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# Enhanced $H_\infty$ Filtering for Continuous-time State-delayed Systems

Ying Zhang[1,2,*] Ai-Guo Wu[1,3] Guang-Ren Duan[1,3]

[1]Institute of Information and Control, Shenzhen Graduate School, Harbin Institute of Technology, Shenzhen 518055, PRC
[2]Department of Automation, Harbin University of Science and Technology, Harbin 150080, PRC
[3]Center for Control Theory and Guidance Technology, Harbin Institute of Technology, Harbin 150001, PRC

**Abstract:** The $H_\infty$ filtering problem for continuous-time polytopic uncertain time-delay systems is investigated. Attention is focused on the design of full-order filters guaranteeing a prescribed $H_\infty$ attenuation level for the filtering error system. First, a simple alternative proof is given for an improved linear matrix inequality (LMI) representation of $H_\infty$ performance. Then, based on the performance criterion which keeps Lyapunov matrices out of the product of system dynamic matrices, a sufficient condition for the existence of robust estimators is formulated in terms of LMIs, and the corresponding filter design is cast into a convex optimization problem which can be efficiently handled by using standard numerical algorithms. It is shown that the proposed design strategy allows the use of parameter-dependent Lyapunov functions and hence it is less conservative than some earlier results. A numerical example is employed to demonstrate the feasibility and advantage of the proposed design.

**Keywords:** Conservativeness, parameter-dependence, time-delay systems, linear matrix inequality (LMI).

## 1 Introduction

Over the past decades, considerable attention has been devoted to the $H_\infty$ filtering problem, and many significant results have been reported in the literature[1−4] and the references therein. Compared with the conventional Kalman filtering, the $H_\infty$ filtering has several advantages. First, the noise sources in the $H_\infty$ filtering setting can be arbitrary signals with bounded energy, and no exact statistics are required to be known. Second, the filter has been shown to be much more robust to parameter uncertainties. On the other hand, time-delays are often encountered in various engineering systems such as long transmission line, chemical processes, nuclear reactors, and so on. The characteristics of dynamic systems can be significantly affected by the presence of uncertainties and time delays, even to the extent of instability or poor performance[5−7]. Many results on the $H_\infty$ filtering for time-delay systems have been reported, for example [8–11]. Most of the reported results are based on quadratic Lyapunov functions[12], which have been largely used for robust analysis and synthesis in the past decades. However, this method can produce conservative results since the same parameter independent Lyapunov function must be used for the entire uncertain domain. One possible way to overcome this conservatism was recognized in considering a parameter-dependent Lyapunov function proposed in [13]. This new condition keeps Lyapunov matrices out of any product with system matrices by introducing an additional matrix variable and hence allows the Lyapunov function to be vertex-dependent. Based on this idea, some other $H_\infty$ analysis conditions with the separation property between Lyapunov matrices and system matrices were presented in [14, 15].

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*Corresponding author. E-mail address: zhangying@hit.edu.cn

The paper is organized as follows. Section 2 states the class of delay systems for which filters will be designed. Section 3 gives a new $H_\infty$ performance criterion, which exhibits a kind of decoupling between Lyapunov matrices and system dynamic matrices by introducing two slack matrices. Section 4 presents a sufficient condition for the existence of robust $H_\infty$ estimators in terms of linear matrix inequalities (LMIs) based on the preliminary formulation of Section 3. It is shown that the proposed design strategy allows the use of vertex-dependent Lyapunov functions and hence it is less conservative than some earlier results. A simulation example is used to illustrate the procedure and performance of the proposed approaches in Section 5, which is followed by conclusions in Section 6.

The notation used throughout the paper is fairly standard. $\mathbf{R}^n$ denotes the $n$-dimensional Euclidean space, $\mathbf{R}^{m\times n}$ is the set of all $m\times n$ real matrices, and the notation $P>0$ means that $P$ is symmetric and positive definite. In addition, in symmetric block matrices or long matrix expressions, we use "*" as an ellipsis for the terms that are introduced by symmetry, and diag(·) stands for a block-diagonal matrix.

## 2 Problem description

Consider the following nominal linear system with time-delay:

$$\left\{\begin{array}{l}\dot{x}(t)=Ax(t)+A_dx(t-d(t))+Bw(t)\\ y(t)=Cx(t)+C_dx(t-d(t))+Dw(t)\\ z(t)=Lx(t)+L_dx(t-d(t))+Hw(t)\\ x(t)=\phi(t),\quad t\in[-\overline{d},0]\end{array}\right. \tag{1}$$

where $x(t)\in\mathbf{R}^n$ is the state vector, $w(t)\in\mathbf{R}^p$ is the disturbance input, $y(t)\in\mathbf{R}^m$ is the measurement output, and $z(t)\in\mathbf{R}^q$ is the signal to be estimated. $d(t)$ is a time-varying delay satisfying $0<d(t)\leqslant\overline{d}<\infty$ and $\dot{d}(t)\leqslant\tau<1$, where $\overline{d}$ and $\tau$ are real constant scalars, $\phi(t)$ is a real-valued initial vector function that is continuous on the interval $[-\overline{d}, 0]$. The state-space data are assumed to be subject to uncertainties in the form of a polytopic model

$$\begin{bmatrix} A & C & L \\ A_d & C_d & L_d \\ B & D & H \end{bmatrix} \in \left\{ \begin{bmatrix} A(\alpha) & C(\alpha) & L(\alpha) \\ A_d(\alpha) & C_d(\alpha) & L_d(\alpha) \\ B(\alpha) & D(\alpha) & H(\alpha) \end{bmatrix} = \sum_{i=1}^{r} \alpha_i \begin{bmatrix} A_i & C_i & L_i \\ A_{di} & C_{di} & L_{di} \\ B_i & D_i & H_i \end{bmatrix}, \ \alpha \in \Gamma \right\} \quad (2)$$

where $\Gamma$ is the unit simplex

$$\Gamma = \left\{ (\alpha_1, \alpha_2, \cdots, \alpha_r) : \sum_{i=1}^{r} \alpha_i = 1, \alpha_i \geqslant 0 \right\}.$$

Consider an estimator or filter described by

$$\begin{cases} x_F(t) = A_F x_F(t) + B_F y(t) \\ z_F(t) = C_F x_F(t) + D_F y(t). \end{cases} \quad (3)$$

Augmenting model (1) to include the states of the filter, we obtain the following system (called filtering error system):

$$\begin{cases} \dot{\xi}(t) = \overline{A}\xi(t) + \overline{A}_d \xi(t - d(t)) + \overline{B} w(t) \\ e(t) = \overline{C}\xi(t) + \overline{C}_d \xi(t - d(t)) + \overline{D} w(t) \\ \xi(t) = \begin{bmatrix} \phi^{\mathrm{T}}(t) & 0 \end{bmatrix}^{\mathrm{T}}, \quad t \in [-\overline{d}, 0] \end{cases} \quad (4)$$

where

$$\xi(t) = \begin{bmatrix} x^{\mathrm{T}}(t) & x_F^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}, \quad e(t) = z(t) - z_F(t),$$

$$\overline{A} = \begin{bmatrix} A & 0 \\ B_F C & A_F \end{bmatrix}, \overline{A}_d = \begin{bmatrix} A_d & 0 \\ B_F C_d & 0 \end{bmatrix}, \overline{B} = \begin{bmatrix} B \\ B_F D \end{bmatrix},$$

$$\overline{C} = \begin{bmatrix} L - D_F C & -C_F \end{bmatrix}, \overline{C}_d = \begin{bmatrix} L_d - D_F C_d & 0 \end{bmatrix},$$

$$\overline{D} = H - D_F D.$$

Our objective is to develop a robust $H_\infty$ filter of form (3) such that for all admissible uncertainties:

1) the filtering error system (4) is asymptotically stable;

2) the filtering error system (4) guarantees, under zero-initial condition, $\|e(t)\|_2 \leqslant \gamma \|w(t)\|_2$ for all nonzero $w(t) \in L_2[0, +\infty)$ and a given positive scalar $\gamma$.

## 3 $H_\infty$ performance analysis

In this section, an improved LMI representation for $H_\infty$ performance analysis is presented. As a preliminary, we first introduce the following lemma that will play an important role in our derivation.

**Lemma 1.** Given a scalar $\gamma > 0$, the error system (4) is asymptotically stable with $\|e\|_2 \leqslant \gamma \|w\|_2$ if there exist positive definite matrices $P$ and $Q$ satisfying

$$\begin{bmatrix} \overline{A}^{\mathrm{T}} P + P\overline{A} + Q & P\overline{A}_d & P\overline{B} & \overline{C}^{\mathrm{T}} \\ \overline{A}_d^{\mathrm{T}} P & -(1-\tau)Q & 0 & \overline{C}_d^{\mathrm{T}} \\ \overline{B}^{\mathrm{T}} P & 0 & -\gamma I & \overline{D}^{\mathrm{T}} \\ \overline{C} & \overline{C}_d & \overline{D} & -\gamma I \end{bmatrix} < 0. \quad (5)$$

**Theorem 1.** System (4) is asymptotically stable with $\|e\|_2 \leqslant \gamma \|w\|_2$ if there exist positive definite matrices $P$ and $Q$ and matrices $F$ and $G$ satisfying

$$\begin{bmatrix} \Phi_1 & \Phi_2 & F^{\mathrm{T}}\overline{A}_d & F^{\mathrm{T}}\overline{B} & \overline{C}^{\mathrm{T}} \\ \Phi_2^{\mathrm{T}} & -G - G^{\mathrm{T}} & G^{\mathrm{T}}\overline{A}_d & G^{\mathrm{T}}\overline{B} & 0 \\ \overline{A}_d^{\mathrm{T}} F & \overline{A}_d^{\mathrm{T}} G & -(1-\tau)Q & 0 & \overline{C}_d^{\mathrm{T}} \\ \overline{B}^{\mathrm{T}} F & \overline{B}^{\mathrm{T}} G & 0 & -\gamma I & \overline{D}^{\mathrm{T}} \\ \overline{C} & 0 & \overline{C}_d & \overline{D} & -\gamma I \end{bmatrix} < 0 \quad (6)$$

where

$$\Phi_1 = \overline{A}^{\mathrm{T}} F + F^{\mathrm{T}}\overline{A} + Q, \ \ \Phi_2 = P - F^{\mathrm{T}} + \overline{A}^{\mathrm{T}} G.$$

**Proof.** Based on Lemma 1, we need only prove the equivalence between (6) and (5). Due to the strictness of inequality (5), there must exist a sufficiently small positive scalar $\alpha$ satisfying

$$\begin{bmatrix} \overline{A}^{\mathrm{T}} P + P\overline{A} + Q & P\overline{A}_d & P\overline{B} & \overline{C}^{\mathrm{T}} \\ \overline{A}_d^{\mathrm{T}} P & -(1-\tau)Q & 0 & \overline{C}_d^{\mathrm{T}} \\ \overline{B}^{\mathrm{T}} P & 0 & -\gamma I & \overline{D}^{\mathrm{T}} \\ \overline{C} & \overline{C}_d & \overline{D} & -\gamma I \end{bmatrix} + \frac{\alpha}{2} \begin{bmatrix} \overline{A}^{\mathrm{T}} \\ \overline{A}_d^{\mathrm{T}} \\ \overline{B}^{\mathrm{T}} \\ 0 \end{bmatrix} \begin{bmatrix} \overline{A} & \overline{A}_d & \overline{B} & 0 \end{bmatrix} < 0. \quad (7)$$

By the Schur complement lemma and congruent transformation, the above condition is equivalent to

$$\begin{bmatrix} \overline{A}^{\mathrm{T}} P + P\overline{A} + Q & \alpha\overline{A}^{\mathrm{T}} & P\overline{A}_d & P\overline{B} & \overline{C}^{\mathrm{T}} \\ \alpha\overline{A} & -2\alpha I & \alpha\overline{A}_d & \alpha\overline{B} & 0 \\ \overline{A}_d^{\mathrm{T}} P & \alpha\overline{A}_d^{\mathrm{T}} & -(1-\tau)Q & 0 & \overline{C}_d^{\mathrm{T}} \\ \overline{B}^{\mathrm{T}} P & \alpha\overline{B}^{\mathrm{T}} & 0 & -\gamma I & \overline{D}^{\mathrm{T}} \\ \overline{C} & 0 & \overline{C}_d & \overline{D} & -\gamma I \end{bmatrix} < 0.$$

By selecting $F = F^{\mathrm{T}} = P$ and $G = G^{\mathrm{T}} = \alpha P$, we can obtain (6).

In addition, since

$$T = \begin{bmatrix} I & \overline{A}^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & \overline{A}_d^{\mathrm{T}} & I & 0 & 0 \\ 0 & \overline{B}^{\mathrm{T}} & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$

has full row rank, pre- and post-multiplying both sides of (6) by $T$ and its transpose, respectively, gives (5). □

**Remark 1.** Theorem 1 gives an enhanced LMI representation of $H_\infty$ performance criterion. It is proven to be equivalent to Lemma 1. Moreover, LMI (6) realizes the elimination of the products coupling Lyapunov matrices and system dynamic matrices by introducing two slack matrices. Hence, this new criterion allows Lyapunov functions to be vertex-dependent for polytopic uncertain systems.

**Remark 2.** To obtain the $H_\infty$ performance criterion in the above theorem, we introduce a small scalar $\alpha$. For convenience, we call such an approach "small scalar method". This method was used in [9] to give an alternative proof of the equivalence between two LMI representations for $H_\infty$

performance of linear continuous-time systems. The so-called "small scalar method" can also be extended to obtain improved LMI performance criteria, such as $H_2$, dissipation, etc., for hybrid systems, stochastic systems, and so on.

## 4 Robust $H_\infty$ filtering

By virtue of the property of polytopic uncertainties, the following conclusion is readily obtained from Theorem 1.

**Theorem 2.** The error system (4) is asymptotically stable with $\|e\|_2 \leqslant \gamma \|w\|_2$ if there exist positive definite matrices $P_i$ and $Q_i$, $i = 1, 2, \cdots, r$, and matrices $F$ and $G$ such that for $i = 1, 2, \cdots, r$,

$$\begin{bmatrix} \Phi_{1i} & \Phi_{2i} & F^{\mathrm{T}}\overline{A}_{di} & F^{\mathrm{T}}\overline{B}_i & \overline{C}_i^{\mathrm{T}} \\ \Phi_{2i}^{\mathrm{T}} & -G-G^{\mathrm{T}} & G^{\mathrm{T}}\overline{A}_{di} & G^{\mathrm{T}}\overline{B}_i & 0 \\ \overline{A}_{di}^{\mathrm{T}}F & \overline{A}_{di}^{\mathrm{T}}G & -(1-\tau)Q_i & 0 & \overline{C}_{di}^{\mathrm{T}} \\ \overline{B}_i^{\mathrm{T}}F & \overline{B}_i^{\mathrm{T}}G & 0 & -\gamma I & \overline{D}_i^{\mathrm{T}} \\ \overline{C}_i & 0 & \overline{C}_{di} & \overline{D}_i & -\gamma I \end{bmatrix} < 0 \tag{8}$$

where

$$\Phi_{1i} = \overline{A}_i^{\mathrm{T}}F + F^{\mathrm{T}}\overline{A}_i + Q_i, \quad \Phi_{2i} = P_i - F^{\mathrm{T}} + \overline{A}_i^{\mathrm{T}}G,$$

$$\overline{A}_i = \begin{bmatrix} A_i & 0 \\ B_F C_i & A_F \end{bmatrix}, \overline{A}_{di} = \begin{bmatrix} A_{di} & 0 \\ B_F C_{di} & 0 \end{bmatrix}, \overline{B}_i = \begin{bmatrix} B_i \\ B_F D_i \end{bmatrix},$$

$$\overline{C}_i = \begin{bmatrix} L_i - D_F C_i & -C_F \end{bmatrix}, \overline{C}_{di} = \begin{bmatrix} L_{di} - D_F C_{di} & 0 \end{bmatrix},$$

$$\overline{D}_i = H_i - D_F D_i.$$

While the above is useful for evaluating the $H_\infty$ norm bound for the error system (4) when an estimator (3) is given, it may not be directly applicable to the robust $H_\infty$ filtering design problem due to the presence of the products of $F$ with $\overline{A}_i$ and $\overline{A}_{di}$, $G$ with $\overline{A}_i$ and $\overline{A}_{di}$. To enable the sub-optimal $H_\infty$ filtering design, the matrix is specialized as

$$F = \Lambda G \tag{9}$$

where $\Lambda = \mathrm{diag}(\lambda_1 I_n, \lambda_2 I_n)$ with $\lambda_1$ and $\lambda_2$ being real scalars. Using the above $F$, (8) can be rewritten as

$$\begin{bmatrix} \Psi_{1i} & \Psi_{2i} & G^{\mathrm{T}}\Lambda\overline{A}_{di} & G^{\mathrm{T}}\Lambda\overline{B}_i & \overline{C}_i^{\mathrm{T}} \\ \Psi_{2i}^{\mathrm{T}} & -G-G^{\mathrm{T}} & G^{\mathrm{T}}\overline{A}_{di} & G^{\mathrm{T}}\overline{B}_i & 0 \\ \overline{A}_{di}^{\mathrm{T}}\Lambda G & \overline{A}_{di}^{\mathrm{T}}G & -(1-\tau)Q_i & 0 & \overline{C}_{di}^{\mathrm{T}} \\ \overline{B}_i^{\mathrm{T}}\Lambda G & \overline{B}_i^{\mathrm{T}}G & 0 & -\gamma I & \overline{D}_i^{\mathrm{T}} \\ \overline{C}_i & 0 & \overline{C}_{di} & \overline{D}_i & -\gamma I \end{bmatrix} < 0 \tag{10}$$

where

$$\Psi_{1i} = \overline{A}_i^{\mathrm{T}}\Lambda G + G^{\mathrm{T}}\Lambda\overline{A}_i + Q_i, \quad \Psi_{2i} = P_i - G^{\mathrm{T}}\Lambda + \overline{A}_i^{\mathrm{T}}G.$$

The following result gives a solution to the robust $H_\infty$ filtering problem.

**Theorem 3.** Consider system (1) subject to polytopic uncertainties (2) and let $\gamma > 0$ be a given constant. Then, for all admissible uncertainties, a robust $H_\infty$ filter of the form (3) exists if there exist positive definite matrices $P_{11i} \in \mathbf{R}^{n\times n}$, $P_{22i} \in \mathbf{R}^{n\times n}$, $Q_{11i} \in \mathbf{R}^{n\times n}$, $Q_{22i} \in \mathbf{R}^{n\times n}$ and matrices $P_{12i} \in \mathbf{R}^{n\times n}$, $Q_{12i} \in \mathbf{R}^{n\times n}$, $X \in \mathbf{R}^{n\times n}$, $R \in \mathbf{R}^{n\times n}$, $U \in \mathbf{R}^{n\times n}$, $\overline{A}_F \in \mathbf{R}^{n\times n}$, $\overline{B}_F \in \mathbf{R}^{n\times m}$, $\overline{C}_F \in \mathbf{R}^{m\times n}$, $\overline{D}_F \in \mathbf{R}^{m\times p}$, and scalars $\lambda_1$ and $\lambda_2$ such that for $i = 1, 2, \cdots, r$, matrix inequalities (11), (12), and (13) hold.

$$\begin{bmatrix} P_{11i} & P_{12i} \\ P_{12i}^{\mathrm{T}} & P_{22i} \end{bmatrix} > 0 \tag{11}$$

$$\begin{bmatrix} Q_{11i} & Q_{12i} \\ Q_{12i}^{\mathrm{T}} & Q_{22i} \end{bmatrix} > 0 \tag{12}$$

where

$$\Omega_{11i} = \lambda_1(X^{\mathrm{T}}A_i + A_i^{\mathrm{T}}X) + \lambda_2(\overline{B}_F C_i + C_i^{\mathrm{T}}\overline{B}_F^{\mathrm{T}}) + Q_{11i}$$

$$\Omega_{21i} = \lambda_1(R^{\mathrm{T}}A_i + A_i^{\mathrm{T}}X) + \lambda_2(C_i^{\mathrm{T}}\overline{B}_F^{\mathrm{T}} + \overline{A}_F^{\mathrm{T}}) + Q_{12i}^{\mathrm{T}}$$

$$\Omega_{31i} = P_{11i} - \lambda_1 X + X^{\mathrm{T}}A_i + \overline{B}_F C_i$$

$$\Omega_{41i} = P_{12i}^{\mathrm{T}} - \lambda_1 X - \lambda_2 U^{\mathrm{T}} + R^{\mathrm{T}}A_i$$

$$\Omega_{51i} = \lambda_1 A_{di}^{\mathrm{T}}X + \lambda_2 C_{di}^{\mathrm{T}}\overline{B}_F^{\mathrm{T}}$$

$$\Omega_{61i} = \lambda_1 A_{di}^{\mathrm{T}}X + \lambda_2 C_{di}^{\mathrm{T}}\overline{B}_F^{\mathrm{T}}$$

$$\Omega_{71i} = \lambda_1 B_i^{\mathrm{T}}X + \lambda_2 D_i^{\mathrm{T}}\overline{B}_F^{\mathrm{T}}$$

$$\Omega_{22i} = \lambda_1(R^{\mathrm{T}}A_i + A_i^{\mathrm{T}}R) + Q_{22i}$$

$$\Omega_{32i} = P_{12i} - \lambda_1 R + X^{\mathrm{T}}A_i + \overline{B}_F C_i + \overline{A}_F$$

$$\Omega_{42i} = P_{22i} - \lambda_1 R + R^{\mathrm{T}}A_i$$

$$\Omega_{82i} = L_i - \overline{D}_F C_i - \overline{C}_F$$

$$\Omega_{85i} = L_{di} - \overline{D}_F C_{di}$$

$$\Omega_{86i} = L_{di} - \overline{D}_F C_{di}.$$

In addition, an admissible estimator with the form of (3) can be given by

$$A_F = U^{-1}\overline{A}_F, B_F = U^{-1}\overline{B}_F, C_F = \overline{C}_F, D_F = \overline{D}_F. \tag{14}$$

$$\begin{bmatrix} \Omega_{11i} & * & * & * & * & * & * & * \\ \Omega_{21i} & \Omega_{22i} & * & * & * & * & * & * \\ \Omega_{31i} & \Omega_{32i} & -X-X^{\mathrm{T}} & * & * & * & * & * \\ \Omega_{41i} & \Omega_{42i} & -R^{\mathrm{T}}-X-U^{\mathrm{T}} & -R-R^{\mathrm{T}} & * & * & * & * \\ \Omega_{51i} & \lambda_1 A_{di}^{\mathrm{T}}R & A_{di}^{\mathrm{T}}X + C_{di}^{\mathrm{T}}\overline{B}_F^{\mathrm{T}} & A_{di}^{\mathrm{T}}R & -(1-\tau)Q_{11i} & * & * & * \\ \Omega_{61i} & \lambda_1 A_{di}^{\mathrm{T}}R & A_{di}^{\mathrm{T}}X + C_{di}^{\mathrm{T}}\overline{B}_F^{\mathrm{T}} & A_{di}^{\mathrm{T}}R & -(1-\tau)Q_{12i}^{\mathrm{T}} & -(1-\tau)Q_{22i} & * & * \\ \Omega_{71i} & \lambda_1 B_i^{\mathrm{T}}R & B_i^{\mathrm{T}}X + D_i^{\mathrm{T}}\overline{B}_F^{\mathrm{T}} & B_i^{\mathrm{T}}R & 0 & 0 & -\gamma I & * \\ L_i - \overline{D}_F C_i & \Omega_{82i} & 0 & 0 & \Omega_{85i} & \Omega_{86i} & H_i - \overline{D}_F D_i & -\gamma I \end{bmatrix} < 0. \tag{13}$$

**Proof.** Since (13) implies

$$\begin{bmatrix} X+X^{\rm T} & * \\ X+U^{\rm T}+R^{\rm T} & R+R^{\rm T} \end{bmatrix} > 0$$

where $X$ and $R$ are also nonsingular, we can construct the matrices $G$ and $G^{-1}$ as

$$G = \begin{bmatrix} X & X_1 \\ X_2 & X_3 \end{bmatrix}, \quad G^{-1} = \begin{bmatrix} R^{-1} & Y_1 \\ Y_2 & Y_3 \end{bmatrix}.$$

Introduce matrices

$$\Psi = \begin{bmatrix} I & 0 \\ 0 & R \end{bmatrix}, \Pi_1 = \begin{bmatrix} I & R^{-1} \\ 0 & Y_2 \end{bmatrix}, \Pi_2 = \begin{bmatrix} X & I \\ X_2 & 0 \end{bmatrix}.$$

Then, we have $G\Pi_1 = \Pi_2$. Without loss of generality, it is assumed that both $Y_2$ and $X_2$ are nonsingular. Therefore, $\Pi_1\Psi$ is also nonsingular. Let

$$J = \Pi_1\Psi = \begin{bmatrix} I & I \\ 0 & Y_2R \end{bmatrix}.$$

By some algebraic operations, we can obtain

$$J^{\rm T}P_iJ = \begin{bmatrix} P_{11i} & P_{12i} \\ P_{12i}^{\rm T} & P_{22i} \end{bmatrix}, \quad J^{\rm T}Q_iJ = \begin{bmatrix} Q_{11i} & Q_{12i} \\ Q_{12i}^{\rm T} & Q_{22i} \end{bmatrix}$$

$$J^{\rm T}GJ = \begin{bmatrix} X & R \\ X+R^{\rm T}Y_2^{\rm T}X_2 & R \end{bmatrix}$$

$$\overline{B}_i^{\rm T}GJ = \begin{bmatrix} B_i^{\rm T}X + D_i^{\rm T}B_F^{\rm T}X_2 & B_i^{\rm T}R \end{bmatrix}$$

$$J^{\rm T}G^{\rm T}\overline{A}_{di}J = \begin{bmatrix} X^{\rm T}A_{di} + X_2^{\rm T}B_FC_{di} & X^{\rm T}A_{di} + X_2^{\rm T}B_FC_{di} \\ R^{\rm T}A_{di} & R^{\rm T}A_{di} \end{bmatrix}$$

$$J^{\rm T}G^{\rm T}\overline{A}_iJ = \begin{bmatrix} X^{\rm T}A_i + X_2^{\rm T}B_FC_i & \Delta_i \\ R^{\rm T}A_i & R^{\rm T}A_i \end{bmatrix}$$

$$\overline{C}_iJ = \begin{bmatrix} L_i - D_FC_i & L_i - D_FC_i - C_FY_2R \end{bmatrix}$$

$$\overline{C}_{di}J = \begin{bmatrix} L_{di} - D_FC_{di} & L_{di} - D_FC_{di} \end{bmatrix}$$

where

$$\Delta_i = X^{\rm T}A_i + X_2^{\rm T}B_FC_i + X_2^{\rm T}A_FY_2R.$$

Based on the above relations, and define

$$U = X_2^{\rm T}Y_2R, \quad \overline{A}_F = X_2^{\rm T}A_FY_2R,$$

$$\overline{B}_F = X_2^{\rm T}B_F, \quad \overline{C}_F = C_FY_2R, \quad \overline{D}_F = D_F \tag{15}$$

it can be readily established that (13) reads as (16).

By virtue of the nonsingularity of $J$, performing congruence transformations on (16) by $\text{diag}(J^{-1}, J^{-1}, J^{-1}, I, I)$ yields (10). In addition, the conditions that $P_i$ and $Q_i$ are positive definite are equivalent to LMIs (11) and (12), respectively. Therefore, from Theorem 2 and (9), we can conclude that an estimator can be given from (15). Denote the filter transfer function from $y(t)$ to $z_F(t)$ by $T_{z_Fy} = C_F(sI - A_F)^{-1}B_F + D_F$. By substituting the filter matrices with (15), we have

$$\begin{aligned} T_{z_Fy} = &\overline{C}_FR^{-1}Y_2^{-1}(sI - X_2^{-\rm T}\overline{A}_FR^{-1}Y_2^{-1})^{-1}X_2^{-\rm T}\overline{B}_F + \\ &\overline{D}_F = \overline{C}_F(sI - U^{-1}\overline{A}_F)^{-1}U^{-1}\overline{B}_F + \overline{D}_F. \end{aligned} \tag{17}$$

Therefore, we can conclude from (17) that the parameters of the filter to be specified in (3) can be constructed by (14). □

The $H_\infty$ filtering can be determined by solving a certain convex optimization problem. This is the following corollary.

**Corollary 1.** A suboptimal full-order $H_\infty$ filter for (1) can be found by solving the following optimization problem:

$$\min \ \gamma \ \text{subject to} \ (11)\text{–}(13), \quad i = 1, 2, \cdots, r. \tag{18}$$

**Remark 3.** Note that for the given $\lambda_1$ and $\lambda_2$, (11)–(13) are linear with respect to $P_{11i}$, $P_{22i}$, $Q_{11i}$, $Q_{22i}$, $P_{12i}$, $Q_{12i}$, $X$, $R$, $U$, $\overline{A}_F$, $\overline{B}_F$, $\overline{C}_F$, $\overline{D}_F$, and hence can be solved by LMI Toolbox. The problem is then how to find the optimal values of $\lambda_1$ and $\lambda_2$. This can be accomplished by using the Matlab command Fminsearch.

## 5 Illustrative example

Consider a continuous-time delay uncertain system, similar to [3], with the following parameters:

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 3+\rho \\ -4 & -5 \end{bmatrix} x(t) + \begin{bmatrix} -0.1 & 0 \\ 0.2 & -0.2+\sigma \end{bmatrix} x(t-\tau) + \\ \qquad \begin{bmatrix} -0.4545 \\ 0.9090 \end{bmatrix} w(t) \\ y(t) = \begin{bmatrix} 0 & 100 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 10 \end{bmatrix} x(t-\tau) + w(t) \\ z(t) = \begin{bmatrix} 0 & 100 \end{bmatrix} x(t) \end{cases}$$

where $\rho$ and $\sigma$ are uncertain real parameters satisfying

$$|\rho| \leqslant 0.3, \qquad |\sigma| \leqslant 0.1. \tag{19}$$

Using the proposed approach in this paper, the scaling parameters are obtained as $(\lambda_1, \lambda_2) = (24.3756, 68.3524)$, the minimum noise attenuation level obtained from Theorem 3 is $\gamma^* = 2.7383$. The estimator matrices are given by

$$A_f = \begin{bmatrix} 1.1117 & 23.9669 \\ -6.0736 & -42.3858 \end{bmatrix}, \quad B_f = \begin{bmatrix} -0.2195 \\ 0.3939 \end{bmatrix},$$

$$C_f = \begin{bmatrix} 3.2651 & 31.5084 \end{bmatrix}, \quad D_f = 0.6708.$$

$$\begin{bmatrix} J^{\rm T}(\overline{A}_i^{\rm T}\Lambda G + G^{\rm T}\Lambda\overline{A}_i + Q_i)J & J^{\rm T}(P_i - G^{\rm T}\Lambda + \overline{A}_i^{\rm T}G)J & J^{\rm T}G^{\rm T}\Lambda\overline{A}_{di}J & J^{\rm T}G^{\rm T}\Lambda\overline{B}_i & J^{\rm T}\overline{C}_i^{\rm T} \\ J^{\rm T}(P_i - \Lambda G + G^{\rm T}\overline{A}_i)J & J^{\rm T}(-G - G^{\rm T})J & J^{\rm T}G^{\rm T}\overline{A}_{di}J & J^{\rm T}G^{\rm T}\overline{B}_i & 0 \\ J^{\rm T}\overline{A}_{di}^{\rm T}\Lambda GJ & J^{\rm T}\overline{A}_{di}^{\rm T}GJ & -(1-\tau)J^{\rm T}Q_iJ & 0 & J^{\rm T}\overline{C}_{di}^{\rm T} \\ \overline{B}_i^{\rm T}\Lambda GJ & \overline{B}_i^{\rm T}GJ & 0 & -\gamma I & \overline{D}_i^{\rm T} \\ \overline{C}_iJ & 0 & \overline{C}_{di}J & \overline{D}_i & -\gamma I \end{bmatrix} < 0. \tag{16}$$

A strictly proper filter is also designed by imposing the constraint $D_f = 0$ on the conditions of Theorem 3. Considering the uncertain parameters $\rho$ and $\sigma$ as in (19), from Theorem 3, the minimum achievable noise attenuation level is given by $\gamma^* = 2.7522$, and the corresponding estimator matrices are given by

$$A_f = \begin{bmatrix} -0.2628 & 53.9540 \\ -2.7490 & -113.0849 \end{bmatrix}, \quad B_f = \begin{bmatrix} -0.5125 \\ 1.0854 \end{bmatrix},$$

$$C_f = \begin{bmatrix} -0.0281 & 100.0557 \end{bmatrix}.$$

By the approach proposed in [3], the minimum $H_\infty$ noise attenuation level $\gamma^* = 3.0175$.

From the example given above, it can be seen that the robust $H_\infty$ filter design method proposed in this paper produces a less conservative result.

## 6 Conclusions

The robust $H_\infty$ filtering problem of polytopic uncertain linear time-delay systems is studied in this paper. A new $H_\infty$ performance criterion is proposed, which exhibits a kind of decoupling between Lyapunov matrices and system dynamic matrices. Based on this, a sufficient condition for the existence of a robust estimator is provided in terms of LMIs. It is shown that the proposed method is less conservative than some existing ones by introducing some additional matrices. A numerical example is given to demonstrate the feasibility and advantage of the proposed criterion.

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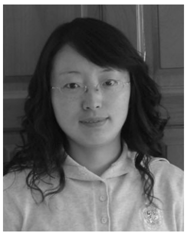
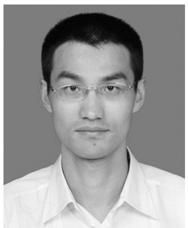


**Ying Zhang** received the M. Eng. degree in control theory and control engineering from Harbin University of Science and Technology, PRC, in 2003, and Ph. D. degree in control science and engineering in 2007 from Harbin Institute of Technology, PRC. In April 2003, she joined Harbin University of Science and Technology as an assistant teacher. She is currently a post-doctoral student in Shenzhen Graduate School, Harbin Institute of Technology.

Her research interests include robust control, filter theory, and networked control systems.



**Ai-Guo Wu** received the B. Eng. degree in automation in 2002, M. Eng. degree in navigation, guidance, and control in 2004, and Ph. D. degree in control science and engineering in 2008 all from Harbin Institute of Technology, PRC. In October 2008, he joined Harbin Institute of Technology Shenzhen Graduate School as an assistant professor. He has served as a general secretary of the 25th Chinese Control Conference at Harbin, PRC, in 2006.

His research interests include observer design, descriptor linear systems, and robust control and estimation.

**Guang-Ren Duan** received the B. Sc. degree in applied mathematics, and both the M. Sc. and Ph. D. degrees in control systems theory. From 1989 to 1991, he was a post-doctoral researcher at Harbin Institute of Technology, where he became a professor of control systems theory in 1991. He visited the University of Hull, UK, and the University of Sheffield, UK, from December 1996 to October 1998, and worked at the Queen′s University of Belfast, UK, from October 1998 to October 2002. Since August 2000, he has been elected specially employed professor at Harbin Institute of Technology, PRC, sponsored by the Cheung Kong Scholars Program of the Chinese government. He is currently the director of the Center for Control Theory and Guidance Technology at Harbin Institute of Technology. He is a charted engineer in the UK, a senior member of IEEE, and a fellow of IEE.

His research interests include robust control, eigenstructure assignment, descriptor systems, missile autopilot design, and magnetic bearing control.