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Review

On History of Mathematical Economics: Application of Fractional Calculus

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Abstract: Modern economics was born in the Marginal revolution and the Keynesian revolution. These revolutions led to the emergence of fundamental concepts and methods in economic theory, which allow the use of differential and integral calculus to describe economic phenomena, effects, and processes. At the present moment the new revolution, which can be called "Memory revolution", is actually taking place in modern economics. This revolution is intended to "cure amnesia" of modern economic theory, which is caused by the use of differential and integral operators of integer orders. In economics, the description of economic processes should take into account that the behavior of economic agents may depend on the history of previous changes in economy. The main mathematical tool designed to "cure amnesia" in economics is fractional calculus that is a theory of integrals, derivatives, sums, and differences of non-integer orders. This paper contains a brief review of the history of applications of fractional calculus in modern mathematical economics and economic theory. The first stage of the Memory Revolution in economics is associated with the works published in 1966 and 1980 by Clive W. J. Granger, who received the Nobel Memorial Prize in Economic Sciences in 2003. We divide the history of the application of fractional calculus in economics into the following five stages of development (approaches): ARFIMA; fractional Brownian motion; econophysics; deterministic chaos; mathematical economics. The modern stage (mathematical economics) of the Memory revolution is intended to include in the modern economic theory new economic concepts and notions that allow us to take into account the presence of memory in economic processes. The current stage actually absorbs the Granger approach based on ARFIMA models that used only the Granger–Joyeux–Hosking fractional differencing and integrating, which really are the well-known Grunwald–Letnikov fractional differences. The modern stage can also absorb other approaches by formulation of new economic notions, concepts, effects, phenomena, and principles. Some comments on possible future directions for development of the fractional mathematical economics are proposed.

Keywords: mathematical economics; economic theory; fractional calculus; fractional dynamics; long memory; non-locality

1. Introduction: General Remarks about Mathematical Economics

Mathematical economics is a theoretical and applied science, whose purpose is a mathematically formalized description of economic objects, processes, and phenomena. Most of the economic theories are presented in terms of economic models. In mathematical economics, the properties of these models are studied based on formalizations of economic concepts and notions. In mathematical economics, theorems on the existence of extreme values of certain parameters are proved, properties of equilibrium states and equilibrium growth trajectories are studied, etc. This creates the impression that the proof of the existence of a solution (optimal or equilibrium) and its calculation is the main aim of mathematical economics. In reality, the most important purpose is to formulate economic notions and concepts in mathematical form, which will be mathematically adequate and self-consistent, and then, on their basis to construct mathematical models of economic processes and phenomena. Moreover, it is not enough to prove the existence of a solution and find it in an analytic or numerical form, but it is necessary to give an economic interpretation of these obtained mathematical results.

We can say that modern mathematical economics began in the 19th century with the use of differential (and integral) calculus to describe and explain economic behavior. The emergence of modern economic theory occurred almost simultaneously with the appearance of new economic concepts, which were actively used in various economic models. "Marginal revolution" and "Keynesian revolution" in economics led to the introduction of the new fundamental concepts into economic theory, which allow the use of mathematical tools to describe economic phenomena and processes. The most important mathematical tools that have become actively used in mathematical modeling of economic processes are the theory of derivatives and integrals of integer orders, the theory of differential and difference equations. These mathematical tools allowed economists to build economic models in a mathematical form and on their basis to describe a wide range of economic processes and phenomena. However, these tools have a number of shortcomings that lead to the incompleteness of descriptions of economic processes. It is known that the integer-order derivatives of functions are determined by the properties of these functions in an infinitely small neighborhood of the point, in which the derivatives are considered. As a result, differential equations with derivatives of integer orders, which are used in economic models, cannot describe processes with memory and non-locality. In fact, such equations describe only economic processes, in which all economic agents have complete amnesia and interact only with the nearest neighbors. Obviously, this assumption about the lack of memory among economic agents is a strong restriction for economic models. As a result, these models have drawbacks, since they cannot take into account important aspects of economic processes and phenomena.

2. A Short History of Fractional Mathematical Economics

"Marginal revolution" and "Keynesian revolution" introduced fundamental economic concepts, including the concepts of "marginal value", "economic multiplier", "economic accelerator", "elasticity" and many others. These revolutions led to the use of mathematical tools based on the derivatives and integrals of integer orders, and the differential and difference equations. As a result, the economic models with continuous and discrete time began to be mathematically described by differential equations with derivatives of integer orders or difference equations of integer orders.

It can be said that at the present moment new revolutionary changes are actually taking place in modern economics. These changes can be called a revolution of memory and non-locality. It is becoming increasingly obvious in economics that when describing the behavior of economic agents, we must take into account that their behavior may depend on the history of previous changes in the economy. In economic theory, we need new economic concepts and notions that allow us to take into account the presence of memory in economic agents. New economic models and methods are needed, which take into account that economic agents may remember the changes of economic indicators and factors in the past, and that this affects the behavior of agents and their decision making. To describe this behavior we cannot use the standard mathematical apparatus of differential (or difference) equations of integer orders. In fact, these equations describe only such economic processes, in which agents actually have an amnesia. In other words, economic models, which use only derivatives of integer orders, can be applied when economic agents forget the history of changes of economic indicators and factors during an infinitesimally small period of time. At the moment it is becoming clear that this restriction holds back the development of economic theory and mathematical economics.

In modern mathematics, derivatives and integrals of arbitrary order are well known [1–5]. The derivative (or integral), order of which is a real or complex number and not just an integer, is called fractional derivative and integral. Fractional calculus as a theory of such operators has a long history [6–15]. There are different types of fractional integral and differential operators [1–5]. For fractional differential and integral operators, many standard properties are violated, including such properties as the standard product (Leibniz) rule, the standard chain rule, the semi-group property for orders of derivatives, the semi-group property for dynamic maps [16–21]. We can state that the violation of the standard form of the Leibniz rule is a characteristic property of derivatives of non-integer orders [16]. The most important application of fractional derivatives and integrals of non-integer order is fading memory and spatial non-locality.

The new revolution ("Memory revolution") is intended to include in the modern economic theory and mathematical economics different processes with long memory and non-locality. The main mathematical tool designed to "cure amnesia" in economics is the theory of derivatives and integrals of non-integer order (fractional calculus), fractional differential and difference equations [1–5]. This revolution has led to the emergence of a new branch of mathematical economics, which can be called "fractional mathematical economics."

Fractional mathematical economics is a theory of fractional dynamic models of economic processes, phenomena and effects. In this framework of mathematical economics, the fractional calculus methods are being developed for application to problems of economics and finance. The field of fractional mathematical economics is the application of fractional calculus to solve problems in economics (and finance) and for the development of fractional calculus for such applications. Fractional mathematical economics can be considered as a branch of applied mathematics that deals with economic problems. However, this point of view is obviously a narrowing of the field of research, goals and objectives of this area. An important part of fractional mathematical economics is the use of fractional calculus to formulate new economic concepts, notions, effects and phenomena. This is especially important due to the fact that the fractional mathematical economics is now only being formed as an independent science. Moreover, the development of the fractional calculus itself and its generalizations will largely be determined precisely by such goals and objectives in economics, physics and other sciences.

This "Memory revolution" in the economics, or rather the first stage of this revolution, can be associated with the works, which were published in 1966 and 1980 by Clive W. J. Granger [22–26], who received the Nobel Memorial Prize in Economic Sciences in 2003 [27].

The history of the application of fractional calculus in economics can be divided into the following stages of development (approaches): ARFIMA; fractional Brownian motion; econophysics; deterministic chaos; mathematical economics. The appearance of a new stage obviously does not mean the cessation of the development of the previous stage, just as the appearance of quantum theory did not stop the development of classical mechanics.

Further in Sections 2.1–2.5, we briefly describe these stages of development, and then in Section 3 we outline possible ways for the further development of fractional mathematical economics.

2.1. ARFIMA Stage (Approach)

ARFIMA Stage (Approach): This stage is characterized by models with discrete time and application of the Grunwald–Letnikov fractional differences.

More than fifty years ago, Clive W. J. Granger (see preprint [22], paper [23], the collection of the works [24,25]) was the first to point out long-term dependencies in economic data. The articles demonstrated that spectral densities derived from the economic time series have a similar shape. This fact allows us to say that the effect of long memory in the economic processes was found by Granger. Note that, he received the Nobel Memorial Prize in Economics in 2003 "for methods of analyzing economic time series with general trends (cointegration)" [27].

Then, Granger and Joyeux [26], and Hosking [28] proposed the fractional generalization of ARIMA(p,d,q) models (the ARFIMA (p,d,q) models) that improved the statistical methods for researching of processes with memory. As the main mathematical tool for describing memory, fractional differencing and integrating (for example, see books [29–34] and reviews [35–38]) were proposed for discrete time case. The suggested generalization of the ARIMA(p,d,q) model is realized by considering non-integer (positive and negative) order d instead of positive integer values of d. The Granger–Joyeux–Hosking (GJH) operators were proposed and used without relationship with the fractional calculus. As was proved in [39,40], these GJH operators are actually the Grunwald–Letnikov fractional differences (GLF-difference), which have been suggested more than a hundred and fifty years ago and are used in the modern fractional calculus [1,3]. We emphasize that in the continuous limit these GLF-differences give the GLF-derivatives that coincide with the Marchaud fractional derivatives (see Theorem 4.2 and Theorem 4.4 of [1]).

Among economists, the approach proposed by Gravers (and based on the discrete operators proposed by them) is the most common and is used without an explicit connection with the development of fractional calculus. It is obvious that the restriction of mathematical tools only to the Grunwald–Letnikov fractional differences significantly reduces the possibilities for studying processes with memory and non-locality. The use of fractional calculus in economic models will significantly expand the scope and allows us to obtain new results.

2.2. Fractional Brownian Motion (Mathematical Finance) Stage (Approach)

Fractional Brownian Motion Stage (Approach): This stage is characterized by financial models and the application of stochastic calculus methods and stochastic differential equations.

Andrey N. Kolmogorov, who is one of the founders of modern probability theory, was the first who considered in 1940 [41] the continuous Gaussian processes with stationary increments and with the self-similarity property A.N. Kolmogorov called such Gaussian processes "Wiener Spirals". Its modern name is the fractional Brownian motion that can be considered as a continuous self-similar zero-mean Gaussian process and with stationary increments.

Starting with the article by L.C.G. Rogers [42], various authors began to consider the use of fractional Brownian motion to describe different financial processes. The fractional Brownian motion is not a semi-martingale and the stochastic integral with respect to it is not well-defined in the classical Ito's sense. Therefore, this approach is connected with the development of fractional stochastic calculus [43–45]. For example, in the paper [43] a stochastic integration calculus for the fractional Brownian motion based on the Wick product was suggested.

At the present time, this stage (approach), which can be called as a fractional mathematical finance, is connected with the development of fractional stochastic calculus, the theory fractional stochastic differential equations and their application in finance. The fractional mathematical finance is a field of applied mathematics, concerned with mathematical modeling of financial markets by using the fractional stochastic differential equations.

As a special case of fractional mathematical finance, we can note the fractional generalization of the Black–Scholes pricing model. In 1973, Fischer Black and Myron Scholes [46] derived the famous theoretical valuation formula for options. In 1997, the Royal Swedish Academy of Sciences has decided to award the Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel [47] to Myron S. Scholes, for the so-called Black–Scholes model published in 1973: "Robert C. Merton and Myron S. Scholes have, in collaboration with the late Fischer Black, developed a pioneering formula for the valuation of stock options." [47].)

For the first time a fractional generalization of the Black–Scholes equation was proposed in [48] by Walter Wyss in 2000. Wyss [48] considered the pricing of option derivatives by using the time-fractional Black–Scholes equation and derived a closed form solution for European vanilla options. The Black–Scholes equation is generalized by replacing the first derivative in time by a fractional derivative in time of the order $\alpha \in (0,1)$. The solution of this fractional Black–Scholes equation is considered. However, in the Wyss paper, there are no financial reasons to explain why a time-fractional derivative should be used.

The works of Cartea and Meyer-Brandis [49] and Cartea [50] proposed a stock price model that uses information about the waiting time between trades. In this model the arrival of trades is driven by a counting process, in which the waiting-time between trades processes is described by the Mittag–Leffler survival function (see also [51]). In the paper [50], Cartea proposed that the value of derivatives satisfies the fractional Black–Scholes equation that contains the Caputo fractional derivative with respect to time. It should be noted that, in general, the presence of a waiting time and a delay time does not mean the presence of memory in the process.

In the framework of the fractional Brownian motion Stage, a lot of papers [50–71] and books [72,73] were written on the description of financial processes with memory and non-locality.

As a rule, in fractional mathematical finance, fractional dynamic models are created without establishing links with economic theory and without formulating new economic or financial concepts, taking only observable market prices as input data. In the fractional mathematical finance, the main requirement is the mathematical consistency and the compatibility with economic theory is not the key point.

2.3. *Econophysics Stage (Approach)*

Econophysics Stage (Approach): This stage is characterized by financial models and the application of physical methods and equations.

Twenty years ago, a new branch of the econophysics, which is connected with the application of fractional calculus, has appeared. In fact, this branch, which can be called fractional econophysics, was born in 2000 and it can be primarily associated with the works of Francesco Mainardi, Rudolf Gorenflo, Enrico Scalas, Marco Raberto [74–76] on the continuous-time finance.

In fractional econophysics, the fractional diffusion models [74–76] are used in finance, where price jumps replace the particle jumps in the physical diffusion model. The corresponding stochastic models are called continuous time random walks (CTRWs), which are random walks that also incorporate a random waiting time between jumps. In finance, the waiting times measure delay between transactions. These two random variables (price change and waiting time) are used to describe the long-time behavior in financial markets. The diffusion (hydrodynamic) limit, which is used in physics, is considered for continuous time random walks [74–76]. It was shown that the probability density function for the limit process obeys a fractional diffusion equation [74–76].

After the pioneering works [74–76] that laid the foundation for the new direction of econophysics (fractional econophysics), various papers were written on the application of fractional dynamics methods and physical models to describe processes in finance and economics (for example, see [77–84]). The history and achievements of the econophysics stage in the first five years are described by Enrico Scalas in the article [85] in 2016.

The fractional econophysics, as a branch of econophysics, can be defined as a new direction of research applying methods developed in physical sciences, to describe processes in economics and finance, basically those including power-law memory and spatial non-locality. The mathematical tool of this branch of econophysics is the fractional calculus. For example, application to the study of continuous time finance by using methods and results of fractional kinetics and anomalous diffusion. Another example, which is not related to finance, is the time-dependent fractional dynamics with memory in quantum and economic physics [86].

In this stage, the fractional calculus was applied mainly to financial processes. In the papers on fractional econophysics, generalization of basic economic concepts and principles for economic processes with memory (and non-locality) are not suggested.

Unfortunately, economists do not always understand the analogies with the methods and concepts of modern physics, which restricts the possibilities for economists to use this approach. As a result, it holds back and limits both the development and application of the fractional econophysics approach to describing economic processes with memory and non-locality.

It can be said that the time has come for economists and econophysicists to work together on the formulation of economic analogs of physical concepts and methods used in fractional econophysics, and linking them with existing concepts and methods of economic theory. For the development of a fractional mathematical economics, a translation should be made from the language of physics into the language of economics.

2.4. Deterministic Chaos Stage (Approach)

Deterministic Chaos Stage (Approach): This stage is characterized by financial (and economic) models and application of methods of nonlinear dynamics. Strictly speaking, this approach should be attributed to the econophysics stage/approach.

Nonlinear dynamics models are useful to explain irregular and chaotic behavior of complex economic and financial processes. The complex behaviors of nonlinear economic processes restrict the use of analytical methods to study nonlinear economic models.

In 2008, for the first time, Wei-Ching Chen proposed in [87] a fractional generalization of a financial model with deterministic chaos. Chen [87] studied the fractional-dynamic behaviors and describes fixed points, periodic motions, chaotic motions, and identified period doubling and intermittency routes to chaos in the financial process that is described by a system of three equations with the Caputo fractional derivatives. He demonstrates by numerical simulations that chaos exists when orders of derivatives are less than three and that the lowest order at which chaos exists was 2.35. The work [88] studied the chaos control method of such a kind of system by feedback control, respectively.

In the framework of the deterministic chaos stage, many papers [89–99] have been devoted to the description of financial processes with memory. In some papers [100–105], economic models were considered.

We should note that the various stages/approaches of development of fractional mathematical economics did not develop in complete isolation from each other. For example, for the fractional Chen model of dynamic chaos in the economy, Tomáš Škovránek, Igor Podlubny, Ivo Petráš [106] applied the concept of the state space (the configuration space, the phase space) that arose in physics more than a hundred years ago. As state variables authors consider the gross domestic product, inflation, and unemployment rate. The dynamics of the modeled economy in time, which is represented by the values of these three variables, is described as a trajectory in state-space. The system of three fractional order differential equations is used to describe dynamics of the economy by fitting the available economic data. Then *José* A. Tenreiro Machado, Maria E. Mata, Antonio M. Lopes suggested the development of the state space concept in the papers [107–109]. The economic growth is described by using the multidimensional scaling (MDS) method for visualizing information in data. The state space is used to represent the sequence of points (the fractional state space portrait, FSSP, and pseudo phase plane, PPP) corresponding to the states over time.

2.5. Mathematical Economics Stage (Approach)

Mathematical Economics Stage (Approach): This stage is characterized by macroeconomic and microeconomic models with continuous time and generalization of basic economic concepts and notions.

The fractional calculus approach has been used to describe the concept of memory itself for economic processes in [39,40,110–117], and to define basic concepts of economic processes with memory and non-locality in works [118–139].

From a subjective point of view, this stage began with a proposal of generalizations of the basic economic concepts and notions at the beginning of 2016, when the concept of elasticity for economic processes with memory was proposed [132–135]. Then in 2016, the concepts of the marginal values with memory [118–120], the concept of accelerator and multiplier with memory [123,130] and others were suggested [132–136,138]. These concepts are used in fractional generalizations of some standard economic models [140–167] with the continuous time, which were proposed in 2016 and subsequent years [168–189]. These dynamic models describe fractional dynamics of economic processes with memory.

The fractional calculus approach has been used to define basic concepts of economic processes with memory in works [118–139], and to describe dynamics of economic processes with memory [168–189] in the framework of the continuous time models.

It should be noted that formal replacements of derivatives of integer order by fractional derivatives in standard differential equations, which describe economic processes, and solutions of the obtained fractional differential equations were considered in papers published before 2016. However, these papers were purely mathematical works, in which generalizations of economic concepts and notions were not proposed. In these works, fractional differential equations have not been derived, since a formal replacement of integer-order derivatives by fractional derivatives cannot be recognized as a derivation of the equations. Formulations of economic conclusions and interpretations from the obtained solutions are not usually suggested in these papers. Examples of incorrectness and errors in such generalizations are given in the work [189]. In the paper [189], we formulate five principles of the fractional-dynamic generalization of standard dynamic models and then we illustrate these principles by examples from fractional mathematical economics. We can state that in the works with formal fractional generalizations of standard economic equations the Principles of Derivability and Interpretability [189] were neglected. Let us give a brief formulation of the Principles of Derivability and Interpretability.

Derivability Principle: It is not enough to generalize the differential equations describing the dynamic model. It is necessary to generalize the whole scheme of obtaining (all steps of derivation) these equations from the basic principles, concepts and assumptions. In this sequential derivation of the equations we should take into account the non-standard characteristic properties of fractional derivatives and integrals. If necessary, generalizations of the notions, concepts and methods, which are used in this derivation, should also be obtained.

Interpretability Principle: The subject (physical, economic) interpretation of the mathematical results, including solutions and their properties, should be obtained. Differences, and first of all qualitative differences, from the results based on the standard model should be described.

The most important purpose of the modern stage of development of fractional mathematical economics is the inclusion of memory and non-locality into the economic theory, into the basic economic concepts and methods. The economics should be extended and generalized such that it takes into account the memory and non-locality. Fractional generalizations of standard economic models should be constructed only on this conceptual basis. The most important purpose of studying such generalizations is the search and formulation of qualitatively new effects and phenomena caused by memory and non-locality in the behavior of economic processes. In this case, these results in mathematical economics, which are based on fractional calculus, can be further used in computer simulations of real economic processes and in econometric studies.

Let us list some generalizations of economic concepts and fractional generalizations of economic models that have already been proposed in recent years. Using the fractional calculus approach to describe the processes with memory and non-locality, the generalizations of some basic economic notions were proposed in the works [118–139]. The list of these new notions and concepts primarily include the following:

- The marginal value of non-integer order [118–122,190] with memory and non-locality;
- The economic multiplier with memory [123,124];
- The economic accelerator with memory [123,124];
- The exact discretization of economic accelarators and multiplier [125–128] based on exact fractional differences [129];
- The accelerator with memory and crisis periodic sharp bursts [130,131];
- The duality of the multiplier with memory and the accelerator with memory [123,124];
- The elasticity of fractional order [132–135] for processes with memory and non-locality;
- The measures of risk aversion with non-locality [136] and with memory [137];
- The warranted (technological) rate of growth with memory [112,170,174–176,189];
- The non-local methods of deterministic factor analysis for [138,139];

And some other.

The use of these notions and concepts makes it possible for us to generalize some classical economic models, including those proposed by the following well-known economists:

- Henry Roy F. Harrod [140–142] and Evsey D. Domar, [143,144];
- John M. Keynes [145–148];
- Wassily W. Leontief [149,150];
- Alban W.H. Phillips [151,152];
- Roy G.D. Allen [153–156];
- Robert M. Solow [157,158] and Trevor W. Swan [159];
- Nicholas Kaldor [160–162];

And other scientists.

Valentiva V. Tarasova and the author built various economic models with power-law memory, which are generalizations of the classical models. For example, the following economic models have been proposed.

- The natural growth model [168,169];
- The growth model with constant pace [170,171];
- The Harrod–Domar model [172,173] and [112,174–176];
- The Keynes model [177–179];
- The dynamic Leontief (intersectoral) model [86,180–182];
- The dynamics of fixed assets (or capital stock) [170,171];
- The logistic growth model with memory [183,189];
- The model of logistic growth with memory and crises [183];
- The time-dependent dynamic intersectoral model with memory [86,182];
- The Phillips model with memory and lag [185];
- The Harrod–Domar growth model with memory and distributed lag [186];
- The dynamic Keynesian model with memory and lag [187];
- The model of productivity with fatigue and memory [188];
- The Solow–Swan model [189];
- The Kaldor-type model of business cycles (the Van der Pol model) [189];

And some other economic models.

Let us also note works, in which fractional dynamic generalizations of economic models were proposed without introducing new economic notions and concepts.

(1) Michele Caputo proposed some fractional dynamic model of economy [191–200]:
    - In the standard relaxation equation, which describes the relaxation economy to equilibrium, the memory has been introduced in the reactivity of investment to the interest rate.
    - The continuous-time IS–LM model with memory [192];
    - The tax version of the Fisher model with memory for stock prices and inflation rates [199] that can be used to predict nominal and real interest rate behavior with memory.

(2) Mathematical description of some fractional generalization of economic models was proposed by the Kabardino–Balkarian group: Adam M. Nakhushev [201,202], Khamidbi Kh. Kalazhokov [203], Zarema A. Nakhusheva [204].

(3) Mathematical description of some fractional generalization of economic models were proposed by the Kamchatka group: Viktoriya V. Samuta, Viktoriya A. Strelova, and Roman I. Parovik [205], Yana E. Shpilko, Anastasiya E. Solomko., Roman I. Parovik [206] Danil M. Makarov [207].

(4) Shiou-Yen Chu and Christopher Shane proposed the hybrid Phillips curve model with memory to describe the dynamic process of inflation with memory in the work [208].

(5) Rituparna Pakhira, Uttam Ghosh and Susmita Sarkar derived [209–213] some inventory models with memory.

(6) Computer simulation for modeling the national economies in the framework of the fractional generalizations of the Gross domestic product (GDP) model was proposed by Inés Tejado, Duarte Valério, Nuno Valério, Pedro Pires [214–220] in 2014–2019, and by Dahui Luo, JinRong Wang, Michal Feckan in 2018 in the paper [221].

(7) In addition, we may note the works with economic models that were proposed in [100–105] that are related to the deterministic chaos stage.

Let us note that the problems and difficulties arising in the construction of fractional-dynamic analogs of standard economic models by using the fractional calculus are described in [189] with details.

Some of proposed models can be considered as econophysics approach, which are based on fractional generalization of the standard damped harmonic oscillator equation, where the memory has been introduced in the frictional term by using fractional derivative instead of first-orderr derivative.

New principles, effects and phenomena have been suggested for fractional economic dynamics with memory and non-locality (for example, see [174–176,189,222]). Qualitatively new effects due to the presence of memory in the economic process are described in the works [174–176,189,222].

In my opinion, this stage of the development of fractional mathematical economics actually includes (absorbs) approaches based on the ARFIMA model using only the Granger–Joyeux–Hosking fractional differencing and integrating, which in really are the well-known Grunwald–Letnikov fractional differences [39]. This opinion is based on the obvious fact that the new stage allows the AFRIMA approach to go beyond the restrictions of the Grunwald–Letnikov operators, and use different types of fractional finite differences and fractional derivatives of non-integer orders.

Moreover, this stage can include (absorbs) approaches based on the fractional econophysics and deterministic chaos. For the econophysics approach, new opportunities are opening up on the way to formulating economic analogues of physical concepts and notions that will be more understandable to economists. This will significantly simplify the implementation of the concepts and methods of fractional econophysics in economic theory and application.

The most important element in the construction of the fractional mathematical economics as a new theory is the emergence and the formation of new notions, concepts, effects, phenomena, principles and methods, which are specific only to this theory. This gives rise to a new scientific direction (the fractional mathematical economics), since there is something of their own that others do not have.

We have now entered the stage of forming a new direction in mathematical economics and economic theory, when concepts and methods are not borrowed from other sciences and areas, but their own are created.

3. New Future Stages and Approaches

There is a natural question. What stages and approaches will appear and develop in the future? In this section we propose some assumptions about the future use of fractional calculus in economics and discuss the direction of development of fractional mathematical economics.

3.1. Self-Organization in Fractional Economic Dynamics

Self-organization processes play an important role in both the natural and social sciences. In the description of self-organization in economic (and social) processes, it is necessary to abandon the assumption that all economic agents suffer from amnesia. They should be considered as agents with memory that interact with each other. We can consider self-organization with memory [222] in economics and social sciences. Therefore, the fractional mathematical economics and economic theory can be developed by considering the generalization of different economic models of self-organization that are described in the books [223–225].

3.2. Distributed Lag Fractional Calculus

The continuously distributed lag has been considered in economics starting with the works of Michal A. Kalecki [163] and Alban W.H. Phillips [151,152]. The continuous uniform distribution of delay time is considered by M.A. Kalecki in 1935 [163], (see also Section 8.4 of [154], (pp. 251–254)) for dynamic models of business cycles. The continuously distributed lag with the exponential distribution of delay time is considered by A.W.H. Phillips [151,152] in 1954. In the Phillips growth models, generalization of the economic concepts of accelerator and multipliers were proposed by taking into account the distributed lag. The operators with continuously distributed lag were considered by Roy G.D. Allen [153,154], (pp. 23–29), in 1956. Currently, economic models with delay are actively used to describe the processes in the economy.

The time delay is caused by finite speeds of processes and therefore it cannot be considered as processes with long memory (for some details about concept of memory, see Section 3.12 of this paper and [29–35,112,113,116,117]). For example, in physics the propagation of the electromagnetic field with finite speed in a vacuum does not mean the presence of memory in this process. In economics and electrodynamics, processes with time delay (lag) are not referred to as processes with memory and effects of time delay are not interpreted as a memory.

Note that the operators with exponentially distributed lag, which were defined in the works of Caputo and Fabrizio [226,227], cannot be interpreted as fractional derivatives of non-integer orders and cannot describe processes with memory. Note that exponential distribution is the continuous analogue of the geometric distribution that has the key property of being memoryless. The Caputo–Fabrizio operators are integer-order derivatives with the exponentially distributed delay time [228]. The fractional generalizations of the Caputo–Fabrizio operator are proposed in [185–187,228].

The distributed lag fractional calculus was proposed by the author and Svetlana S. Tarasova in [228]. To take into account the memory and lag in economic and physical models, the fractional differential and integral operators with continuously distributed lag (time delay) were proposed in [228]. The distributed lag fractional operators are compositions of fractional differentiation or integration and continuously distributed translation (shift). The kernels of these operators are the Laplace convolution of probability density function and the kernels of fractional derivatives or integrals. The random variable is the delay time that is distributed by probability law (distribution) on positive semiaxis. Examples of economic application of the lag-distributed fractional operators have been suggested in the works [185–187], where the economic concepts of accelerator and multipliers with distributed lag and memory were proposed.

3.3. Distributed Order Fractional Calculus

In general, the order parameter α, which can be interpreted as the parameter of memory fading [112,113], can be distributed on the interval $[\alpha_1, \alpha_2]$, where the distribution is described by a weight function $\rho(\alpha)$. The functions $\varrho(\alpha)$ describes distribution of the parameter of the memory fading on a set of economic agents. This is important for the economics, since various types of economic agents may have different parameters of memory fading. In this case, we should consider the fractional operators, which depends on the weight function $\rho(\alpha)$ and the interval $[\alpha_1, \alpha_2]$.

The concept of the integrals and derivatives with distributed orders was first proposed by Michele Caputo in [229] in 1995, and then these operators are applied and developed in different works (for example see [230–234]).

Let us note the following three cases.

(1) The simplest distribution of the order of fractional derivatives and integrals is the continuous uniform distribution. The fractional operators with uniform distribution were proposed in [112,113,129] and were called as the Nakhushev operators. Adam M. Nakhushev [235,236] proposed the continual fractional derivatives and integrals in 1998. The fractional operators, which are inversed to the continual fractional derivatives and integrals, were suggested by Arsen V.

Pskhu [237,238]. In papers [112,113,129], we proved that the fractional integrals and derivatives of the uniform distributed order could be expressed (up to a numerical factor) through the continual fractional integrals and derivatives, which have been suggested by A. M. Nakhushev [235,236]. The proposed fractional integral and derivatives of uniform distributed order have been called in our paper [129] as the Nakhushev fractional integrals and derivatives. The corresponding inverse operators, which contains the two-parameter Mittag–Leffler functions in the kernel, were called as the Pskhu fractional integrals and derivatives [129].

(2) In the papers [112,113], we proposed the concept of "weak" memory and the distributed order fractional operators with the truncated normal distribution of the order. The truncated normal distribution with integer mean and small variance can be used to describe economic processes with memory, which is distributed around the classical case.

(3) As a special case of the general fractional operators, which were proposed by Anatoly N. Kochubei, the fractional derivatives and integrals of distributed order are investigated in the works [239,240].

Fractional differential equations of distributed orders are actively used to describe physical processes. However, at the present time, equations with distributed order operators have not yet been used to describe economic processes. We hope that new interesting effects in economics can be described by using order-distributed fractional operators.

3.4. Generalized Fractional Calculus in Economics

Generalized fractional calculus was proposed by Virginia Kiryakova and described in detail in the book [2] in 1994. The brief history of the generalized fractional calculus is given in the paper [10]. Operators of generalized fractional calculus [2,10] can be use to describe complex processes with memory and non-locality in real economy. In the application of the generalized fractional operators, an important question arises about the correct economic (and physical) interpretation of these operators. It is important to emphasize that not all fractional operators can describe the processes with memory. It is important to clearly understand what type of phenomena can be described by a given operator. For example, among these types of phenomena, in addition to memory, we can specify the time delay (lag) and the scaling. Let us give a few examples to clarify this problem.

Example 1. The Abel-type fractional integral (and differential) operator with Kummer function in the kernel, which is described in the classic book [1] (see equation 37.1 in [1], (p. 731)) can be interpreted as the Riemann–Liouville fractional integral (and derivatives) with gamma distribution of delay time [187,228]. Some Prabhakar fractional operators with the three-parameter Mittag–Leffler functions in the kernel can also be interpreted as a Laplace convolution of the Riemann–Liouville (or Caputo) fractional operators with continuously distributed lag (time delay) [186,228].

Example 2. We can state that the Kober fractional integration of non-integer order [1,2,4] can be interpreted as an expected value of a random variable up to a constant factor [241] (see also Section 9 in [228]). In this interpretation, the random variable describes dilation (scaling), which has the gamma distribution. The Erdelyi–Kober fractional integration also has a probabilistic interpretation. Fractional differential operators of Kober and the Erdelyi–Kober type have analogous probabilistic interpretation, i.e., these operators cannot describe the memory. These operators describe integer-order operator with continuously distributed dilation (scaling). The fractional generalizations of the Kober and Erdelyi–Kober operators, which can be used to describe memory and distributed dilation (scaling), were proposed in [228].

Example 3. The Riesz fractional integro-differentiation (See Section 2.10 of [4]) cannot be used to describe memory since this operator violates the causality principle, if it is written in the standard form. For economic and physical processes with memory, the causality can be described by the

Kramers–Kronig relations [116]. The Riesz fractional integro-differentiation can be used to describe power-law non-locality and power-law spatial dispersion.

Therefore, an important part of the application of generalized fractional calculus is a clear understanding of what types of processes and phenomena can describe fractional operators of non-integer order.

3.5. General Fractional Calculus

The concept of general fractional calculus was suggested by Anatoly N. Kochubei [239,240] by using the differential-convolution operators. The works [239,240] describe the conditions under which the general operator has a right inverse (a kind of a fractional integral) and produce, as a kind of a fractional derivative, equations of evolution type. A solution of the relaxation equations with the Kochubei general fractional derivative with respect to the time variable is described [239]. In the works [239,240] the Cauchy problem (A) is considered for the relaxation equation $\left(D_{(k)}X\right)(t) = \lambda X(t)$, where $\lambda < 0$. This Cauchy problem has a solution $X(\lambda, t)$, which is continuous on $\mathbb{R}_+$, infinitely differentiable and completely monotone on $\mathbb{R}_+$, if the Kochubei conditions (*) are satisfied.

In the economics, various growth models are used to describe real processes in economy. Therefore, it is very important to describe conditions, for which the Cauchy problem (A) for the growth equation $\left(D_{(k)}X\right)(t) = \lambda X(t)$, where $\lambda > 0$, has a solution $X(\lambda, t)$.

The growth equation is considered in [242] for the special case of a distributed order derivative, where it was proved that a smooth solution exists and is monotone increasing. In addition, the solution of the growth equation has been proposed for the case of fractional differential operators with distributed lag in [185–187,228]. The existence of a solution in the growth case has been also proved in 2018 by Chung-Sik Sin [243] for nonlinear equation with a generalized derivative like the Kochubei fractional derivative. The growth equation for physics is discussed in [244].

Solving the problem in the general case will allow us to accurately describe the conditions on the operator kernels (the memory functions), under which equations for models of economic growth with memory have solutions. A paper dedicated to solving this mathematical problem was written by Anatoly N. Kochubei and Yuri Kondratiev [245] in 2019 for the Special Issue "Mathematical Economics: Application of Fractional Calculus" of Mathematics. The application of these mathematical results in economics and their economic interpretation is an open question at the moment.

To understand the warranted (technological) growth rate of the economy, it is important to obtain the asymptotic behavior for solutions of the general growth equation.

In the application of the general fractional operators, it is also important to have correct economic and physical interpretations that will connect the types of operator kernels with the types of phenomena. For example, it is obvious that the kernels of general fractional operators satisfying the normalization condition will describe distributed delays in time (lag), and not memory.

3.6. Partial Differential Equations in Economics

Usually the mathematical formulation of macroeconomic models is reduced to systems of difference equations or ordinary differential equations that describe the dynamics of a relatively small number of macroeconomic aggregates. However, it is known that, in macroeconomics, partial differential equations (PDEs) naturally arise, and they are used in macroeconomics [246,247]. Accounting for non-locality (for example, a power type) in the state space leads us to the necessity of using fractional partial differential equations.

3.7. Fractional Variational Calculus in Economics

In mathematical economics, theorems on the existence of extreme values of certain parameters are proved, the properties of equilibrium points and equilibrium growth trajectories are actively studied. The existence of optimal solutions for fractional differential equations should be considered for economic processes with memory and non-locality.

Methods of the fractional calculus of variations are actively developing [248,249]. However, at the present time, none of the variational problems, which are well known in economics, has been generalized to the case of processes with memory using fractional calculus.

In the variation approach, there are some problems that restrict the possibilities of its application. One of the problems associated with the property of integration in parts, which actually turns the left-second fractional derivative into a right-sided derivative. As a result, we will obtain equations in which, in addition to being dependent on the past, there is a dependence on the future, that is, the principle of causality is violated.

We assume that this problem cannot be solved within the framework of using the principle of stationarity of the holonomic functional (action). It is necessary to use non-holonomic functionals. We can also consider non-holonomic constraints with fractional derivatives of non-integer orders [250]. We can also consider variations of non-integer order [251] and fractional variational derivatives [252].

Another problem is the mathematical interpretation and the economic (and physical) interpretations of extreme values. The non-holonomic constraints and variations of fractional orders should also have a correct economic (and physical) interpretation.

However, we emphasize that for the economics, finding the optimality and stability of the solution is very important.

3.8. Fractional Differential Games in Economics

Models of differential games in which derivatives of non-integer orders are used and, thereby, the power-fading memory is taken into account were proposed in the works of Arkadiy A. Chikriy (Arkadii Chikrii), Ivan I. Matychyn and Alexander G. Chentsov [253–257] (see also [258–260]), which are clearly not related to economy. Note that the construction of models of economic behavior, using differential games with power memory, instead of games with full memory, currently remains an open question. The construction of such models requires further research on economic behavior within the framework of game theory. The basis for such constructions can be the methods and results described in [253–260].

3.9. Economic Data and Fractional Calculus in Economic Modelling

We note the importance of using fractional calculus in computer simulations and modeling of real economic data, including data related to both macroeconomics and microeconomics.

The first works that can be attributed to the mathematical economics stage/approach are works published in 2014–2016 by Inés Tejado, Duarte Valério, Nuno Valério, [214–216]. An application of fractional calculus for modeling the national economies in the framework of the fractional generalizations of the Gross domestic product (GDP) model, which are described by the fractional differential equations are used, were considered by Inés Tejado, Duarte Valério, Nuno Valério, Pedro Pires [214–220] in 2014–2019, and by Dahui Luo, JinRong Wang, Michal Feckan in 2018 in the paper [221]. The fractional differential equation used in the fractional GDP model was obtained by replacing first-order derivatives with fractional derivatives. Therefore, this model requires theoretical justification and consistent derivivation.

To describe the real economic processes in the framework of fractional dynamic models, it is necessary to combine theoretical constructions and computer modeling.

3.10. Big Data

It is obvious that Big Data that describes behavior of peoples and other economic agents should contain information that can be considered as “Traces of People’s Memory”. It would be strange if these Big Data neglected memory, since people have memory if they don’t suffer from amnesia. We can assume that economic modeling in the era of Big Data will describe the memory effects in microeconomics and macroeconomics. The Big Data will give us a possibility to take into account the effects of memory and non-locality in those economic and financial processes in which they were not even suspected.

3.11. Fractional Econometrics

Economic theory is a branch of economics that employs mathematical models and abstractions of economic processes and objects to rationalize, explain and predict economic phenomena.

One of the main goals of economic theory and mathematical economics is to explain the processes and phenomena in economy and make predictions. To achieve this goal, within the framework of economic theory and mathematical economics, new notions, concepts, tools and methods should be developed for describing and interpreting economic processes with memory and non-locality. Obviously, it is impossible to explain an economy with memory without having adequate concepts.

Economic theory and mathematical economics are branches of economics in which the creation of concepts and theoretical (primarily mathematical) models of phenomena and their comparison with reality is used as the main method of understanding economy. Economic theory is a separate way of studying economy, although its content is naturally formed taking into account the observations of economy. The methodology of economic theory consists in constructing main economic concepts; in formulating (in mathematical language) the principles and laws of economics connecting these concepts; in explaining observable economic phenomena and effects by using the formulated concepts and laws; in predicting new phenomena that may be discovered.

Mathematical economics, when viewed in a narrow sense as a branch of mathematics, is reduced only to the study of the properties of economic models at the mathematical level of rigor. In this approach, mathematical economics is often denied in the choice of economic concepts, interpretation and comparison of models with economic reality. For fractional mathematical economics, to describe processes with memory this view is erroneous. Obviously, it is impossible to explain economic processes with memory without having adequate concepts, since many standard concepts are not applicable.

Now the economics is undergoing a new revolution, "Memory revolution". A mathematic tool in the revolution is a new mathematics (fractional calculus), which was not previously used in mathematical economics. As a result, the present stage is a stage of the formation of new concepts and methods. In this stage, the mathematical economics cannot be isolated from the process of formation of new concepts and methods. Non-standard properties of fractional operators should be reflected in new economic concepts that take into account memory effects. Economic theory and mathematical economics can explain and predict processes with memory in real economy only if they create a solid foundation of new adequate economic concepts and principles.

To explain and predict the processes and phenomena with memory in economy, we must have a good instrument for conducting observations and their adequate description. This instrument is econometrics, or rather fractional econometrics. Econometrics is a link that connects economic theory and mathematical economics with the phenomena and processes in real economy.

Econometrics mainly based on statistics for formulating and testing models and hypotheses about economic processes or estimating parameters for them. Theoretical econometrics considers the statistical properties of assessments and tests, while applied econometrics deals with the use of econometric methods for evaluating economic models. Theoretical econometrics develops tools and methods, and also studies the properties of econometric methods. Applied econometrics uses theoretical econometrics and economic data to evaluate economic theories, develop econometric models, analyzing economic dynamics, and forecasting.

Fractional econometrics is based on statistics of long-memory processes [29–32]. Currently, fractional econometrics methods are actually related to ARFIMA models and the well-known Grunwald–Letnikov fractional differences in the form of the Granger–Joyeux–Hosking fractional differencing and integrating.

Fractional econometrics can reach new opportunities in the development of new econometric methods and their use in describing economic reality by applying methods of modern fractional calculus, various types of fractional finite differences, differential and integral operators of non-integer order.

As a result, we can state that the main goals of fractional economics such as explain the economy and to make predictions for processes with memory, and correctly describe economic events, data, processes, and to give adequate predictions, we should have fractional econometrics.

One of the main goals of fractional mathematical economics and economic theory is to explain the processes and phenomena with memory in economy and make predictions. However, to explain the economic processes with memory, it is necessary to understand what memory is and how to describe it. Note that a clear understanding of the memory does not even exist within the framework of fractional calculus approach.

3.12. Development Concept of Memory

Further development of the concept of memory for economic processes in the framework of fractional calculus is of great importance. It should be emphasized that not all types of fractional derivatives and integrals of non-integer order can describe processes with memory. The concept of memory itself for economic processes is discussed in [29–35], and [39,40,110–117]. One of the possible criteria of memory proposed in the work [116]. Let us give some comments about memory concept.

From mathematical point of view, economic models may be classified as stochastic or deterministic and as discrete or continuous. For simplification, let us consider the deterministic approach with continuous time. In this case, memory can be defined as a property of the process, when there exists at least one endogenous variable $Y(t)$, and an associated exogenous (or endogenous) variable $X(t)$, such that the variable $Y(t)$ at the time $t > t_0$ depends on the history of the change of $X(\tau)$ on the interval $\tau \in (t_0, t)$. However, not all such dependencies are due to the memory effect.

To describe processes with memory, this dependence of one variable on another should satisfy the causality principle. For economic and physical processes with memory, the causality can be described by the Kramers–Kronig relations [116].

An important property of memory is described by the principle of memory fading that was proposed by Ludwig Boltzmann in 1874 and 1876. Then, it was significantly developed by Vito Volterra in 1928 and 1930. The principle of memory fading states that the increasing of the time interval leads to a decrease in the corresponding contribution to the variable $Y(t)$.

Note that in physics the concept of fading memory assumes a set of stronger restrictions on memory. For example, it is often assumed that the memory is described by functions, which tends to zero monotonically with increasing the time variable. In this form, the principle of fading memory assumes that it is less probable to expect strengthening of the memory with respect to the more distant events. However, in some economic processes, it should be taken into account that the economic agents may remember sharp and significant changes of the exogenous variable $X(\tau)$, despite the fact that these changes were a more distant past compared to weaker changes in the near past. For this reason, in economics we can use memory functions that are not monotonic decrease.

For a simple case, we can consider the dependence in the form

$$Y(t) = \int_{t_0}^{t} M(t,\tau) X(\tau) d\tau,$$

where the kernel $M(t,\tau)$ of this integral operator is called the memory function (or the linear response function). Obviously, the derivative of the integer orders of some variable can be considered as an associated variable $X(\tau)$. We also can consider integer-order derivatives of $Y(t)$ as endogenous variables.

It is obvious that not every kernels $M(t,\tau)$ can be used to describe the memory in the economic processes. Possible restrictions on the memory function are discussed in paper [112,113].

In this paper [112,113], we describe some general restrictions that can be imposed on the structure and properties of memory. In addition, to the causality principle [116], these restrictions include the following three principles:

- The principle of fading memory;
- The principle of memory homogeneity on time (the principle of non-aging memory);
- The principle of memory reversibility (the principle of memory recovery).

Mathematically, the principle of non-aging memory means that the memory function has a property $M(t,\tau) = M(t-\tau)$. In this case, the integral operator can be described by the convolution $Y(t) = (M * X)(t)$, [112,113].

The principle of memory reversibility is connected with the principle of duality of accelerator with memory and multiplier with memory, which is proposed in [123,124]. In general, fractional calculus, which was proposed in [239,240] and based on the use of differential-convolution operators, the principle of memory reversibility means that the general operators should have a right inverse (a kind of a fractional integral).

Note that the Kober and Erdelyi–Kober fractional operators, which are interpreted in [112,113] as operators that describe memory with generalized power-law fading, really are integer-order operators with continuously distributed scaling or dilation (see [241] and Section 9 in [228]), and therefore, these operators cannot describe the memory.

Note that time delay, which is sometimes interpreted as a complete (perfect, ideal) memory [112,113], cannot describe memory. In economics and electrodynamics, processes with time delay (lag) are not referred to as processes with memory and time delay is not interpreted as a memory. The interpretation of the time delay, which is usually called a lag in economics, as some kind of memory seems to be incorrect for the following reasons.

From economic and physical points of view, the time delay is caused by finite speeds of processes, i.e., the change of one variable does not lead to instant changes of another variable. Therefore, the time delay cannot be considered as memory in processes. This fact is well-known in physics as the retarded potential of an electromagnetic field, when a change in the electromagnetic field at the observation point is delayed with respect to the change in the sources of the field located at another point. The processes of propagation of the electromagnetic field in a vacuum are not interpreted in physics as presence of memory in these processes.

From a mathematical point of view, the kernels of integral operators for distributed time delay (lag) and fading memory are distinguished by the fact that the normalization condition holds for the time delay case. Note that the probability distribution functions as kernels, which are usually called the weighting function in economics, are actively used for macroeconomic models with distributed delay time. Equivalent differential equations of integer orders in economics are usually used instead of equations with integro-differential operators, in which the weighting function in the kernels. It is known that under certain conditions, equations with continuously distributed lag are equivalent to differential equations with standard derivatives of integer orders. Mathematically, this means that processes with time delay can be described by equations containing only a finite number of derivatives of integer orders. The integer-order derivatives of functions are determined by the properties of these functions in small neighborhood of the considered point. As a result, differential equations of integer orders cannot describe a memory. To describe processes with fading memory and distributed time delay, we should use the distributed lag fractional calculus [228], (see also [185–187]).

As a result, within the framework of fractional calculus, it is necessary to distinguish between fractional operators that describe distributed time delay and distributed scaling from operators describing memory, and the combination of memory with these phenomena. However, there are open questions about what types of memory we can describe by using fractional calculus (for example, see [116,117]), and in what directions the concept of memory for economic processes will develop.

4. Conclusions

In this brief historical description, an attempt was made to draw a sketch picture of the development of fractional calculus applications in economics, the birth of a new direction in mathematical economics, a new revolution in economic theory. Due to brevity and schematics, this picture obviously cannot reflect the fullness and complexity of the development of a fractional mathematical economics. As a result, it was possible that some directions and approaches, results and works close in the described history were missed. One can hope that the written short history will be perceived with understanding and will be supplemented in the future with new works on the history of the use of fractional calculus in the economics.

We can hope that the further development of the use of fractional calculus to describe economic phenomena and processes will take an important place with modern mathematical economics and economic theory. Generally speaking, it is strange to neglect memory in the economics, since the most important actors are people with memory.

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