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Hereditary Coreflective Subcategories in Certain Categories of Abelian Semitopological Groups

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Received: 29 June 2019; Accepted: 22 July 2019; Published: 24 July 2019

Abstract: Let **A** be an epireflective subcategory of the category of all semitopological groups that consists only of abelian groups. We describe maximal hereditary coreflective subcategories of **A** that are not bicoreflective in **A** in the case that the **A**-reflection of the discrete group of integers is a finite cyclic group, the group of integers with a topology that is not T_0, or the group of integers with the topology generated by its subgroups of the form $\langle p^n \rangle$, where $n \in \mathbb{N}$, $p \in P$ and P is a given set of prime numbers.

Keywords: semitopological group; abelian group; coreflective subcategory; hereditary subcategory

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1. Introduction

By **STopGr** we denote the category of all semitopological groups and continuous homomorphisms. All subcategories of **STopGr** are assumed to be full and isomorphism-closed. All homomorphisms are assumed to be continuous. It is well-known that a subcategory **A** of **STopGr** is epireflective in **STopGr** if and only if it is closed under the formation of subgroups and products. A coreflective subcategory **B** of **A** is called monocoreflective (bicoreflective) if every **B**-coreflection is a monomorphimsm (a bimorphism, i.e., simultaneously a monomorphism and an epimorphism). A subcategory **B** of **A** is monocoreflective in **A** if and only if it is closed under the formation of coproducts and extremal quotient objects. It is interesting to investigate coreflective subcategories of **A** closed under additional constructions, namely products or subgroups. Productive (closed under the formation of arbitrary products) coreflective subcategories were studied in [1–3]. In [4] the author investigated hereditary (closed under the formation of subgroups) coreflective subcategories of **A**. It is shown that in the categories **STopGr** and **QTopGr** (the category of all quasitopological groups), every hereditary coreflective subcategory that contains a group with a non-indiscrete topology is bicoreflective. Maximal hereditary coreflective subcategories of **A** that are not bicoreflective in **A** are described in the case that **A** is extremal epireflective (closed under the formation of products, subgroups and semitopological groups with finer topologies) in **STopGr**, it contains only abelian groups and the **A**-reflection $r(\mathbb{Z})$ of the discrete groups of integers is a finite discrete cyclic group $\mathbb{Z}_n$. In this paper we describe the maximal hereditary coreflective, not bicoreflective subcategories in other epireflective subcategories of **STopGr**.

2. Preliminaries and Notation

Recall that a semitopological group is a group with such topology that the group operation is separately continuous. A quasitopological group is a semitopological group with a continuous inverse. A paratopological group is a group with such topology that the group operation is continuous. The category of all paratopological groups will be denoted by **PTopGr**. The category of all topological groups will be denoted by **TopGr**. The subcategory of all abelian semitopological (paratopological) groups will be denoted by **STopAb** (**PTopAb**).

Let **A** be an epireflective subcategory of **STopGr**. Note that every hereditary coreflective subcategory of **A** is monocoreflective in **A** (see [4]). Hence a subcategory of **A** is hereditary and coreflective in **A** if and only if it is closed under the formation of coproducts, extremal quotients and subgroups.

Let **A** be an epireflective subcategory of **STopGr** consisting only of abelian groups and $\{G_i\}_{i\in I}$ be a family of groups from **A**. By $\bigoplus^*_{i\in I} G_i$ we denote the direct sum with the cross topology (see [5] (Example 1.2.6)). Let $i_0 \in I$, $H_{i_0} = G_{i_0}$ and $H_i = \{g_i\}$, where $g_i \in G_i$, for $i \neq i_0$. A subset U is open in $\bigoplus^*_{i\in I} G_i$ if and only if $U \cap \bigoplus^*_{i\in I} H_i$ is open in $\bigoplus^*_{i\in I} H_i$ for every choice of i_0 and g_i. The groups $\bigoplus^*_{i\in I} G_i$ and $\coprod^{\mathbf{A}}_{i\in I} G_i$ (the coproduct of the family $\{G_i\}_{i\in I}$ in **A**) have the same underlying set and the identity considered as a map $\bigoplus^*_{i\in I} G_i \to \coprod^{\mathbf{A}}_{i\in I} G_i$ is continuous.

Note that monomorphisms in **A** are precisely the injective homomorphisms. However, epimorphisms do not need to be surjective.

3. Results

Let **A** be an epireflective subcategory of **STopGr** that contains only abelian groups. Our goal is to describe maximal hereditary coreflective subcategories of **A** that are not bicoreflective in **A**. It is well known that if a coreflective subcategory **B** of **A** contains the **A**-reflection $r(\mathbb{Z})$ of the discrete group of integers, then it is bicoreflective in **A** (see [6] (Proposition 16.4)). It is easy to see that also the converse holds if $r(\mathbb{Z})$ is a discrete group (see [4]). Now we show that it holds also in other cases. The case of discrete groups is included for the sake of completeness.

Lemma 1. *Let* **A** *be an epireflective subcategory of* **STopGr** *such that the* **A***-reflection of the discrete group of integers is one of the following:*

1. *a finite cyclic group,*
2. *the discrete group of integers,*
3. *the indiscrete group of integers,*
4. *the group of integers with the topology generated by its subgroups of the form $\langle p^n \rangle$, where $n \in \mathbb{N}$, $p \in P$ and P is a given set of prime numbers.*

Then a coreflective subcategory **B** *of* **A** *is bicoreflective in* **A** *if and only if it contains the group $r(\mathbb{Z})$.*

Proof. Let **B** be a bicoreflective subcategory of **A**. We show that the **B**-coreflection of the group $r(\mathbb{Z})$ is homeomorphic to $r(\mathbb{Z})$. Let $r(\mathbb{Z})$ be the group Z_n for some $n \in \mathbb{N}$ and $c : cr(\mathbb{Z}) \to r(\mathbb{Z})$ be the **B**-coreflection of $r(\mathbb{Z})$. Assume it is not surjective. Then $c(1) = k$ for some $k \in \mathbb{N}$, $k > 1$, $k|n$. Let $\langle \frac{n}{k} \rangle$ be the subgroup of $r(\mathbb{Z})$ generated by $\frac{n}{k}$. There exists a continuous homomorphism $f : r(\mathbb{Z}) \to \langle \frac{n}{k} \rangle$ such that $f(1) = \frac{n}{k}$. Let $g : r(\mathbb{Z}) \to \langle \frac{n}{k} \rangle$ be the trivial homomorphism. Then $f \neq g$ but $f \circ c = g \circ c$. Hence c is not an epimorphism, a contradiction. It follows that c is bijective. The identity considered as a map $r(\mathbb{Z}) \to cr(\mathbb{Z})$ is continuous, hence $r(\mathbb{Z}) \cong cr(\mathbb{Z})$.

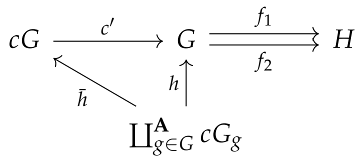
Now let $r(\mathbb{Z})$ be the group of integers with one of the topologies specified in the lemma. Consider the **B**-coreflection $c : cr(\mathbb{Z}) \to r(\mathbb{Z})$. The image of $cr(\mathbb{Z})$ under c is a non-trivial subgroup of $r(\mathbb{Z})$ (otherwise c would not be an epimorphism). Note that the topologies on $r(\mathbb{Z})$ specified in the lemma (part 2–4) have the property that all the non-trivial subgroups of $r(\mathbb{Z})$ are homeomorphic to $r(\mathbb{Z})$. Hence the image of $cr(\mathbb{Z})$ is homeomorphic to $r(\mathbb{Z})$. It follows from the definition of reflection that the topology on $r(\mathbb{Z})$ is the finest topology on the group of integers in the subcategory **A**, therefore also $cr(\mathbb{Z})$ is homeomorphic to $r(\mathbb{Z})$. □

Corollary 1. *Let* **A** *be an epireflective subcategory of* **STopGr** *such that* $\mathbf{A} \subseteq$ **STopAb** *and $r(\mathbb{Z})$ is the group of integers with the indiscrete topology. Let* **B** *be the subcategory of* **A** *consisting of all torsion groups from* **A**. *Then* **B** *is the largest hereditary coreflective subcategory of* **A** *that is not bicoreflective in* **A**.

We will need also the following lemma:

Lemma 2. *Let* **A** *be an epireflective subcategory of* **STopGr** *and* **B** *be a monocoreflective subcategory of* **A**. *Then* **B** *is bicoreflective in* **A** *if and only if the* **B***-coreflection of* $r(\mathbb{Z})$ *is an* **A***-epimorphism.*

Proof. Clearly, if **B** is bicoreflective in **A**, then the **B**-coreflection of $r(\mathbb{Z})$ is an **A**-epimorphism. Assume that the **B**-coreflection $c : cr(\mathbb{Z}) \to r(\mathbb{Z})$ of $r(\mathbb{Z})$ is an epimorphism. We will show that the **B**-coreflection $c' : cG \to G$ for an arbitrary group G from **A** is an epimorphism. Let H be a group from **A** and $f_1, f_2 : G \to H$ be homomorphisms such that $f_1 \circ c' = f_2 \circ c'$. For every $g \in G$ let G_g be a group isomorphic to $r(\mathbb{Z})$ and $i_g : r(\mathbb{Z}) \cong G_g \to G$ be the homomorphism given by $i_g(1) = g$. Moreover, let $cG_g \to G_g$ be the **B**-coreflection of G_g. Then $h : \coprod^{\mathbf{A}}_{g \in G} cG_g \to \coprod^{\mathbf{A}}_{g \in G} G_g \to G$ is an epimorphism. There exists a unique homomorphism $\bar{h} : \coprod^{\mathbf{A}}_{g \in G} cG_g \to cG$ such that the following diagram commutes:



We have $f_1 \circ h = f_1 \circ c' \circ \bar{h} = f_2 \circ c' \circ \bar{h} = f_2 \circ h$. But h is an epimorphism, therefore $f_1 = f_2$ and c' is an epimorphism. □

In the following example we show that Lemma 1 does not hold in general.

Example 1. *Let Z be the group of integers with the topology generated by the subgroup $\{2n : n \in Z\}$ and* **A** *be the smallest epireflective subcategory containing Z. Then* **A** *consists of subgroups of products of the form $\prod_{i \in I} G_i$, where each G_i is isomorphic to the group Z. Let* **B** *be the subcategory consisting of all indiscrete groups from* **A**. *The* **B***-coreflection of $r(\mathbb{Z}) \cong Z$ is $c : cr(\mathbb{Z}) \to r(\mathbb{Z})$, where $cr(\mathbb{Z})$ is the indiscrete group of integers and $c(1) = 2$. Clearly, c is an* **A***-epimorphism. Hence, by Lemma 2,* **B** *is bicoreflective in* **A**, *but it does not contain the group $r(\mathbb{Z})$.*

Consider a finite cyclic semitpological group Z_n. The closure of $\{0\}$ in Z_n is a subgroup of Z_n and it is the smallest (with respect to inclusion) open neighborhood of 0. The same holds for the group of integers with a non-T_0 topology. Moreover, we have the following simple fact:

Lemma 3. *Let G and H be cyclic semitopological groups, either finite or infinite and non-T_0. Let $n, k \in \mathbb{N}$ be such that $\overline{\{0\}} = \langle n \rangle$ in G and $\overline{\{0\}} = \langle k \rangle$ in H. Consider the subgroup $\langle (1,1) \rangle$ of $G \times^* H$. Then $\overline{\{(0,0)\}} = \langle (m,m) \rangle$, where m is the least common multiple of n and k.*

Proof. Let U be an open neighborhood of $(0,0)$ in $G \times^* H$. Then $V = U \cap G \times^* \{0\}$ is open in $G \times^* \{0\}$. Therefore V (and hence also U) contains $\langle n \rangle \times^* \{0\}$. Analogously, U contains $\{0\} \times^* \langle k \rangle$. Hence U contains $\langle n \rangle \times^* \langle k \rangle$. Therefore every neighborhood of $(0,0)$ in $\langle (1,1) \rangle$ contains $\langle (m,m) \rangle$. The subgroup $\langle (m,m) \rangle$ is open in $\langle (1,1) \rangle$, since $\langle n \rangle \times \langle k \rangle$ is open in $G \times^* H$. □

Clearly, the above lemma can be generalized to any finite number of groups.

The following proposition is a generalization of [4] (Proposition 4.9).

Proposition 1. *Let* **A** *be an epireflective subcategory of* **STopGr** *such that* **A** $\subseteq$ **STopAb** *and $r(\mathbb{Z}) = Z_n$, where $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the prime factorization of n. For $i \in \{1, \ldots, k\}$, consider the group $Z_{p_i^{\alpha_i}}$ with the subspace topology induced from $r(\mathbb{Z})$. Let m_i be the natural number such that $\overline{\{0\}} = \langle p_i^{m_i} \rangle$ in $Z_{p_i^{\alpha_i}}$. We define the subcategories $\mathbf{B}_i$ and $\mathbf{C}_i$ of* **A** *as follows:*

1. *If every cyclic group from* $\mathbf{A}$ *of order* $p_i^{\alpha_i}$ *is homeomorphic to* $Z_{p_i^{\alpha_i}}$ *with the subspace topology induced from* $r(\mathbb{Z})$ *or there exists a cyclic group* $Z_{p_i^{\beta_i}}$ *(where* $\beta_i < \alpha_i$*) from* $\mathbf{A}$ *such that* $\overline{\{0\}} = \langle p_i^{m_i} \rangle$ *in* $Z_{p_i^{\beta_i}}$*, then let* $\mathbf{B}_i$ *be the subcategory consisting precisely of those groups from* $\mathbf{A}$ *that do not have an element of order* $p_i^{\alpha_i}$*.*
2. *If the subgroup* $Z_{p_i^{\alpha_i}}$ *of* $r(\mathbb{Z})$ *is not indiscrete, let* $\mathbf{C}_i$ *be the subcategory consisting precisely of such groups* G *from* $\mathbf{A}$ *that if* H *is a cyclic subgroup of* G *of order* $p_i^{\beta_i}$*, where* $\beta_i \leq \alpha_i$*, then the index of* $\overline{\{e_H\}}$ *in* H *is less than* $p_i^{m_i}$*.*

Then $\mathbf{B}_i$ *and* $\mathbf{C}_i$ *are maximal hereditary coreflective subcategories of* $\mathbf{A}$ *that are not bicoreflective in* $\mathbf{A}$*.*

Note that there does not need to be a subcategory $\mathbf{B}_i$ or $\mathbf{C}_i$ for every $i \in \{1, \ldots, k\}$.

Proof. Clearly, the subcategories $\mathbf{B}_i$ and $\mathbf{C}_i$ are hereditary and, by Lemma 1, they are not bicoreflective in $\mathbf{A}$. The subcategories $\mathbf{B}_i$ are coreflective in $\mathbf{A}$.

We need to show that also the subcategories $\mathbf{C}_i$ are coreflective in $\mathbf{A}$. Let $\{G_j\}_{j\in I}$ be a family of groups from $\mathbf{C}_i$ for some $i \in \{1, \ldots, k\}$, $\coprod_{j\in I} G_j \to G$ be an extremal $\mathbf{A}$-epimorphism and f be the homomorphism $\bigoplus^*_{j\in J} G_j \to \coprod_{j\in I} G_j \to G$. Assume that G has a subgroup H homeomorphic to $Z_{p_i^{\alpha_i}}$ with the subspace topology induced from $r(\mathbb{Z})$. Let x be an element of $\bigoplus^*_{j\in I} G_j$ such that $\langle f(x) \rangle = H$. Then the subgroup $\langle x \rangle$ is also homeomorphic to $Z_{p_i^{\alpha_i}}$. Without loss of generality we may assume that $\langle x \rangle = \langle (x_1, \ldots, x_m) \rangle$ is a subgroup of $\langle x_1 \rangle \times^* \cdots \times^* \langle x_m \rangle$, where each x_l belongs to some $G_{j_l} \in \{G_j\}_{j\in I}$. By Lemma 3, the topology of $\langle x \rangle$ is coarser then the topology of $Z_{p_i^{\alpha_i}}$, a contradiction.

Lastly we show that every hereditary coreflective subcategory of $\mathbf{A}$ that is not bicoreflective in $\mathbf{A}$ is contained in one of the subcategories $\mathbf{B}_i$ or $\mathbf{C}_i$. If a subcategory $\mathbf{D}$ is hereditary and coreflective in $\mathbf{A}$, but not bicoreflective in $\mathbf{A}$, then it does not contain the group $r(\mathbb{Z})$. Therefore it does not contain one of its subgroups $Z_{p_i^{\alpha_i}}$. Hence, it either does not contain a cyclic group of order $p_i^{\alpha_i}$ (and then $\mathbf{D} \subseteq \mathbf{B}_i$) or it does not contain the group $Z_{p_i^{\alpha_i}}$ with $\overline{\{0\}} = \langle p_i^{m_i} \rangle$. Then it also does not contain a cyclic group $Z_{p_i^{\beta_i}}$, where $\beta_i \leq \alpha_i$, with $\overline{\{0\}} = \langle p_i^{m_i} \rangle$, and then $\mathbf{D} \subseteq \mathbf{C}_i$. □

In [4] we presented examples of such epireflective subcategories $\mathbf{A}$ of **STopGr** that every hereditary coreflective subcategory of $\mathbf{A}$ that contains a group with a non-indiscrete topology is bicoreflective in $\mathbf{A}$. Here we give another example of subcategories of **STopGr** with this property. Note that if the subcategory $\mathbf{A}$ from the following example consists only of abelian groups, then we easily obtain the following result from the above proposition.

Example 2. *Let* $\mathbf{A}$ *be an epireflective subcategory of* **STopGr** *such that* $r(\mathbb{Z})$ *is the discrete cyclic group* $\mathbb{Z}_p$*, where* p *is a prime number. Then every hereditary coreflective subcategory* $\mathbf{B}$ *of* $\mathbf{A}$ *that contains a group with a non-indiscrete topology is bicoreflective in* $\mathbf{A}$*. Let* G *be a non-indiscrete group from* $\mathbf{B}$ *and* U *be an open neighborhood of* e_G *such that* $U \neq G$*. Choose an element* $x \in G \setminus U$*. The order of* x *is* p *and the subgroup* $\langle x \rangle$ *of* G *is discrete, therefore* $\langle x \rangle \cong \mathbb{Z}_p$ *belongs to* $\mathbf{B}$ *and* $\mathbf{B}$ *is bicoreflective in* $\mathbf{A}$*.*

Proposition 2. *Let* $\mathbf{A}$ *be an epireflective subcategory of* **STopGr** *such that* $\mathbf{A} \subseteq$ **PTopAb** *and* $r(\mathbb{Z})$ *is the group of integers with the topology generated by its subgroups of the form* $\langle p^n \rangle$*, where* $n \in \mathbb{N}$*,* $p \in P$ *and* P *is a given set of prime numbers. Let* $p \in P$ *and* $\mathbf{B}_p$ *be the subcategory of* $\mathbf{A}$ *consisting precisely of such groups* G *from* $\mathbf{A}$ *that if* H *is an infinite cyclic subgroup of* G *then there exists an* $n \in \mathbb{N}$ *such that the subgroup of index* p^n *is not open in* H*. Then those* $\mathbf{B}_p$ *that contain a group with an element of infinite order are maximal hereditary coreflective subcategories of* $\mathbf{A}$ *that are not bicoreflective in* $\mathbf{A}$*. If all* $\mathbf{B}_p$ *contain only torsion groups, then they are all equal to the subcategory* $\mathbf{B}$ *of all torsion groups from* $\mathbf{A}$ *and* $\mathbf{B}$ *is the largest hereditary coreflective subcategory of* $\mathbf{A}$ *that is not bicoreflective in* $\mathbf{A}$*.*

Proof. Obviously, the subcategory $\mathbf{B}$ is hereditary and coreflective, but not bicoreflective in $\mathbf{A}$. If all subcategories $\mathbf{B}_p$ contain only torsion groups, then for every group G from $\mathbf{A}$ and every element $g \in G$

of infinite order we have $\langle g\rangle \cong r(\mathbb{Z})$. Therefore every hereditary coreflective subcategory of $\mathbf{A}$ that is not bicoreflective in $\mathbf{A}$ is contained in $\mathbf{B}$ and the subcategory $\mathbf{B}$ is maximal with this property.

Now assume that at least one of the subcategories $\mathbf{B}_p$ contains a group with an element of infinite order. Clearly, every subcategory $\mathbf{B}_p$ is hereditary. It does not contain the group $r(\mathbb{Z})$, and therefore, by Lemma 1, it is not bicoreflective in $\mathbf{A}$.

We show that the subcategories $\mathbf{B}_p$ are coreflective in $\mathbf{A}$. Let $p \in P$, $\{G_i\}_{i\in I}$ be a family of groups from $\mathbf{B}_p$ and $f : \coprod^{\mathbf{A}}_{i\in I} G_i \to G$ be an extremal $\mathbf{A}$-epimorphism. Let x be an element of $\coprod^{\mathbf{A}}_{i\in I} G_i$ such that the subgroup of index p^n is open in $\langle f(x)\rangle$ for every $n \in \mathbb{N}$. Then also the subgroup of index p^n of $\langle x\rangle$ is open in $\langle x\rangle$ for every $n \in \mathbb{N}$. Without loss of generality we may assume that $\langle x\rangle = \langle (x_1,\dots,x_k)\rangle$ is a subgroup of $\langle x_1\rangle \sqcup \cdots \sqcup \langle x_k\rangle = \langle x_1\rangle \times \cdots \times \langle x_k\rangle$, where $\langle x_1\rangle \times \cdots \times \langle x_k\rangle$ is the product with the usual topology and each x_j belongs to some $G_{i_j} \in \{G_i\}_{i\in I}$. For every $j \in \{1,\dots,k\}$ there exists a natural number n_j such that the subgroup of index p^{n_j} is not open in $\langle x_j\rangle$. Then the subgroup of $\langle x\rangle$ of index $p^{n_{j_0}}$, where n_{j_0} is the largest from $n_1,\dots,n_k$, is not open in $\langle x\rangle$, a contradiction.

Next we show that every hereditary coreflective subcategory of $\mathbf{A}$ that is not bicoreflective in $\mathbf{A}$ is contained in some $\mathbf{B}_p$. Let $\mathbf{C}$ be a hereditary coreflective subcategory of $\mathbf{A}$ that is not bicoreflective in $\mathbf{A}$. Then $\mathbf{C}$ does not contain the group $r(\mathbb{Z})$. If $\mathbf{C}$ contains only torsion groups, then $\mathbf{C} \subseteq \mathbf{B}_p$ for every $p \in P$. Otherwise $\mathbf{C}$ contains the group of integers with a topology such that its subgroup of index p^n is not open for some $p \in P$ and $n \in \mathbb{N}$. Therefore $\mathbf{C}$ is contained in $\mathbf{B}_p$. □

Proposition 3. *Let $\mathbf{A}$ be an epireflective subcategory of* **STopGr** *such that $\mathbf{A} \subseteq$* **STopAb** *and $r(\mathbb{Z})$ is the group of integers with a non-T_0 topology. Let the closure of $\{0\}$ in $r(\mathbb{Z})$ be the subgroup $\langle n\rangle$. Then the following holds:*

1. *If the embedding $\langle n\rangle \to r(\mathbb{Z})$ is an $\mathbf{A}$-epimorphism, then the subcategory $\mathbf{B}$ of all torsion groups from $\mathbf{A}$ is the largest hereditary coreflective subcategory of $\mathbf{A}$ that is not bicoreflective in $\mathbf{A}$.*
2. *For every minimal natural number k such that $k|n$ and the embedding $\langle k\rangle \to r(\mathbb{Z})$ is not an $\mathbf{A}$-epimorphism let $\mathbf{B}_k$ be the subcategory consisting of such groups G from $\mathbf{A}$ that if H is a cyclic subgroup of G then the index of $\overline{\{e_H\}}$ in H is at most $\frac{n}{k}$. The subcategories $\mathbf{B}_k$ are maximal hereditary coreflective subcategories of $\mathbf{A}$ that are not bicoreflective in $\mathbf{A}$.*

   Assume that for every minimal natural number k such that $k|n$ and the embedding $\langle k\rangle \to r(\mathbb{Z})$ is not an $\mathbf{A}$-epimorphism, $\mathbf{A}$ contains a finite cyclic group G_k such that the index of $\overline{\{e_{G_k}\}}$ in G_k is greater than $\frac{n}{k}$. Then the subcategory $\mathbf{B}$ of all torsion groups from $\mathbf{A}$ is also a maximal hereditary coreflective subcategory of $\mathbf{A}$ that is not bicoreflective in $\mathbf{A}$.

Proof. Assume that the closure of $\{0\}$ in $r(\mathbb{Z})$ is the subgroup $\langle n\rangle$ and the embedding $i : \langle n\rangle \to r(\mathbb{Z})$ is an $\mathbf{A}$-epimorphism. Clearly, the subcategory $\mathbf{B}$ is hereditary and coreflective, but not bicoreflective in $\mathbf{A}$. We need to show that it is maximal with this property. Let $\mathbf{C}$ be a hereditary coreflective subcategory of $\mathbf{A}$ that contains the group Z with some topology and $c : cr(\mathbb{Z}) \to r(\mathbb{Z})$ be the $\mathbf{B}$-coreflection of $r(\mathbb{Z})$. Assume that $c(1) = k$. Let $f : Z \to r(\mathbb{Z})$ be the homomorphism given by $f(1) = n$. Then f is continuous. There exists a unique homomorphism $\bar{f} : Z \to cr(\mathbb{Z})$ such that $f = c \circ \bar{f}$. Hence $k > 0$ and $k|n$. Therefore c is an $\mathbf{A}$-epimorphism and, by Lemma 2, the subcategory $\mathbf{B}$ is bicoreflective in $\mathbf{A}$.

Now assume that the embedding $i : \langle n\rangle \to r(\mathbb{Z})$ is not an $\mathbf{A}$-epimorphism. The subcategory $\mathbf{B}$ is hereditary and coreflective in $\mathbf{A}$, but not bicoreflective in $\mathbf{A}$. Let k be minimal such that $k|n$ and the embedding $\langle k\rangle \to r(\mathbb{Z})$ is not an $\mathbf{A}$-epimorphism. Clearly, the subcategory $\mathbf{B}_k$ is hereditary. For the $\mathbf{B}_k$-coreflection $c : cr(\mathbb{Z}) \to r(\mathbb{Z})$ we have $c(1) \geq k$. Therefore it is not an epimorphism and $\mathbf{B}_k$ is not bicoreflective in $\mathbf{A}$.

We need to show that $\mathbf{B}_k$ is coreflective in $\mathbf{A}$. Let $\{G_i\}_{i\in I}$ be a family of groups from $\mathbf{B}_k$, $\coprod^{\mathbf{A}}_{i\in I} G_i \to G$ be an extremal $\mathbf{A}$-epimorphism and f be the homomorphism $\bigoplus^*_{i\in I} G_i \to \coprod^{\mathbf{A}}_{i\in I} G_i \to G$. Assume that x is an element of $\bigoplus^*_{i\in I} G_i$ such that the index of $\overline{\{e_{\langle f(x)\rangle}\}}$ in $\langle f(x)\rangle$ is greater than $\frac{n}{k}$. Then also the index of $\overline{\{e_{\langle x\rangle}\}}$ in $\langle x\rangle$ is greater than $\frac{n}{k}$. Without loss of generality we may assume that $\langle x\rangle = \langle (x_1,\dots,x_m)\rangle$ is a subgroup of $\langle x_i\rangle \times^* \cdots \times^* \langle x_m\rangle$, where each x_j is an element of some G_{i_j}. The index of $\overline{\{e_{\langle x_j\rangle}\}}$ in

$\langle x_j \rangle$ is a divisor of $\frac{n}{k}$ for every $j \in \{1, \ldots, m\}$. Then, by Lemma 3, the index of $\overline{\{e_{\langle x \rangle}\}}$ in $\langle x \rangle$ is at most $\frac{n}{k}$, a contradiction.

Lastly, we show that every hereditary coreflective subcategory of **A** that is not bicoreflective in **A** is contained in **B** or $\mathbf{B}_k$. Let **C** be a hereditary coreflective subcategory of **A** that is not bicoreflective in **A**. Let $c : cr(\mathbb{Z}) \to r(\mathbb{Z})$ be the **C**-coreflection of $r(\mathbb{Z})$. If it is a trivial homomorphism, then **C** is contained in **B**. Otherwise $c(1) \geq k$, where k is minimal such that $k|n$ and the embedding $\langle k \rangle \to r(\mathbb{Z})$ is not an **A**-epimorphism. Assume that G is a cyclic group from **C** that does not belong to $\mathbf{B}_k$. Then the index of $\overline{\{e_G\}}$ in G is greater than $\frac{n}{k}$. Then **C** contains the group of integers Z with a topology such that the index of $\overline{\{0\}}$ in Z is greater than $\frac{n}{k}$ (a subgroup of $cr(\mathbb{Z}) \sqcup G$). Then there exists a homomorphism $f : Z \to r(\mathbb{Z})$ such that $f(1) < k$. Then also $c(1) < k$, a contradiction. Therefore **C** is contained in $\mathbf{B}_k$. Note that if for some minimal natural number k such that $k|n$ and the embedding $\langle k \rangle \to r(\mathbb{Z})$ is not an **A**-epimorphism, **A** does not contain a finite cyclic group G such that the index of $\overline{\{e_G\}}$ in G is greater than $\frac{n}{k}$, then the subcategory **B** is contained in $\mathbf{B}_k$, and therefore it is not maximal. □

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflict of interest.

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