (A.20) must be true. In fact, it is only when $-\varphi_i(f_+(t))$ coincides with an endpoint of the arc E_+ that the functions φ_1 and φ_2 take on the same values (in this case we set $j = 1$).

Continuity of the maps f_+, φ_1 and φ_2 implies that Φ_h is also continuous. We now argue that Φ_h is a homeomorphism, that is, a continuous bijection. First, we show that the map is onto; that is, given any point $(\tau, j) \in [0,1] \times \{1,2\}$ we can find $(t,i) \in [0,1] \times \{1,2\}$ such that $\Phi_h(t,i) = (\tau, j)$. To see this, let $t = f_+^{-1}(\varphi_j^{-1}(f_-(\tau))$ and note that either $\Phi_h(t,1) = (\tau, j)$, or $\Phi_h(t,2) = (\tau, j)$. That Φ_h is 1-to-1 follows from f_-, f_+ and φ_i being 1-to-1. In fact, the map Φ_h is of class C^2.

Consider the set $[0,1] \times \{1,2\}$; by identifying, or gluing together, the point $(0,1)$ with $(0,2)$ and the point $(1,1)$ with $(1,2)$ we can view the set $[0,1] \times \{1,2\}$ as a circle S^1 and the map Φ_h as a homeomorphism from S^1 to S^1. Let s_1, s_2, s_3 be three points on the circle S_1 and suppose that as we move clockwise on the circle starting from s_1 we encounter first s_2 and then s_3. We say that the map Φ_h is *orientation preserving* if as we move clockwise on the circle starting from $\Phi_h(s_1)$ we encounter first $\Phi_h(s_2)$ and then $\Phi_h(s_3)$. The map Φ_h is *orientation reversing* if as we move clockwise on the circle starting from $\Phi_h(s_1)$ we encounter first $\Phi_h(s_3)$ and then $\Phi_h(s_2)$. Since it is a homeomorphism, Φ_h must be either orientation preserving, or orientation reversing. In fact, if Φ_1 is orientation preserving, then Φ_2 is orientation reversing and vice versa. In the reminder of the proof we will use the orientation preserving map and denote it simply as Φ.

Recall that the covering space of a circle S^1 is the real line; that is, we can find a homeomorphism $h : [0,1) \to S^1$ with $\lim_{x \to 1} h(x) = h(0)$ and then define the map $H : \mathbb{R} \to S^1$ by letting $H(x+z) = h(x)$ for all $x \in [0,1)$ and all integers $z \in Z$. The *lift* of the orientation preserving map Φ is the function $\widetilde{\Phi} : \mathbb{R} \to \mathbb{R}$ defined by $\widetilde{\Phi}(x+z) = h^{-1}(\Phi(H(x+z)))$ for all $x \in [0,1)$ and all integers $z \in Z$. (We could add any integer q to $\widetilde{\Phi}$; the lift of Φ is uniquely defined up to the addition of an integer.) Let $\widetilde{\Phi}^n = \widetilde{\Phi} \circ \widetilde{\Phi}^{n-1}$ and define the following limit:

$$r(\Phi, x) = \lim_{n \to \infty} \frac{1}{n}\bigl(\widetilde{\Phi}^n(x) - x\bigr) \quad \text{for } x \in \mathbb{R}.$$

This definition was proposed by Poincare; he showed that this limit exists and is independent of x (e.g., see Milnor, 1999) (if we added an integer to the lift then the limit would only be unique up to addition of an integer); that is, $r(\Phi, x) = r(\Phi)$ for all $x \in \mathbb{R}$. We call $r(\Phi)$ the rotational number of the map Φ. Except for a zero measure set of cases, the rotational number of Φ is irrational. In fact for fixed β_1 there are only countably many choices of β_2 that yield a rational rotational number.

A *rotation by* α is a map $r_\alpha : S^1 \to S^1$ whose lift $\tilde{r}_\alpha : \mathbb{R} \to \mathbb{R}$ is $\tilde{r}_\alpha(x) = x + \alpha$. Let $r_\alpha^n = r_\alpha \circ r_\alpha^{n-1}$. If α is an irrational number, then for all $t \in S^1$ the set of points in the infinite sequence $r_\alpha(t), r_\alpha^2(t), r_\alpha^3(t), \ldots$ is dense in S^1. A theorem by Denjoy (1932) implies that a C^2 homeomorphism Φ with an irrational rotation number $r(\Phi) = \alpha$ is conjugate to a rotation by α; that is, there exists a homeomorphism $g : S^1 \to S^1$ such that $\Phi = g^{-1} \circ r_\alpha \circ g$. This implies that for all $t \in S^1$ the set of points in the infinite sequence $\Phi(t), \Phi^2(t), \Phi^3(t), \ldots$ is dense in S^1, where $\Phi^n = \Phi \circ \Phi^{n-1}$. As a consequence, given any $a^0 \in \mathbb{R}^2$ and any $\varepsilon > 0$, we can find a finite sequence $a^0, a^1, \ldots, a^m$ of payoff neutral single-player moves from a^0 to a^m where $|a^m + a^0| < \varepsilon$; that is, the points a^0 and a^m are almost symmetric with respect to the origin $(0, 0)$. By continuity and quasi-concavity of the players' payoffs, given any $\delta > 0$, we can then find a finite sequence of single-player improvements $a^0, \hat{a}^1, \ldots, \hat{a}^m$ such that $|\hat{a}^h - a^h| < \delta$ for all $h = 1, \ldots, m$; that is, the finite sequence of single-player improvements can be chosen to be as close as desired to the sequence of payoff neutral single-player moves. By choosing $\delta = \varepsilon - |a^m + a^0|$ we obtain that $|\hat{a}^m + a^0| < \varepsilon$; that is, we can construct a finite sequence of single-player improvements from any point a^0 to a point arbitrarily close to $-a^0$. Think of this sequence as a sequence of horizontal and vertical steps, for $\theta \in (-1, 1)$ at least one of this steps must cross the line $x_1 + x_2 = \theta(a_1^0 + a_2^0)$, say it crosses at a^* in the step from a^h to a^{h+1}. Quasi-concavity of the payoff functions then implies that the sequence $a^0, a^1, \ldots, a^h, a^*$ is a finite sequence of single-player improvements. To conclude the proof, we only need to show that if $\theta = 0$ we can reach the origin. This simply follows from the fact that there will be a first step, let say from a^k to a^{k+1}, when the sequence from a^0 to a point arbitrarily close to $-a^0$ must cross one of the axis. Let b^{k+1} be the point on the intersection of line segment with endpoints a^k, a^{k+1} and one of the axis. Again, quasi-concavity of the payoff functions implies that $a^0, a^1, \ldots, a^k, b^{k+1}, (0, 0)$ is a finite sequence of single-player improvements. This concludes the proof of this case.

Case 2. If we graph the best-reply functions in the $a_1 - a_2$ plane, the line corresponding to the best-reply function of player 1 passes through the second and fourth quadrant, whereas that of player 2 goes through the first and third quadrant. It is sufficient to show that from any starting point a^0 it is possible to construct a sequence of single-player improvements that spirals away from the equilibrium $(0, 0)$, since this implies that such a sequence crosses the region between the lines $x_1 + x_2 = a_1^0 + a_2^0$ and $x_1 + x_2 = -(a_1^0 + a_2^0)$. Then, there is another sequence that reaches a point on any line $x_1 + x_2 = \theta(a_1^0 + a_2^0)$, with $-1 < \theta < 1$. There is no loss of generality in choosing a starting point $a^0 = (a_1^0, \beta_2 a_1^0)$ on the best reply of player 2 (it is always possible to reach such a point with a finite sequence of single-player improvements). Let $0 < \mu < 1$ and consider the sequence $a^0, a^1, a^2, a^3, a^4, \ldots$ where

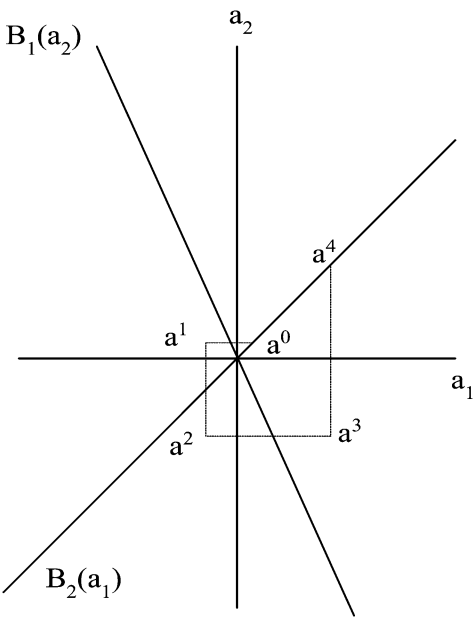

Fig. 4. A sequence of single-player improvements.

(1) $a_1^1 = \mu(2\beta_1 a_2^0 - a_1^0) + (1-\mu)a_2^0$, and $a_2^1 = a_2^0 = a_1^0\beta_2$,
(2) $a_1^2 = a_1^1$ and $a_2^2 = \mu(2\beta_2 a_1^1 - a_2^1) + (1-\mu)a_1^1$,
(3) $a_1^3 = \mu(2\beta_1 a_2^2 - a_1^2) + (1-\mu)a_2^2$ and $a_2^3 = a_2^2$,
(4) $a_1^4 = a_1^3$ and $a_2^4 = \beta_2 a_1^3$.

This is a sequence of single-player improvements, since each time a player moves it changes action in the direction of his best reply by an amount less than twice the distance between its current action and his best reply, see Fig. 4. For μ sufficiently close to 1, a^4, which is on player 2's best-reply function, is further away from $(0,0)$ than a^0. To see this note that $\lim_{\mu\to 1}|a_1^3| = [2\beta_1\beta_2(2\beta_1\beta_2 - 2) + (2\beta_1\beta_2 - 1)^2]|a_1^0| > |a_1^0|$. By iterating this construction we can obtain a sequence with any finite number of steps, spiraling away from the origin.

Finally, from any starting point a^0 the equilibrium $(0,0)$ can be reached by two single-player improvements; first a player moves to one of the axis and then the other moves from the axis to $(0,0)$.

Case 3. Change the choice variable of player 2 from a_2 to $-a_2$. More precisely, let $x_1 = a_1$, $x_2 = -a_2$, and $\alpha_i = -\beta_i$ for $i = 1, 2$, and view the game g as one in which player i chooses x_i. The best-reply functions in this game are $B_1(x_2) = \alpha_1 x_2$ and $B_2(x_1) = \alpha_2 x_1$. Since we now have $0 < \alpha_2 < \alpha_1$, we are in the same situation as in Case 1, modulo a permutation of player 1 with player 2, and we can use its proof. (Note that after the change of variable: (i) the numbers corresponding to C_1 and C_2 are $\alpha_1/(1+\alpha_1)$ and $\alpha_2/(1+\alpha_2)$; (ii) $0 < \alpha_2/(1+\alpha_2) < \alpha_1/(1+\alpha_1) < 1$; (iii) $\alpha_1/(1+\alpha_1) + \alpha_2/(1+\alpha_2) < 1$). This concludes the proof. □

Proof of Theorem 5. First note that by Lemma 3 a finite number of single-player improvements are sufficient to move the n players from an arbitrary starting point a^0 to a ball of any given radius $r > 0$ around an isolated Nash equilibrium a^*. We now proceed by induction. We know from Theorem 4 that every transversal, 2-person, aggregative game has the weak FIP. More precisely, the proof in Friedman and Mezzetti (2000) shows that from any small neighborhood of a Nash equilibrium that is asymptotically stable under the dynamics defined by (7) there is a finite, single-player, improvement path leading to the equilibrium. Suppose that almost all transversal, $(n-1)$-person, aggregative games have this property. We will show then that the property must also hold for n-person games.

Consider a transversal, n-person, aggregative game g. By Lemma 3, from any starting point a^0, we will reach a point a^r that lies in a neighborhood V_r around an asymptotically stable Nash equilibrium a^*. By changing coordinates we can assume, without loss of generality, that the Nash equilibrium a^* is at the origin: $a^* = (0, \ldots, 0)$. Since r is arbitrary, we can choose it small enough so that the players' payoff functions are closely approximated by quadratic functions. This implies that the best-reply functions are of the form

$$B_i(\Sigma_{-i}) \simeq \beta_i \Sigma_{-i} \tag{A.21}$$

which in turn implies that

$$\frac{\mathrm{d}U_i(a)}{\mathrm{d}a_i} = D_i(a_i, \Sigma) \simeq \gamma_i\big(\beta_i \Sigma - (1+\beta_i)a_i\big)$$

where γ_i is a constant. Then, by Eq. (6), the M_i functions are

$$M_i(\Sigma) = C_i \Sigma, \quad \text{where } C_i = \frac{\beta_i}{1+\beta_i}. \tag{A.22}$$

Quasi-concavity of U_i (condition (4)) implies $\gamma_i > 0$, while condition (5) requires $\gamma_i(1+\beta_i) > 0$. Hence we have $\beta_i > -1$ and $1 - C_i = 1/(1+\beta_i) > 0$, or, equivalently, $C_i < 1$. Furthermore, since a^* is asymptotically stable under the dynamics defined by Eq. (7), it must be $\sum_{i=1}^n C_i \leqslant 1$.

We need to show that from a^r there is a finite sequence of single-player improvements leading to $(0, \ldots, 0)$. Note that $\sum_{i=1}^n C_i \leqslant 1$ implies that there are at least two players i and j such that $C_i + C_j \leqslant 1$; without loss of generality, we will assume that $C_{n-1} + C_n \leqslant 1$. Furthermore, since transversality of the game g implies $\beta_{n-1}\beta_n \neq 1$, or equivalently $C_{n-1} + C_n \neq 1$, it must be $C_{n-1} + C_n < 1$. Define a new game $\tilde{g} = \langle \{1, 2, \ldots, n-1\}, \widetilde{A}_i, \widetilde{U}_i \rangle$ with $(n-1)$ players as follows. The first $n-2$ players are as in game g; that is, for $i = 1, 2, \ldots, n-2$ the strategy sets are $\widetilde{A}_i = A_i$ and the payoff functions are $\widetilde{U}_i(\tilde{a}) = \phi_i(\tilde{a}_i, \sum_{j=1}^{n-1} \tilde{a}_j)$, where ϕ_i is i's payoff function in g. Player $(n-1)$ in $\tilde{g}$ has the strategy set $\widetilde{A}_{n-1} = A_{n-1} + A_n$, and his payoff function is:

$$\widetilde{U}_{n-1}(\tilde{a}) \simeq \tilde{\mu}_{n-1} - \left(\frac{C_{n-1} + C_n}{1 - C_{n-1} - C_n} \sum_{j=1}^{n-2} \tilde{a}_j - \tilde{a}_{n-1}\right)^2$$

where $\tilde{\mu}_{n-1}$ is a constant. Letting $\widetilde{\Sigma}_{-(n-1)} = \sum_{j=1}^{n-2} \tilde{a}_j$, the best-reply function of player $(n-1)$ in $\tilde{g}$ is given by:

$$\widetilde{B}_{n-1}\big(\widetilde{\Sigma}_{-(n-1)}\big) = \frac{C_{n-1} + C_n}{1 - C_{n-1} - C_n} \widetilde{\Sigma}_{-(n-1)}.$$

Let $\tilde{a}^r$ be the strategy profile in $\tilde{g}$ corresponding to a^r in g: $\tilde{a}^r = (\tilde{a}_1^r, \dots, \tilde{a}_{n-2}^r, \tilde{a}_{n-1}^r) = (a_1^r, \dots, a_{n-2}^r, a_{n-1}^r + a_n^r)$. Note that $\tilde{a}^r$ is in a small neighborhood of the $(n-1)$-dimensional zero vector, which is a Nash equilibrium of the game $\tilde{g}$. Hence, by the induction hypothesis, there is a finite sequence $\widetilde{S}$ of single-player improvements in $\tilde{g}$ starting at $\tilde{a}^r$ and leading to $(\tilde{a}_1^*, \dots, \tilde{a}_{n-1}^*) = (0, \dots, 0)$. Observe that each step in this sequence in which the improving player is $i < n-1$ also corresponds to an improvement for player i in game g. Next, consider a step, say from $\tilde{a}$ to $\tilde{b} = \tilde{a} \setminus \tilde{b}_{n-1}$, in the sequence $\widetilde{S}$ in which the improving player in $\tilde{g}$ is $(n-1)$. Let $\tilde{a}_{n-1} = a_{n-1}^0 + a_n^0$. We will show that we can find a finite sequence S in g going from $(\tilde{a}_1, \dots, \tilde{a}_{n-2}, a_{n-1}^0, a_n^0)$ to $(\tilde{a}_1, \dots, \tilde{a}_{n-2}, a_{n-1}^T, a_n^T)$, where $a_{n-1}^T + a_n^T = \tilde{b}_{n-1}$, in which at each step either player $(n-1)$ or player n improves her payoff.

First, note that the payoff of player $(n-1)$ in game $\tilde{g}$ must have improved in moving from $\tilde{a}$ to $\tilde{b}$; that is, $\tilde{b}_{n-1}$ must be closer to player $(n-1)$'s best reply $\widetilde{B}_{n-1}(\widetilde{\Sigma}_{-(n-1)})$ than $\tilde{a}_{n-1}$. This implies that

$$\begin{aligned}
\tilde{b}_{n-1} &= \lambda \tilde{a}_{n-1} + (1-\lambda)\big(2\widetilde{B}_{n-1}\big(\widetilde{\Sigma}_{-(n-1)}\big) - \tilde{a}_{n-1}\big) \\
&= \lambda\big(a_{n-1}^0 + a_n^0\big) + (1-\lambda)\left(2\frac{C_{n-1}+C_n}{1-C_{n-1}-C_n}\widetilde{\Sigma}_{-(n-1)} - \big(a_{n-1}^0 + a_n^0\big)\right) \\
&= (2\lambda - 1)\big(a_{n-1}^0 + a_n^0\big) + 2(1-\lambda)\frac{C_{n-1}+C_n}{1-C_{n-1}-C_n}\widetilde{\Sigma}_{-(n-1)} \qquad \text{(A.23)}
\end{aligned}$$

for some $\lambda \in (0, 1)$, where $2\widetilde{B}_{n-1}(\widetilde{\Sigma}_{-(n-1)}) - \tilde{a}_{n-1}$ is the point on the line going through $\tilde{a}_{n-1}$ and $\widetilde{B}_{n-1}(\widetilde{\Sigma}_{-(n-1)})$ whose distance from $\widetilde{B}_{n-1}(\widetilde{\Sigma}_{-(n-1)})$ is the same as $\tilde{a}_{n-1}$. Next, consider the 2-person game $\hat{\hat{g}} = \langle\{n-1, n\}, X_{n-1} \times X_n, \{U_{n-1}, U_n\}\rangle$ derived from g by forcing players $i = 1, \dots, n-2$ to play actions $\tilde{a}_i$ and by changing the $n-1$ and n coordinate as follows:

$$x_{n-1} = a_{n-1} - \frac{C_{n-1}\widetilde{\Sigma}_{-(n-1)}}{1-C_{n-1}-C_n}; \qquad x_n = a_n - \frac{C_n\widetilde{\Sigma}_{-(n-1)}}{1-C_{n-1}-C_n}. \qquad \text{(A.24)}$$

The strategy spaces in $\hat{\hat{g}}$ are $X_i = A_i - C_i\widetilde{\Sigma}_{-(n-1)}/(1-C_{n-1}-C_n)$. Using (A.21), (A.22) and (A.24), simple algebra shows that the best reply and the M_i functions in $\hat{\hat{g}}$ are:

$$\begin{aligned}
&B_i(x_j) = \frac{C_i}{1-C_i}x_j \quad i, j = n-1, n, \quad i \neq j, \\
&M_i(x_{n-1} + x_n) = C_i(x_{n-1} + x_n) \quad i, j = n-1, n, \quad i \neq j.
\end{aligned}$$

Let x_{n-1}^0 and x_n^0 be the actions corresponding to a_{n-1}^0 and a_n^0 under the new coordinates, and let $\tilde{y}_{n-1}$ correspond to $\tilde{b}_{n-1}$. By Eqs. (A.23) and (A.24),

$$\tilde{y}_{n-1} = \tilde{b}_{n-1} - \frac{C_{n-1}+C_n}{1-C_{n-1}-C_n}\widetilde{\Sigma}_{-(n-1)} = (2\lambda-1)\big(x_{n-1}^0 + x_n^0\big).$$

Since $(2\lambda - 1) \in (-1, 1)$, Lemma 6 implies that for almost all games $\hat{\hat{g}}$ there exists a finite sequence $\hat{\hat{S}}$ of single-player improvements from (x_{n-1}^0, x_n^0) to (x_{n-1}^T, x_n^T), where $x_{n-1}^T + x_n^T = (2\lambda - 1)(x_{n-1}^0 + x_n^0)$. This sequence corresponds to a sequence S in game g going from $(\tilde{a}_1, \dots, \tilde{a}_{n-2}, a_{n-1}^0, a_n^0)$ to $(\tilde{a}_1, \dots, \tilde{a}_{n-2}, a_{n-1}^T, a_n^T)$, where $a_{n-1}^T + a_n^T = \tilde{b}_{n-1}$.

The profile $(\tilde{a}_1^*, \ldots, \tilde{a}_{n-1}^*)$ in $\tilde{g}$ corresponds to the profile $(0, \ldots, 0, a_{n-1}, a_n)$, with $a_{n-1} + a_n = 0$, in g. Thus, combining the sequences $\widetilde{S}$ and S we obtain a finite sequence of single-player improvements in the game g going from a^r to some profile $(0, \ldots, 0, a_{n-1}, a_n)$, where the projection (a_{n-1}, a_n) of this profile on the last two coordinates is in a small neighborhood of $(0, 0)$. Since $(0, 0)$ is a Nash equilibrium of the 2-person game $\hat{g}$ derived from g by forcing players $i = 1, \ldots, n-2$ to play action $a_i = 0$, we know from Theorem 4 that there is a finite sequence $\widehat{S}$ of single-player improvements in $\hat{g}$ leading to $(0, 0)$. Each step of the sequence $\widehat{S}$ corresponds to an improvement by either player $(n-1)$ or player n in game g and thus there is a finite sequence of single-player improvements in g going from $(0, \ldots, 0, a_{n-1}, a_n)$ to the Nash equilibrium $(0, \ldots, 0)$. This concludes the proof. □

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