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Properties of the forgotten index in bipolar fuzzy graphs and applications

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Topological indices are the numbers that remain constant under graph automorphism. Topological indices describe a network's connectivity, structure, and topological characteristics. These indices have many applications in crisp graphs. However, in many cases, it is observed that some situations can't be described using the idea of crisp graphs. So, to overcome this issue, the need to define topological indices for fuzzy and bipolar fuzzy graphs arises. The F-index, or the Forgotten Index, is a significant topological index. A bipolar fuzzy graph with two opposite-sided opinions of both the edges and vertices measures the impreciseness or uncertainties of the edges and vertices along the positive and negative sides. In this article, we have presented the Forgotten Index for bipolar fuzzy graphs. Then, we have proved some theorems regarding the F-index of numerous types of bipolar fuzzy graphs, such as regular bipolar fuzzy graphs, complete bipolar fuzzy graphs, etc., the bounds of the F-index in bipolar fuzzy graphs, and the relationships of the F-index with other topological indices in bipolar fuzzy graphs. We have applied the proposed topological index, the F-index for bipolar fuzzy graphs, to matrimonial websites to find potential life partners based on compatibility and discussed the application of the Forgotten Index in gene regulatory networks.

Keywords Topological index, Forgotten index, Bipolar fuzzy graph, Matrimonial websites, Gene regulatory networks

Background and literature review

A crisp set is a well-defined group of distinct objects. In real life, we come across many situations where we cannot describe every situation with the concept of crisp sets. To overcome this issue, Zadeh[1] came up with the concept of fuzzy sets in 1965, revolutionizing the idea of set theory. Rosenfield[2] presented the notion of fuzzy graphs in 1975 and discussed their applications. The notion of bipolar fuzzy sets was first developed by Zhang[3] in 1994. An expansion of Zadeh's fuzzy set theory with a membership value range of [-1, 1] is called a bipolar fuzzy set. In a bipolar fuzzy set, an element with membership degree 0 indicates that it does not relate to the associated property; an element with membership degree (0,1] indicates that it partially satisfies the property; and an element with membership degree [-1,0) indicates that it partially satisfies the underlying counter-property. Akram[4] proposed the concept of bipolar fuzzy graphs in 2011. When there are positive and negative thinking sides as effect and side effect, gain and loss, friendship and animosity, collaborative and competitive, etc., then bipolar fuzzy sets and bipolar fuzzy graphs play a significant role in real-life decision-making problems. The mathematical definition of a bipolar fuzzy graph is provided in Definition 2.4. in Sect. "Preliminaries". For better understanding, let us consider an example of a graphical representation of a students' network. A small group of students can form a network among them. There exist two conflicting types of characteristics among them. One is friendship, and the other one is rivalry or animosity. A bipolar fuzzy graph can easily demonstrate this network and the two contrasting characteristics. Such a bipolar fuzzy graph demonstrating a students' network formed by a small group of students is provided in Fig. 1. Here, the vertices and edges represent the students and their relationships, respectively. The positive and negative membership of a particular vertex demonstrates the specific

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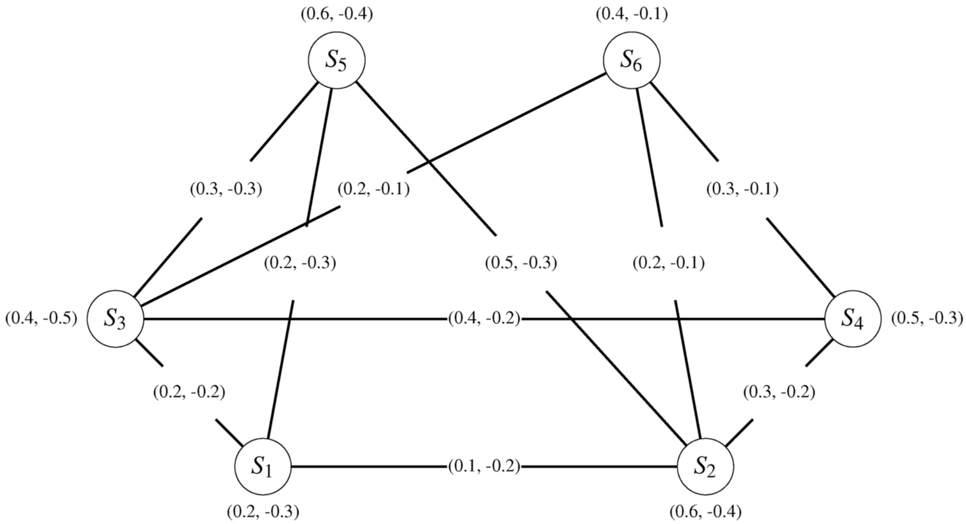

Fig. 1. A bipolar fuzzy graph representing a students' network formed by a small group of students.

student's nature of friendship and rivalry. Similarly, the positive and negative membership value of a particular edge depicts the friendship and animosity or rivalry, respectively, between the two students, represented by the end vertices of that edge. A few other examples of bipolar fuzzy graphs can be the graphical representation of a network formed by different mutual fund companies where growth and risk factors, two opposite-sided characters occur, a graphical demonstration of the network formed by various countries where friendship and rivalry both occur, etc. Thus, using the concept of bipolar fuzzy graphs, one can easily demonstrate two conflicting opinions.

In 2013, Akram[5] defined the cardinality of the edge set, vertex set, neighborhood degree of bipolar fuzzy graphs, and bipolar fuzzy directed graphs. In the same year, Yang et al.[6] presented the generalized idea of different kinds of bipolar fuzzy graphs. Akram and Karunambigai[7] defined distance, eccentricity, diameter, radius, etc. in bipolar fuzzy graphs in 2011. In the same article, they also gave the idea of centered bipolar fuzzy graphs and discussed their properties. In 2013, Akram[5] developed the notion of neighborly irregular, highly irregular, totally irregular, and highly totally irregular bipolar fuzzy graphs and discussed bipolar fuzzy influence graphs in the same article. Poulik and Ghorai[8] defined the degrees of the vertices in bipolar fuzzy graphs and bipolar fuzzy digraphs in 2013. Akram and Farooq[9] introduced cut-vertices, bridges, cycles, and trees in bipolar fuzzy graphs in 2016 and discussed their properties. Akram et al.[10] introduced different graph operations, such as strong product, cartesian product, etc., on m-polar fuzzy graphs in the same year. Akram[11] defined several kinds of bipolar and m-polar fuzzy graphs and discussed their applications in 2018. In the same year, Ghorai and Pal[12] introduced various degrees and defined regular bipolar fuzzy graphs. In 2020, Poulik and Ghorai[13] proposed the Connectivity index for bipolar fuzzy graphs and proved some theorems. Akram et al.[14] discussed decision-making methods in bipolar fuzzy graphs and discussed their applications in real-life scenarios in 2021. Poulik and Ghorai[15] introduced the Wiener Index and Wiener Absolute Index and compared them to the Connectivity Index the same year. Binu et al.[16] discussed the connectivity status of bipolar fuzzy graphs and several bipolar fuzzy subgraphs in 2021.

A mathematical invariant that provides information on a network or graph's connectivity, construction, or other topological features is called a topological index. It is a measure that helps us comprehend the network's characteristics and behavior. Topological indices have been developed using various network features, including nodes' degrees, nodes' distances, eccentricity, and other structural metrics. During his research on the boiling point of paraffin, Wiener[17] first developed a distance-based topological index, the Wiener Index, in 1947. In mathematics, numerous vertex-degree-based topological indices have been developed, and their general formula is $TI(\Delta) = \sum_{xy\in\Lambda} f(d(x), d(y))$. The 1st and 2nd Zagreb Indices are two of the former topological indices that were presented by Gutman and Trinajstic[18] in 1972. The Randic Index was developed by Randic[19] in 1975. Shirdel et al.[20] proposed another degree-based topological index, the Hyper Zagreb Index, in 2013. Gutman et al.[21] presented the idea of the Reciprocal Randic Index in 2014. The Forgotten Index, or F-index, was developed by Furtula and Gutman[22] in 2015. In 2021, Gutman[23] presented the idea of the Sombor Index and gave a geometrical interpretation of the Sombor index. In many real-life situations, many circumstances cannot be handled using crisp graphs. So, topological indices had to be introduced for fuzzy and bipolar fuzzy graphs. Binu et al.[24] introduced the Wiener Index for fuzzy graphs in 2020. Islam et al. commented on the Wiener Index[25] for fuzzy graphs and discussed its application in the same year. Poulik and Ghorai[13] presented the Connectivity Index for bipolar fuzzy graphs in 2020 and proved some properties. The same authors[15] developed the Wiener Index and Wiener Absolute Index the following year and compared them to the Connectivity Index. Islam and Pal presented the idea of the 1st Zagreb Index[26] for fuzzy graphs in the same year. The Randic Index for bipolar fuzzy graphs was developed by Poulik et al.[27] in 2022. Islam and Pal discussed the properties of the F-index[28] for different graph operations and transformations for fuzzy graphs in 2023. Islam and Pal[29] applied the Second

Zagreb Index for fuzzy graphs in mathematical chemistry in 2023. Ahmed et al.[30] utilized the idea of fuzzy topological indices for application in cybercrime problems in the same year. In the following year, Islam and Pal[31] introduced a multiplicative version of the 1st Zagreb index in fuzzy graphs and applied it to crime analysis. Gutman et al.[32] introduced the concept of the Elliptic Sombor Index and presented its geometric approach in 2024. In the same year, Lal et al.[33] used graph entropies and topological indices to apply to the Y-junctions of carbon nanotubes.

Research gaps and motivation

The application of topological indices in molecular and chemical graph theory, network theory, spectral graph theory, etc., is enormous. One of the earliest and most significant topological indices are the 1st and 2nd Zagreb Indices, which Gutman and Trinajstic[18] developed in 1972. They used these degree-based topological indices to determine a conjugate system's π-electron energy. These topological indices were followed by the F-index (Forgotten Index) introduction by Furtula and Gutman[22] in 2015. The authors demonstrated that the 1st Zagreb Index and the F-index (Forgotten Index) had nearly identical entropy, acentric factor, and predictive ability, and the Forgotten Index obtained correlation coefficients higher than 0.95. As a result, this topological index is extremely helpful in molecular chemistry and network theory. These indices were defined for only crisp graphs. However, in real-life scenarios, there are some situations where the idea of crisp graphs cannot be implemented. So, the need to use fuzzy graphs or bipolar fuzzy graphs arises. So, researchers had to develop topological indices for fuzzy and bipolar fuzzy graphs. Binu et al.[24] developed the Wiener Index for fuzzy graphs and provided various applications related to illegal immigration in 2020. The same authors[16] discussed the connectivity status of fuzzy graphs in the following year. Researchers have introduced several topological indices for fuzzy graphs and are still working on this topic. Islam and Pal[34] presented the Forgotten Index for fuzzy graphs in 2021. In situations with two contrasting opinions, like gain and loss, growth and risk factors, fortune and misfortune, likes and dislikes, etc., bipolar fuzzy sets and graphs become essential. Although much research has been done on topological indices for crisp and fuzzy graphs, not so much research has been done on the topological indices for bipolar fuzzy graphs. Topological indices have many real-life applications. Although topological indices were originally introduced for chemical structures, research is not limited to them. Islam and Pal[34] applied the Forgotten Index to find the most influential researcher in the co-authorship network in 2021. The same authors[28] applied the Forgotten Index to Indian railway crimes and found the most crime-free and crime-centric railway routes in 2023. To overcome the research gap, Poulik et al.[27] proposed the Randic Index for bipolar fuzzy graphs and applied it to the transmission network system between a few cities and a Wi-Fi network in a town. These research works encouraged us to extend the results for the Forgotten Index, to prove some new theorems, and to investigate new application areas. The Forgotten Index for bipolar fuzzy graphs can be applied to friendship networks, investment in stock markets, etc., where two contrasting opinions occur, like likes and dislikes, growth and risk factors, etc. These vast areas of application of topological indices and the large research gaps motivated us to develop the Forgotten Index for bipolar fuzzy graphs, discuss its properties, and apply it to matrimonial websites and gene regulatory networks.

Objective and significance of the article

As in real-life scenarios, sometimes decision-making cannot be performed using the concept of crisp graphs, so bipolar fuzzy graphs are of huge importance. These graphs use the degree of belongingness for the vertices and the edges to depict imprecise situations. Bipolar fuzzy graphs are in huge demand where there is an opposite or contrasting opinion. Our main objective is to introduce the Forgotten Index in bipolar fuzzy graphs, discuss its properties, and apply it to matrimonial websites and gene regulatory networks. In reality, finding potential partners is a very time-consuming and laborious job. We aim to apply the Forgotten Index to bipolar fuzzy graphs obtained from a group of males and females seeking life partners on matrimonial websites to find potential life partners based on compatibility, making the process easy and less time-consuming. The significance of this article is that the newly introduced Forgotten Index for bipolar fuzzy graphs can be applied to many real-life problems. Here, we have proved several theorems on regular bipolar fuzzy graphs, complete bipolar fuzzy graphs, star graphs, etc. We have also proved some theorems regarding the relationships of different topological indices with the F-index in bipolar fuzzy graphs. As we have applied the F-index to matrimonial websites and gene regulatory networks, similarly, one can apply this Forgotten Index to bipolar fuzzy graphs where contrasting indecisive opinions occur. Although this research article focuses on introducing the Forgotten Index in bipolar fuzzy graphs, several other topological indices for bipolar fuzzy graphs are also introduced in Section "Preliminaries". Using these topological indices, some interesting results can be proved in the future and used them in real-life applications.

Framework of the article

In this article, we have developed the Forgotten Topological Index for bipolar fuzzy graphs and proved some theorems on the newly introduced Forgotten Index. We have also applied the Forgotten Index to matrimonial websites to find potential life partners based on compatibility and gene regulatory networks to determine influential genes that play an important part in characterizing the overall activating or inhibiting properties of the network. The article is structured as follows. In Sect. "Introduction", we introduce the article with some background and literature review, discuss the research gaps and motivation, and talk about the significance and objectives of the article. In Sect. "Preliminaries", some basic definitions are discussed. Section "Forgotten index and its properties in bipolar fuzzy graphs" introduces the Forgotten Index in bipolar fuzzy graphs and provides proof of some exciting theorems regarding the Forgotten Index in bipolar fuzzy graphs. In Sect. "Applications", we have applied the newly presented Forgotten Index to matrimonial websites and gene regulatory networks, and in Sect. "Conclusion and future works", we have concluded the article with limitations and future works.

Preliminaries

Here, we have provided some definitions that are needed to understand this article. We've also proposed the First, Second, and Hyper Zagreb Index, Sombor Index, and Reciprocal Randic Index for bipolar fuzzy graphs.

Definition 2.1 An ordered pair of two components is called a graph, where the first and second components are the vertex set and the edge set of the graph, respectively.

Definition 2.2 A bipolar fuzzy set S on a non-empty set W is given as, $S = \{(w, \mu_S^P(w), \mu_S^N(w)) : w \in W\}$, where $\mu_S^P : W \to [0,1]$ and $\mu_S^N : W \to [-1,0]$ are two mappings.[5]

Definition 2.3 Let $W(\neq \phi)$ be a set. Then the mapping $D = (\mu_D^P, \mu_D^N) : W \times W \to [0,1] \times [-1,0]$ is said to be a bipolar fuzzy relation on W such that $\mu_D^P(b_ib_j) \in [0,1]$ and $\mu_D^N(b_ib_j) \in [-1,0], \forall b_i, b_j \in W$.

Let $C = (\mu_C^P, \mu_C^N)$ be a bipolar fuzzy set on $W(\neq \phi)$. If $D = (\mu_D^P, \mu_D^N)$ is a bipolar fuzzy relation on W, then $D = (\mu_D^P, \mu_D^N)$ is a bipolar fuzzy relation on $C = (\mu_C^P, \mu_C^N)$ if $\mu_D^P(b_ib_j) \leq min\{\mu_C^P(b_i), \mu_C^P(b_j)\}$, and $\mu_D^N(b_ib_j) \geq max\{\mu_C^N(b_i), \mu_C^N(b_j)\}, \forall b_i, b_j \in W$.[5]

Definition 2.4 A *BFG* Δ of a graph $\Delta^* = (\Gamma, \Lambda)$ is a triplet (Γ, C, D), where $C = (\mu_C^P, \mu_C^N)$ is a bipolar fuzzy set in Γ and $D = (\mu_D^P, \mu_D^N)$ is a bipolar fuzzy set in $\Lambda \subset \Gamma \times \Gamma$ such that $\mu_D^P(b_ib_j) \leq min\{\mu_C^P(b_i), \mu_C^P(b_j)\}$, and $\mu_D^N(b_ib_j) \geq max\{\mu_C^N(b_i), \mu_C^N(b_j)\}, \forall b_ib_j \in \Lambda$. Here, C and D are bipolar fuzzy vertex set of Γ and bipolar fuzzy edge set of Λ, respectively.[5]

Let $\Delta = (\Gamma, C, D)$ be a *BFG* of the graph $\Delta^* = (\Gamma, \Lambda)$. Δ is called a bipolar fuzzy cycle if Δ^* is a cycle and there is no edge $b_ib_j \in \Delta$ for which $\mu_D^P(b_ib_j)$ is minimum and $\mu_D^N(b_ib_j)$ is maximum.[27]

Let $\Delta = (\Gamma, C, D)$ be a *BFG* and $\Gamma = \{b_1, b_2, ..., b_n\}$. Then Δ is called a star if b_i has edges with every vertex of $\{b_1, b_2, ..., b_{i-1}, b_{i+1}, ..., b_n\}$ and each vertex of $\{b_1, b_2, ..., b_{i-1}, b_{i+1}, ..., b_n\}$ is a pendant vertex. b_i is called the center of the star. Note that positive and negative membership values of an edge in a star can not be zero simultaneously.

Definition 2.5 A *BFG* $\Delta' = (\Gamma', C', D')$ is called a bipolar fuzzy subgraph of a *BFG* $\Delta = (\Gamma, C, D)$ if $\Gamma' \subseteq \Gamma, \Lambda' \subseteq \Lambda$ such that $\mu_C^P(b_i) = \mu_{C'}^P(b_i), \mu_C^N(b_i) = \mu_{C'}^N(b_i), \forall b_i \in \Gamma'$ and $\mu_D^P(b_ib_j) = \mu_{D'}^P(b_ib_j), \mu_D^N(b_ib_j) = \mu_{D'}^N(b_ib_j)$, for every edge b_ib_j of Δ'.[27]

Definition 2.6 The open neighborhood degree or degree of a vertex b_i in a *BFG* $\Delta = (\Gamma, C, D)$ is defined as $d(b_i) = \left(d^P(b_i), d^N(b_i)\right)$,

where $d^P(b_i) = \sum_{b_j : b_i \neq b_j, b_ib_j \in \Lambda} \mu_D^P(b_ib_j)$ and $d^N(b_i) = \sum_{b_j : b_i \neq b_j, b_ib_j \in \Lambda} \mu_D^N(b_ib_j)$.[5]

If $d(b_i) = (p_1, p_2), \forall b_i \in \Gamma$, Δ is called $(p_1, p_2) -$ regular.[27]

If $\mu_D^P(b_ib_j) = \min\left\{\mu_C^P(b_i), \mu_C^P(b_j)\right\}$ and $\mu_D^N(b_ib_j) = \max\left\{\mu_C^N(b_i), \mu_C^N(b_j)\right\}, \forall b_i, b_j \in \Gamma$, then Δ is called a complete *BFG*.[27]

A *BFG* $\Delta = (\Gamma, C, D)$ is called a strong *BFG* if $\mu_D^P(b_ib_j) = \min\left\{\mu_C^P(b_i), \mu_C^P(b_j)\right\}$ and $\mu_D^N(b_ib_j) = \max\left\{\mu_C^N(b_i), \mu_C^N(b_j)\right\}$, for every edge b_ib_j in Δ.[5]

Definition 2.7 The closed neighborhood degree of a vertex $b_i \in \Gamma$ in a *BFG* Δ is denoted by $d[b_i] = \left(d^P[b_i], d^N[b_i]\right)$ and is defined as $d^P[b_i] = d^P(b_i) + \mu_C^P(b_i)$, and $d^N[b_i] = d^N(b_i) + \mu_C^N(b_i)$.[5]

If $d[b_i] = (p_1, p_2)\ \forall b_i \in \Gamma$, then Δ is called (p_1, p_2)-totally regular.[27]

Definition 2.8 The degree of a vertex b_i of a *BFDG* $\tilde{\mathbf{\Delta}} = (\Gamma, C, \vec{\mathbf{D}})$ is $d(b_i) = \left(d^P(b_i), d^N(b_i)\right)$, where $d^P(b_i) = \sum_{j: i \neq j} \left(\mu_D^P\left(\overrightarrow{b_ib_j}\right) + \mu_D^P\left(\overrightarrow{b_jb_i}\right)\right)$ and $d^N(b_i) = \sum_{j: i \neq j} \left(\mu_D^N\left(\overrightarrow{b_ib_j}\right) + \mu_D^N\left(\overrightarrow{b_jb_i}\right)\right)$.[27]

Definition 2.9 Let $\Delta_1 = (\Gamma_1, C_1, D_1)$ and $\Delta_2 = (\Gamma_2, C_2, D_2)$ be two *BFG*s of the graphs $\Delta_1^* = (\Gamma_1, \Lambda_1)$ and $\Delta_2^* = (\Gamma_2, \Lambda_2)$, respectively. Δ_1 and Δ_2 are called isomorphic if there exists a bijective mapping $\psi : \Gamma_1 \to \Gamma_2$, s.t. $\mu_{C_1}^P(b_i) = \mu_{C_2}^P(\psi(b_i))$, $\mu_{C_1}^N(b_i) = \mu_{C_2}^N(\psi(b_i)), \forall b_i \in \Gamma_1$; and $\mu_{D_1}^P(b_ib_j) = \mu_{D_2}^P(\psi(b_i)\psi(b_j)), \mu_{D_1}^N(b_ib_j) = \mu_{D_2}^N(\psi(b_i)\psi(b_j)), \forall b_ib_j \in \Lambda_1$.[27]

Definition 2.10 Let $\Delta = (\Gamma, C, D)$ be a *BFG* of the graph $\Delta^* = (\Gamma, \Lambda)$ such that $|\Gamma| = n$. The First Zagreb Index of the *BFG* Δ is introduced as,

$$FZI_{BF}(\Delta) = (FZI_{BF}^P(\Delta), FZI_{BF}^N(\Delta))$$

where,

$$FZI_{BF}^P(\Delta) = \sum_{b_ib_j \in \Lambda : 1 \leq i \neq j \leq n} \left\{\mu_C^P(b_i)d^P(b_i) + \mu_C^P(b_j)d^P(b_j)\right\} \quad \text{and}$$
$$FZI_{BF}^N(\Delta) = \sum_{b_ib_j \in \Lambda : 1 \leq i \neq j \leq n} \left\{\mu_C^N(b_i)d^N(b_i) + \mu_C^N(b_j)d^N(b_j)\right\}.$$

Definition 2.11 Let $\Delta = (\Gamma, C, D)$ be a *BFG* of the graph $\Delta^* = (\Gamma, \Lambda)$ such that $|\Gamma| = n$. The Second Zagreb Index of the *BFG* Δ is defined as,

$$SZI_{BF}(\Delta) = (SZI^P_{BF}(\Delta), SZI^N_{BF}(\Delta))$$

where,

$$SZI^P_{BF}(\Delta) = \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \left[\left\{\mu^P_C(b_i)d^P(b_i)\right\}\left\{\mu^P_C(b_j)d^P(b_j)\right\}\right] \quad \text{and}$$
$$SZI^N_{BF}(\Delta) = \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \left[\left\{\mu^N_C(b_i)d^N(b_i)\right\}\left\{\mu^N_C(b_j)d^N(b_j)\right\}\right].$$

Definition 2.12 Let $\Delta = (\Gamma, C, D)$ be a *BFG* of the graph $\Delta^* = (\Gamma, \Lambda)$ such that $|\Gamma| = n$. The Hyper Zagreb Index of the *BFG* Δ is given as,

$$HZI_{BF}(\Delta) = (HZI^P_{BF}(\Delta), HZI^N_{BF}(\Delta))$$

where,

$$HZI^P_{BF}(\Delta) = \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \left[\left\{\mu^P_C(b_i)d^P(b_i)\right\} + \left\{\mu^P_C(b_j)d^P(b_j)\right\}\right]^2 \quad \text{and}$$
$$HZI^N_{BF}(\Delta) = \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \left[\left\{\mu^N_C(b_i)d^N(b_i)\right\} + \left\{\mu^N_C(b_j)d^N(b_j)\right\}\right]^2.$$

Definition 2.13 Let $\Delta = (\Gamma, C, D)$ be a *BFG* of the graph $\Delta^* = (\Gamma, \Lambda)$ such that $|\Gamma| = n$. The Sombor Index of the *BFG* Δ is defined as,

$$SO_{BF}(\Delta) = (SO^P_{BF}(\Delta), SO^N_{BF}(\Delta))$$

where,

$$SO^P_{BF}(\Delta) = \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \sqrt{\left[\left\{\mu^P_C(b_i)\, d^P(b_i)\right\}^2 + \left\{\mu^P_C(b_j)\, d^P(b_j)\right\}^2\right]} \quad \text{and}$$
$$SO^N_{BF}(\Delta) = \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \sqrt{\left[\left\{\mu^N_C(b_i)\, d^N(b_i)\right\}^2 + \left\{\mu^N_C(b_j)\, d^N(b_j)\right\}^2\right]}.$$

Definition 2.14 Let $\Delta = (\Gamma, C, D)$ be a *BFG* of the graph $\Delta^* = (\Gamma, \Lambda)$ such that $|\Gamma| = n$. The Reciprocal Randic Index of the *BFG* Δ is provided as,

$$RRI_{BF}(\Delta) = (RRI^P_{BF}(\Delta), RRI^N_{BF}(\Delta))$$

where,

$$RRI^P_{BF}(\Delta) = \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \sqrt{\left\{\mu^P_C(b_i)\, d^P(b_i)\right\}\left\{\mu^P_C(b_j)\, d^P(b_j)\right\}} \quad \text{and}$$
$$RRI^N_{BF}(\Delta) = \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \sqrt{\left\{\mu^N_C(b_i)\, d^N(b_i)\right\}\left\{\mu^N_C(b_j)\, d^N(b_j)\right\}}.$$

Let us consider a bipolar fuzzy undirected graph whose membership values for the vertices and the edges are shown in Table 1. The topological indices defined in Sect. "Preliminaries" are calculated for the graph and demonstrated in Fig. 2.

The notations used in this article are given in Table 2.

Forgotten index and its properties in bipolar fuzzy graphs

Here, we have introduced the Forgotten Index for *BFG*s and proved some theorems.

Forgotten index in bipolar fuzzy graphs

The Forgotten Index, or F-index, is a vital and significant topological index. The application area of the Forgotten Index in molecular chemistry and real-life scenarios is enormous. The concept of topological indices started with the introduction of the Wiener Index[17] way back in 1947. The first degree-based topological indices (First and Second Zagreb Index) were introduced in 1972 by Gutman and Trinajstic[18]. Although being such a significant topological index, the Forgotten Index was unnoticed, overlooked, or forgotten by researchers for many years. That is why, while developing the Forgotten Index in 2015, Furtula and Gutman[22] gave such a name to this topological index.

Vertices	Membership values of the vertices	Edges	Membership values of the edges
1	$(0.2, -0.5)$	(1, 2)	$(0.1, -0.1)$
2	$(0.7, -0.2)$	(1, 3)	$(0.2, -0.4)$
3	$(0.4, -0.6)$	(1, 4)	$(0.1, -0.4)$
4	$(0.3, -0.6)$	(1, 5)	$(0.2, -0.1)$
5	$(0.8, -0.1)$	(2, 3)	$(0.3, -0.2)$
		(2, 4)	$(0.3, -0.1)$
		(2, 5)	$(0.6, -0.1)$
		(3, 4)	$(0.1, -0.4)$
		(3, 5)	$(0.2, -0.1)$
		(4, 5)	$(0.3, -0.1)$

Table 1. Membership values of the vertices and the edges of the undirected graph shown in Fig. 2.

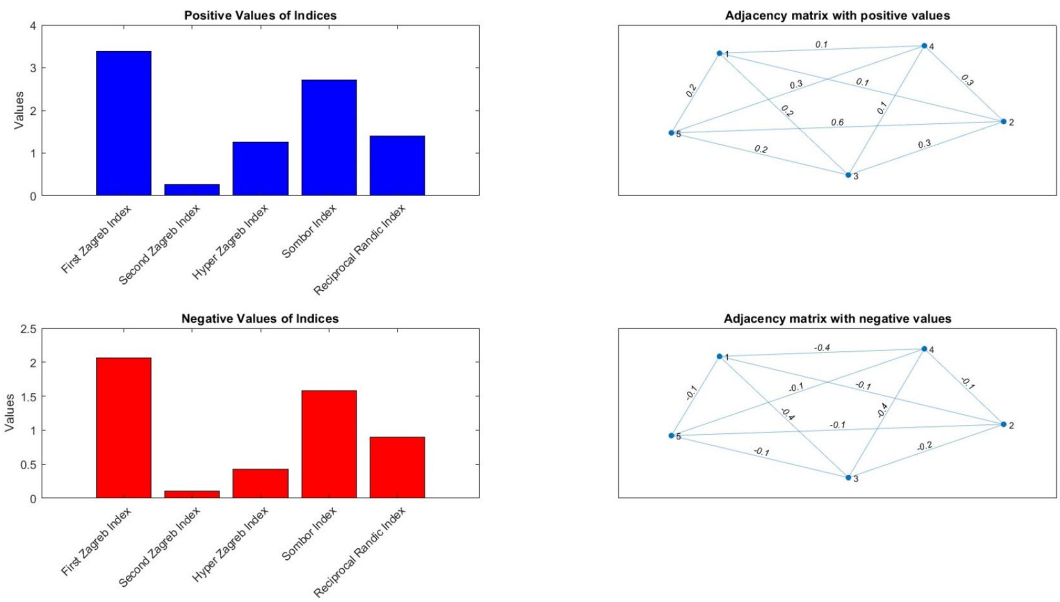

Fig. 2. A bipolar fuzzy graph and the topological indices defined on it.

Furtula and Gutman[22] developed the Forgotten Index for crisp graphs in 2015 and compared it with the First Zagreb Index. Islam and Pal[34] introduced the Forgotten Index for fuzzy graphs in 2021 and applied it to co-authorship networks. In this subsection, we have defined the Forgotten Index for bipolar fuzzy graphs.

Let $\Delta = (\Gamma, C, D)$ be a *BFG* of the graph $\Delta^* = (\Gamma, \Lambda)$ such that $|\Gamma| = n$. The Forgotten Index (F-Index) of the *BFG* Δ is introduced as,

$$FI_{BF}(\Delta) = (FI^P_{BF}(\Delta), FI^N_{BF}(\Delta))$$

where,

$$FI^P_{BF}(\Delta) = \sum_{b_ib_j \in \Lambda : 1 \le i \ne j \le n} [\{\mu^P_C(b_i)d^P(b_i)\}^2 + \{\mu^P_C(b_j)d^P(b_j)\}^2] \quad \text{and}$$
$$FI^N_{BF}(\Delta) = \sum_{b_ib_j \in \Lambda : 1 \le i \ne j \le n} [\{\mu^N_C(b_i)d^N(b_i)\}^2 + \{\mu^N_C(b_j)d^N(b_j)\}^2].$$

Theorems on bounds of the forgotten index in bipolar fuzzy graphs

The bounds of the Forgotten Index are very important aspects as they provide valuable insights regarding networks' character. The bounds, i.e., the lowest and highest values of the Forgotten Index, indicate the capability of the concerned bipolar fuzzy graphs. In this subsection, some theorems regarding the bounds of the Forgotten Index in bipolar fuzzy graphs are proved.

Notations	Meanings
Δ^*	Graph
Γ	Vertex set of the graph Δ^*
Λ	Edge set of the graph Δ^*
$\|\Gamma\|$	Number of elements in Γ
$\|\Lambda\|$	Number of elements in Λ
n	Number of vertices in Δ^*
m	Number of edges in Δ^*
s.t.	Such that
BFG	Bipolar fuzzy graph
BFDG	Bipolar fuzzy directed graph
$d(b_i) = (d^P(b_i), d^N(b_i))$	Degree or open neighborhood degree of the vertex b_i
$d_\Delta(b_i) = (d^P_\Delta(b_i), d^N_\Delta(b_i))$	Degree of the vertex b_i for the graph Δ
$d[b_i] = (d^P[b_i], d^N[b_i])$	Closed neighborhood degree of the vertex b_i
$FZI_{BF}(\Delta) = \left(FZI^P_{BF}(\Delta), FZI^N_{BF}(\Delta)\right)$	First Zagreb Index of the bipolar fuzzy graph Δ
$SZI_{BF}(\Delta) = \left(SZI^P_{BF}(\Delta), SZI^N_{BF}(\Delta)\right)$	Second Zagreb Index of the bipolar fuzzy graph Δ
$HZI_{BF}(\Delta) = \left(HZI^P_{BF}(\Delta), HZI^N_{BF}(\Delta)\right)$	Hyper Zagreb Index of the bipolar fuzzy graph Δ
$SO_{BF}(\Delta) = \left(SO^P_{BF}(\Delta), SO^N_{BF}(\Delta)\right)$	Sombor Index of the bipolar fuzzy graph Δ
$RRI_{BF}(\Delta) = \left(RRI^P_{BF}(\Delta), RRI^N_{BF}(\Delta)\right)$	Reciprocal Randic Index of the bipolar fuzzy graph Δ
$FI_{BF}(\Delta) = \left(FI^P_{BF}(\Delta), FI^N_{BF}(\Delta)\right)$	Forgotten Index or F-index of the bipolar fuzzy graph Δ

Table 2. Notations used in this article and their meanings.

Theorem 1 *Let* $\Delta = (\Gamma, C, D)$ *be a BFG such that* $|\Gamma| = n$, $\delta_1 = \min\left\{d^P(b_i)\right\}, \delta_2 = \min\left\{|d^N(b_i)|\right\}, \lambda_1 = \max\left\{d^P(b_i)\right\}$ *and* $\lambda_2 = \max\left\{|d^N(b_i)|\right\}, \forall b_i \in \Gamma$. *Then,*

$$\delta_1^2\sigma_1 \le FI^P_{BF}(\Delta) \le \lambda_1^2\sigma_1, \quad \text{and} \quad \delta_2^2\sigma_2 \le FI^N_{BF}(\Delta) \le \lambda_2^2\sigma_2,$$

where,

$$\sigma_1 = \sum_{b_ib_j\in\Lambda:1\le i\ne j\le n} [\{\mu^P_C(b_i)\}^2 + \{\mu^P_C(b_j)\}^2], \quad \text{and}$$
$$\sigma_2 = \sum_{b_ib_j\in\Lambda:1\le i\ne j\le n} [\{\mu^N_C(b_i)\}^2 + \{\mu^N_C(b_j)\}^2].$$

Proof We have,

$$d^P(b_i) = \sum_{b_j:b_i\ne b_j, b_ib_j\in\Lambda} \mu^P_D(b_ib_j) \quad \text{and} \quad d^N(b_i) = \sum_{b_j:b_i\ne b_j, b_ib_j\in\Lambda} \mu^N_D(b_ib_j).$$

Again, $\delta_1 = \min\left\{d^P(b_i)\right\}$, $\delta_2 = \min\left\{|\, d^N(b_i)\,|\right\}$, $\lambda_1 = \max\left\{d^P(b_i)\right\}$, and $\lambda_2 = \max\left\{|d^N(b_i)|\right\}, \forall b_i \in \Gamma$.

So,

$$\begin{aligned}
&\delta_1 \le d^P(b_i) \le \lambda_1\\
&\Rightarrow \delta_1\{\mu^P_C(b_i)\} \le d^P(b_i)\,\mu^P_C(b_i) \le \lambda_1\{\mu^P_C(b_i)\}\\
&\Rightarrow \delta_1^2\{\mu^P_C(b_i)\}^2 \le \{\mu^P_C(b_i)d^P(b_i)\}^2 \le \lambda_1^2\{\mu^P_C(b_i)\}^2.
\end{aligned}$$

Similarly, $\delta_1^2\{\mu^P_C(b_j)\}^2 \le \{\mu^P_C(b_j)d^P(b_j)\}^2 \le \lambda_1^2\{\mu^P_C(b_j)\}^2$.

Therefore,

$$\delta_1^2\{\mu_C^P(b_i)\}^2 + \delta_1^2\{\mu_C^P(b_j)\}^2 \le \{\mu_C^P(b_i)d^P(b_i)\}^2 + \{\mu_C^P(b_j)d^P(b_j)\}^2$$
$$\le \lambda_1^2\{\mu_C^P(b_i)\}^2 + \lambda_1^2\{\mu_C^P(b_j)\}^2$$
$$\Rightarrow \delta_1^2[\{\mu_C^P(b_i)\}^2 + \{\mu_C^P(b_j)\}^2] \le \{\mu_C^P(b_i)d^P(b_i)\}^2 + \{\mu_C^P(b_j)d^P(b_j)\}^2$$
$$\le \lambda_1^2[\{\mu_C^P(b_i)\}^2 + \{\mu_C^P(b_j)\}^2]$$
$$\Rightarrow \sum_{b_ib_j\in\Lambda:1\le i\ne j\le n} \delta_1^2[\{\mu_C^P(b_i)\}^2 + \{\mu_C^P(b_j)\}^2]$$
$$\le \sum_{b_ib_j\in\Lambda:1\le i\ne j\le n} [\{\mu_C^P(b_i)d^P(b_i)\}^2 + \{\mu_C^P(b_j)d^P(b_j)\}^2]$$
$$\le \sum_{b_ib_j\in\Lambda:1\le i\ne j\le n} \lambda_1^2[\{\mu_C^P(b_i)\}^2 + \{\mu_C^P(b_j)\}^2]$$
$$\Rightarrow \delta_1^2\sigma_1 \le FI_{BF}^P(\Delta) \le \lambda_1^2\sigma_1, \quad \text{where} \quad \sigma_1 = \sum_{b_ib_j\in\Lambda:1\le i\ne j\le n} [\{\mu_C^P(b_i)\}^2 + \{\mu_C^P(b_j)\}^2]$$

Again, we have

$$\delta_2 \le |d^N(b_i)| \le \lambda_2$$
$$\Rightarrow -\lambda_2 \le d^N(b_i) \le -\delta_2$$
$$\Rightarrow -\delta_2\{\mu_C^N(b_i)\} \le d^N(b_i)\,\mu_C^N(b_i) \le -\lambda_2\{\mu_C^N(b_i)\} \quad [\because \mu_C^N(b_i) \le 0]$$
$$\Rightarrow \delta_2^2\{\mu_C^N(b_i)\}^2 \le \{\mu_C^N(b_i)d^N(b_i)\}^2 \le \lambda_2^2\{\mu_C^N(b_i)\}^2 \quad [\because \text{each side is non-negative}].$$

Similarly, $\delta_2^2\{\mu_C^N(b_j)\}^2 \le \{\mu_C^N(b_j)d^N(b_j)\}^2 \le \lambda_2^2\{\mu_C^N(b_j)\}^2$.

Therefore,

$$\delta_2^2\{\mu_C^N(b_i)\}^2 + \delta_2^2\{\mu_C^N(b_j)\}^2 \le \{\mu_C^N(b_i)d^N(b_i)\}^2 + \{\mu_C^N(b_j)d^N(b_j)\}^2$$
$$\le \lambda_2^2\{\mu_C^N(b_i)\}^2 + \lambda_2^2\{\mu_C^N(b_j)\}^2$$
$$\Rightarrow \delta_2^2[\{\mu_C^N(b_i)\}^2 + \{\mu_C^N(b_j)\}^2] \le \{\mu_C^N(b_i)d^N(b_i)\}^2 + \{\mu_C^N(b_j)d^N(b_j)\}^2$$
$$\le \lambda_2^2[\{\mu_C^N(b_i)\}^2 + \{\mu_C^N(b_j)\}^2]$$
$$\Rightarrow \sum_{b_ib_j\in\Lambda:1\le i\ne j\le n} \delta_2^2[\{\mu_C^N(b_i)\}^2 + \{\mu_C^N(b_j)\}^2]$$
$$\le \sum_{b_ib_j\in\Lambda:1\le i\ne j\le n} [\{\mu_C^N(b_i)d^N(b_i)\}^2 + \{\mu_C^N(b_j)d^N(b_j)\}^2]$$
$$\le \sum_{b_ib_j\in\Lambda:1\le i\ne j\le n} \lambda_2^2[\{\mu_C^N(b_i)\}^2 + \{\mu_C^N(b_j)\}^2]$$
$$\Rightarrow \delta_2^2\sigma_2 \le FI_{BF}^N(\Delta) \le \lambda_2^2\sigma_2, \quad \text{where} \quad \sigma_2 = \sum_{b_ib_j\in\Lambda:1\le i\ne j\le n} [\{\mu_C^N(b_i)\}^2 + \{\mu_C^N(b_j)\}^2]$$

Therefore,

$$\delta_1^2\sigma_1 \le FI_{BF}^P(\Delta) \le \lambda_1^2\sigma_1, \quad \text{and} \quad \delta_2^2\sigma_2 \le FI_{BF}^N(\Delta) \le \lambda_2^2\sigma_2,$$

where,

$$\sigma_1 = \sum_{b_ib_j\in\Lambda:1\le i\ne j\le n} [\{\mu_C^P(b_i)\}^2 + \{\mu_C^P(b_j)\}^2], \quad \text{and}$$
$$\sigma_2 = \sum_{b_ib_j\in\Lambda:1\le i\ne j\le n} [\{\mu_C^N(b_i)\}^2 + \{\mu_C^N(b_j)\}^2].$$

□

Theorem 2 *Let $\Delta = (\Gamma, C, D)$ be a BFG of the graph $\Delta^* = (\Gamma, \Lambda)$. Then,*

$$FI_{BF}^P(\Delta) \le 2m(n-1)^2, \quad \text{and} \quad FI_{BF}^N(\Delta) \le 2m(n-1)^2,$$

where, m be the no. of edges, and n be the no. of vertices in Δ.

Proof As, Δ is a *BFG*, so, $0 \le \mu_C^P(b_i) \le 1, 0 \le \mu_C^P(b_j) \le 1, -1 \le \mu_C^N(b_i) \le 0, \text{and } -1 \le \mu_C^N(b_j) \le 0, \forall b_i, b_j \in \Gamma$.

We also have, $0 \leq \mu_D^P(b_ib_j) \leq 1$, and $-1 \leq \mu_D^N(b_ib_j) \leq 0, \forall b_ib_j \in \Lambda$.
Again, the no. of vertices in Δ is *n*. So, each vertex is adjacent to a maximum of $(n-1)$ vertices.
Now,

$$\begin{aligned} d^P(b_i) &= \sum_{b_j:b_i\neq b_j, b_ib_j\in\Lambda} \mu_D^P(b_ib_j) \leq (n-1), \forall b_i \in \Gamma \\ &\implies \mu_C^P(b_i)d^P(b_i) \leq (n-1) \quad [\because \mu_C^P(b_i) \leq 1] \\ &\implies \{\mu_C^P(b_i)d^P(b_i)\}^2 \leq (n-1)^2 \quad [\because \text{both the sides are non-negative}]. \end{aligned}$$

Similarly, $\{\mu_C^P(b_j)d^P(b_j)\}^2 \leq (n-1)^2$.

So,

$$\begin{aligned} &\{\mu_C^P(b_i)d^P(b_i)\}^2 + \{\mu_C^P(b_j)d^P(b_j)\}^2 \leq (n-1)^2 + (n-1)^2 \\ &\implies \{\mu_C^P(b_i)d^P(b_i)\}^2 + \{\mu_C^P(b_j)d^P(b_j)\}^2 \leq 2(n-1)^2 \\ &\implies \sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n} \left[\{\mu_C^P(b_i)d^P(b_i)\}^2 + \{\mu_C^P(b_j)d^P(b_j)\}^2\right] \leq \sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n} 2(n-1)^2. \\ &\implies FI_{BF}^P(\Delta) \leq 2m(n-1)^2. \end{aligned}$$

Again,

$$\begin{aligned} d^N(b_i) &= \sum_{b_j:b_i\neq b_j, b_ib_j\in\Lambda} \mu_D^N(b_ib_j) \geq -(n-1), \forall b_i \in \Gamma \\ &\implies \mu_C^N(b_i)d^N(b_i) \leq (n-1) \quad [\because \mu_C^N(b_i) \geq -1] \\ &\implies \{\mu_C^N(b_i)d^N(b_i)\}^2 \leq (n-1)^2 \quad [\because \text{both the sides are non-negative}]. \end{aligned}$$

Similarly, $\{\mu_C^N(b_j)d^N(b_j)\}^2 \leq (n-1)^2$.

So,

$$\begin{aligned} &\{\mu_C^N(b_i)d^N(b_i)\}^2 + \{\mu_C^N(b_j)d^N(b_j)\}^2 \leq (n-1)^2 + (n-1)^2 \\ &\implies \{\mu_C^N(b_i)d^N(b_i)\}^2 + \{\mu_C^N(b_j)d^N(b_j)\}^2 \leq 2(n-1)^2 \\ &\implies \sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n} \left[\{\mu_C^N(b_i)d^N(b_i)\}^2 + \{\mu_C^N(b_j)d^N(b_j)\}^2\right] \leq \sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n} 2(n-1)^2. \\ &\implies FI_{BF}^N(\Delta) \leq 2m(n-1)^2. \end{aligned}$$

$\therefore FI_{BF}^P(\Delta) \leq 2m(n-1)^2, \quad \text{and} \quad FI_{BF}^N(\Delta) \leq 2m(n-1)^2. \square$

Theorem 3 *Let $\Delta = (\Gamma, C, D)$ be a BFG having n vertices. Then,*

$$\begin{aligned} FI_{BF}^P(\Delta) \leq (n-1)^2\Psi_1, \quad &\text{and} \quad FI_{BF}^N(\Delta) \leq (n-1)^2\Psi_2, \quad \text{where,} \\ \Psi_1 = \sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n} &[\{\mu_C^P(b_i)\}^4 + \{\mu_C^P(b_j)\}^4], \quad \text{and} \\ \Psi_2 = \sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n} &[\{\mu_C^N(b_i)\}^4 + \{\mu_C^N(b_j)\}^4]. \end{aligned}$$

Proof Let, $\Gamma = \{b_1, b_2, ..., b_n\}$. Clearly, each vertex of Δ has atmost $(n-1)$ adjacent vertices.

Now,

$$\begin{aligned} d^P(b_i) &= \sum_{b_j:b_i\neq b_j, b_ib_j\in\Lambda} \mu_D^P(b_ib_j) \leq (n-1)\mu_C^P(b_i) \quad \text{and} \\ d^N(b_i) &= \sum_{b_j:b_i\neq b_j, b_ib_j\in\Lambda} \mu_D^N(b_ib_j) \geq (n-1)\mu_C^N(b_i). \end{aligned}$$

Now,

$$\begin{aligned}
&FI_{BF}^{P}(\Delta)\\
&=\sum_{b_ib_j\in\Lambda:1\le i\ne j\le n}[\{\mu_C^P(b_i)d^P(b_i)\}^2+\{\mu_C^P(b_j)d^P(b_j)\}^2]\\
&\le\sum_{b_ib_j\in\Lambda:1\le i\ne j\le n}[\{\mu_C^P(b_i)(n-1)\mu_C^P(b_i)\}^2+\{\mu_C^P(b_j)(n-1)\mu_C^P(b_j)\}^2]\\
&=(n-1)^2\sum_{b_ib_j\in\Lambda:1\le i\ne j\le n}[\{\mu_C^P(b_i)\}^4+\{\mu_C^P(b_j)\}^4].
\end{aligned}$$

$\therefore FI_{BF}^{P}(\Delta)\le(n-1)^2\sum_{b_ib_j\in\Lambda:1\le i\ne j\le n}[\{\mu_C^P(b_i)\}^4+\{\mu_C^P(b_j)\}^4]$.

Similarly,

$$FI_{BF}^{N}(\Delta)\le(n-1)^2\sum_{b_ib_j\in\Lambda:1\le i\ne j\le n}[\{\mu_C^N(b_i)\}^4+\{\mu_C^N(b_j)\}^4].$$

Therefore,

$$FI_{BF}^{P}(\Delta)\le(n-1)^2\Psi_1,\quad\text{and}\quad FI_{BF}^{N}(\Delta)\le(n-1)^2\Psi_2,\quad\text{where,}$$

$$\begin{aligned}
\Psi_1&=\sum_{b_ib_j\in\Lambda:1\le i\ne j\le n}[\{\mu_C^P(b_i)\}^4+\{\mu_C^P(b_j)\}^4],\quad\text{and}\\
\Psi_2&=\sum_{b_ib_j\in\Lambda:1\le i\ne j\le n}[\{\mu_C^N(b_i)\}^4+\{\mu_C^N(b_j)\}^4].
\end{aligned}$$

□

Theorem 4 *Let $\Delta=(\Gamma,C,D)$ be a BFG of the graph $\Delta^*=(\Gamma,\Lambda)$ s.t. Δ is a cycle and $|\Gamma|=n$. Then, $FI_{BF}^{P}(\Delta)\le 8n$ and $FI_{BF}^{N}(\Delta)\le 8n$.*

Proof Let, $\Gamma=\{b_1,b_2,b_3,....,b_n\}$.

As Δ is a cycle, so every vertex has exactly two adjacent vertices, and $|\Lambda|=n$.
Now,

$$\begin{aligned}
&d^P(b_i)\\
&=\sum_{b_j:b_i\ne b_j,b_ib_j\in\Lambda}\mu_D^P(b_ib_j)\\
&\le 2.1\quad[\because \mu_D^P(b_ib_j)\le 1,\forall b_ib_j\in\Lambda]\\
&\Rightarrow d^P(b_i)\le 2.
\end{aligned}$$

$\therefore d^P(b_i)\le 2,\forall b_i\in\Gamma$.

Similarly,

$$\begin{aligned}
&d^N(b_i)\\
&=\sum_{b_j:b_i\ne b_j,b_ib_j\in\Lambda}\mu_D^N(b_ib_j)\\
&\ge 2.(-1)\quad[\because \mu_D^N(b_ib_j)\ge(-1),\forall b_ib_j\in\Lambda]\\
&\Rightarrow d^N(b_i)\ge(-2).
\end{aligned}$$

$\therefore d^N(b_i)\ge(-2),\forall b_i\in\Gamma$.

Again, as $\mu_C^N(b_i)\ge(-1)$ and $d^N(b_i)\ge(-2)$, so $\mu_C^N(b_i)d^N(b_i)\le 2,\forall b_i\in\Gamma$.
Now,

$$\begin{aligned}
FI_{BF}^{P}(\Delta)&=\sum_{b_ib_j\in\Lambda:1\le i\ne j\le n}\left[\left\{\mu_C^P(b_i)\,d^P(b_i)\right\}^2+\left\{\mu_C^P(b_j)\,d^P(b_j)\right\}^2\right]\\
&\le\sum_{b_ib_j\in\Lambda:1\le i\ne j\le n}\left[\{1.2\}^2+\{1.2\}^2\right]\\
&=8n.
\end{aligned}$$

$\therefore FI^P_{BF}(\Delta) \leq 8n.$

Similarly, we can prove that, $FI^N_{BF}(\Delta) \leq 8n$.

$$\therefore FI^P_{BF}(\Delta) \leq 8n \quad \text{and} \quad FI^N_{BF}(\Delta) \leq 8n.$$

□

Theorem 5 *Let $\Delta = (\Gamma, C, D)$ be a BFG of the graph $\Delta^* = (\Gamma, \Lambda)$, s.t. Δ is a star, and $|\Gamma| = n$. Then,* $FI^P_{BF}(\Delta) \leq (n-1)(n^2 - 2n + 2)$ and $FI^N_{BF}(\Delta) \leq (n-1)(n^2 - 2n + 2)$.

Proof Let, $\Gamma = \{b_1, b_2, b_3, ..., b_n\}$ and b_1 be the center of the star Δ.

As Δ is a star, the center is adjacent to every other vertex, and every vertex except the center is a pendant vertex, and $|\Lambda| = (n-1)$.
Clearly, $\mu^P_C(b_i) \leq 1, \forall b_i \in \Gamma$, $d^P(b_1) \leq (n-1), d^P(b_i) \leq 1, \forall b_i \in \{b_2, b_3, ..., b_n\}$; and $\mu^N_C(b_i) \geq -1, \forall b_i \in \Gamma$, $d^N(b_1) \geq -(n-1), d^N(b_i) \geq -1, \forall b_i \in \{b_2, b_3, ..., b_n\}$.
Now,

$$\begin{aligned}
FI^P_{BF}(\Delta) = & \sum_{b_ib_j \in \Lambda: 1 \leq i \neq j \leq n} \left[\left\{\mu^P_C(b_i)\, d^P(b_i)\right\}^2 + \left\{\mu^P_C(b_j)\, d^P(b_j)\right\}^2\right] \\
& = [\left\{\mu^P_C(b_1)\, d^P(b_1)\right\}^2 + \left\{\mu^P_C(b_2)\, d^P(b_2)\right\}^2] + \\
& [\left\{\mu^P_C(b_1)\, d^P(b_1)\right\}^2 + \left\{\mu^P_C(b_3)\, d^P(b_3)\right\}^2] + ...+ \\
& [\left\{\mu^P_C(b_1)\, d^P(b_1)\right\}^2 + \left\{\mu^P_C(b_n)\, d^P(b_n)\right\}^2] \quad [(n-1)\text{terms}] \\
& \leq (n-1)[\{1.(n-1)\}^2 + \{1.1\}^2] \\
& = (n-1)(n^2 - 2n + 2).
\end{aligned}$$

$\therefore FI^P_{BF}(\Delta) \leq (n-1)(n^2 - 2n + 2).$

Similarly, $FI^N_{BF}(\Delta) \leq (n-1)(n^2 - 2n + 2)$.

$$\therefore FI^P_{BF}(\Delta) \leq (n-1)(n^2 - 2n + 2) \quad \text{and} \quad FI^N_{BF}(\Delta) \leq (n-1)(n^2 - 2n + 2).$$

□

Theorems on the forgotten index in different types of bipolar fuzzy graphs

Over the years, researchers have developed various characteristics, like completeness, regularity, etc., of bipolar guzzy graphs. Poulik et al.[27] have proved several theorems regarding Randic Index for different types of bipolar fuzzy graphs. In this subsection, we have discussed several theorems on the Forgotten Index for regular *BFG*, complete *BFG*, strong *BFG*, isomorphic *BFG*, etc.

Theorem 6 *Let $\Delta = (\Gamma, C, D)$ be a connected BFG consisting of n vertices, and $\Delta' = (\Gamma', C', D')$ be a BFG obtained by deleting a vertex from Δ. Then,*

$$FI^P_{BF}(\Delta) \geq FI^P_{BF}(\Delta') \quad \text{and} \quad FI^N_{BF}(\Delta) \geq FI^N_{BF}(\Delta').$$

Proof Let, $\Gamma = \{b_1, b_2, ..., b_n\}$ and $\Gamma' = \{b_1, b_2, ..., b_{n-1}\}$.

$\therefore \Gamma' \subset \Gamma \implies \Delta'$ is a bipolar fuzzy subgraph of Δ.
Therefore, $\mu^P_C(b_i) = \mu^P_{C'}(b_i)$, $\mu^N_C(b_i) = \mu^N_{C'}(b_i)$, $\mu^P_D(b_ib_j) = \mu^P_{D'}(b_ib_j)$,
and $\mu^N_D(b_ib_j) = \mu^N_{D'}(b_ib_j)$, $\forall b_i \in \Gamma'$ and $\forall b_ib_j \in \Lambda'$, where Λ' is the set of edges in Δ'.
Now,

$$\begin{aligned}
d^P(b_i) &= \sum_{b_j: b_i \neq b_j, b_ib_j \in \Lambda} \mu^P_D(b_ib_j) \quad \text{and} \quad d^N(b_i) = \sum_{b_j: b_i \neq b_j, b_ib_j \in \Lambda} \mu^N_D(b_ib_j); \\
d'^P(b_i) &= \sum_{b_j: b_i \neq b_j, b_ib_j \in \Lambda'} \mu^P_{D'}(b_ib_j) \quad \text{and} \quad d'^N(b_i) = \sum_{b_j: b_i \neq b_j, b_ib_j \in \Lambda'} \mu^N_{D'}(b_ib_j).
\end{aligned}$$

So,

$$\sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n}\left[\{\mu_C^P(b_i)d^P(b_i)\}^2+\{\mu_C^P(b_j)d^P(b_j)\}^2\right]$$
$$\geq\sum_{b_ib_j\in\Lambda':1\leq i\neq j\leq (n-1)}\left[\{\mu_C^P(b_i)d^P(b_i)\}^2+\{\mu_C^P(b_j)d^P(b_j)\}^2\right]$$
$$=\sum_{b_ib_j\in\Lambda':1\leq i\neq j\leq (n-1)}\left[\{\mu_{C'}^P(b_i){d'}^P(b_i)\}^2+\{\mu_{C'}^P(b_j){d'}^P(b_j)\}^2\right]\quad\text{and}$$
$$\sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n}\left[\{\mu_C^N(b_i)d^N(b_i)\}^2+\{\mu_C^N(b_j)d^N(b_j)\}^2\right]$$
$$\geq\sum_{b_ib_j\in\Lambda':1\leq i\neq j\leq (n-1)}\left[\{\mu_C^N(b_i)d^N(b_i)\}^2+\{\mu_C^N(b_j)d^N(b_j)\}^2\right]$$
$$=\sum_{b_ib_j\in\Lambda':1\leq i\neq j\leq (n-1)}\left[\{\mu_{C'}^N(b_i){d'}^N(b_i)\}^2+\{\mu_{C'}^N(b_j){d'}^N(b_j)\}^2\right].$$

Therefore,

$$FI_{BF}^P(\Delta)\geq FI_{BF}^P(\Delta')\quad\text{and}\quad FI_{BF}^N(\Delta)\geq FI_{BF}^N(\Delta').$$

□

Theorem 7 *Let* $\Delta=(\Gamma,C,D)$ *be a complete BFG, s.t. the function C is constant. Then,*

$$FI_{BF}(\Delta)=(n(n-1)^3v_1^4,n(n-1)^3v_2^4)=n(n-1)^3(v_1^4,v_2^4),$$

where n is the no. of vertices in Δ*, and* $v_1=\mu_C^P(b_i),v_2=\mu_C^N(b_i),b_i\in\Gamma$**.**

Proof As *C* is constant, and $v_1=\mu_C^P(b_i),v_2=\mu_C^N(b_i),b_i\in\Gamma$,

so, $v_1=\mu_C^P(b_i),v_2=\mu_C^N(b_i),\forall b_i\in\Gamma$.
Since Δ is a complete *BFG*, then,

$$\mu_D^P(b_ib_j)=min\{\mu_C^P(b_i),\mu_C^P(b_j)\}=v_1,\quad\text{and}$$
$$\mu_D^N(b_ib_j)=max\{\mu_C^N(b_i),\mu_C^N(b_j)\}=v_2,\forall b_i,b_j\in\Gamma.$$

Again, as Δ is complete, and the no. of vertices in Δ is *n*, so, there are $\binom{n}{2}=\frac{n(n-1)}{2}$ edges, and every vertex in Δ has exactly $(n-1)$ adjacent vertices.

Therefore,

$$d^P(b_i)=\sum_{b_j:b_i\neq b_j,b_ib_j\in\Lambda}\mu_D^P(b_ib_j)=(n-1)v_1,\quad\text{and}$$
$$d^N(b_i)=\sum_{b_j:b_i\neq b_j,b_ib_j\in\Lambda}\mu_D^N(b_ib_j)=(n-1)v_2.$$

Therefore,

$$FI_{BF}^P(\Delta)$$
$$=\sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n}\left[\{\mu_C^P(b_i)d^P(b_i)\}^2+\{\mu_C^P(b_j)d^P(b_j)\}^2\right]$$
$$=\frac{n(n-1)}{2}\left[\{v_1(n-1)v_1\}^2+\{v_1(n-1)v_1\}^2\right]$$
$$=n(n-1)^3v_1^4.$$

So, $FI_{BF}^P(\Delta)=n(n-1)^3v_1^4$

Similarly, $FI_{BF}^N(\Delta)=n(n-1)^3v_2^4$.
Therefore,

$$FI_{BF}(\Delta)=(n(n-1)^3v_1^4,n(n-1)^3v_2^4)=n(n-1)^3(v_1^4,v_2^4).$$

□

Theorem 8 *Let $\Delta = (\Gamma, C, D)$ be a regular and totally regular BFG s.t. $|\Gamma| = n$, and between all pairs of vertices, there is an edge. If $d(b_i) = (p_1, p_2)$; and $\left(\mu_C^P(b_i), \mu_C^N(b_i)\right) = (u_1, u_2), b_i \in \Gamma$, then*

$$FI_{BF}(\Delta) = n(n-1)(u_1^2p_1^2, u_2^2p_2^2).$$

Proof As Δ is regular, so $p_1 = d^P(b_i), p_2 = d^N(b_i), \forall b_i \in \Gamma$. Again, Δ is totally regular *BFG*, so $d^P[b_i] = t_1$ (say), and $d^N[b_i] = t_2$ (say), $\forall b_i \in \Gamma$.

We know that, $d^P[b_i] = d^P(b_i) + \mu_C^P(b_i) \Rightarrow t_1 = p_1 + \mu_C^P(b_i) \Rightarrow \mu_C^P(b_i) = t_1 - p_1$, and $d^N[b_i] = d^N(b_i) + \mu_C^N(b_i) \Rightarrow t_2 = p_2 + \mu_C^N(b_i) \Rightarrow \mu_C^N(b_i) = t_2 - p_2 \, \forall b_i \in \Gamma$.

Hence, *C* is constant function, and $u_1 = \mu_C^P(b_i)$ and $u_2 = \mu_C^N(b_i)$, $\forall b_i \in \Gamma$. As $|\Gamma| = n$, and there exists an edge between all pairs of vertices in Δ, so there are exactly $\binom{n}{2} = \frac{n(n-1)}{2}$ edges in Δ.

Now,

$$\begin{aligned} FI_{BF}^P(\Delta) &= \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \left[\left\{\mu_C^P(b_i)\, d^P(b_i)\right\}^2 + \left\{\mu_C^P(b_j)\, d^P(b_j)\right\}^2\right] \\ &= \frac{n(n-1)}{2}\left[\{u_1p_1\}^2 + \{u_1p_1\}^2\right] = n(n-1)u_1^2p_1^2 \end{aligned}$$

and

$$\begin{aligned} FI_{BF}^N(\Delta) &= \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \left[\left\{\mu_C^N(b_i)\, d^N(b_i)\right\}^2 + \left\{\mu_C^N(b_j)\, d^N(b_j)\right\}^2\right] \\ &= \frac{n(n-1)}{2}\left[\{u_2p_2\}^2 + \{u_2p_2\}^2\right] = n(n-1)u_2^2p_2^2. \end{aligned}$$

So, $FI_{BF}(\Delta) = \left(FI_{BF}^P(\Delta), FI_{BF}^N(\Delta)\right) = n(n-1)\left(u_1^2p_1^2, u_2^2p_2^2\right)$. □

Theorem 9 *Let, $\tilde{\mathbf{\Delta}} = (\Gamma, C, \vec{\mathbf{D}})$ be a BFDG, such that $|\Lambda| = m$, and C is constant. If $d(b_i) = (p_1, p_2)$ and $\left(\mu_C^P(b_i), \mu_C^N(b_i)\right) = (u_1, u_2), \forall b_i \in \Gamma$. Then*

$$FI_{BF}(\tilde{\mathbf{\Delta}}) = 2m(u_1^2p_1^2, u_2^2p_2^2).$$

Proof Since, $|\Lambda| = m$ and C is constant, so the no. of undirected edges is *m*.

Now,

$$\begin{aligned} &FI_{BF}^P(\tilde{\mathbf{\Delta}}) \\ &= \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} [\{\mu_C^P(b_i)d^P(b_i)\}^2 + \{\mu_C^P(b_j)d^P(b_j)\}^2] \\ &= \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} [\{u_1^2p_1^2\} + \{u_1^2p_1^2\}] \\ &= m[2u_1^2p_1^2] \\ &= 2mu_1^2p_1^2. \end{aligned}$$

Similarly, $FI_{BF}^N(\tilde{\mathbf{\Delta}}) = 2mu_2^2p_2^2$.

Therefore,

$$FI_{BF}(\tilde{\mathbf{\Delta}}) = (2mu_1^2p_1^2, 2mu_2^2p_2^2) = 2m(u_1^2p_1^2, u_2^2p_2^2).$$

□

Theorem 10 *Let $\Delta = (\Gamma, C, D)$ be a regular BFG of the graph $\Delta^* = (\Gamma, \Lambda)$ s.t. Δ^* is an odd cycle, and $|\Gamma| = n$. If $\Gamma = \{b_1, b_2, \cdots, b_n\}$ s.t. $\left(\mu_C^P(b_i), \mu_C^N(b_i)\right) = (v_{1i}, v_{2i})$, and $\left(\mu_D^P(b_ib_j), \mu_D^N(b_ib_j)\right) = (e_1, e_2)$, where $b_ib_j \in \Lambda$ and $1 \le i \ne j \le n$. Then,*

$$FI_{BF}(\Delta) = 8(e_1^2\left[{v_{11}}^2 + {v_{12}}^2 + {v_{13}}^2 + \ldots\ldots + {v_{1n}}^2\right], e_2^2\left[{v_{21}}^2 + {v_{22}}^2 + {v_{23}}^2 + \ldots\ldots + {v_{2n}}^2\right]).$$

Proof We have, if $\Delta = (\Gamma, C, D)$ be a *BFG* of an odd cycle $\Delta^* = (\Gamma, \Lambda)$, then Δ is regular if and only if *D* is constant[12].

Here, $\Delta = (\Gamma, C, D)$ is a regular *BFG* of the odd cycle $\Delta^* = (\Gamma, \Lambda)$, so $D = \left(\mu_D^P, \mu_D^N\right)$ is a constant function. But, $\left(\mu_D^P(b_ib_j), \mu_D^N(b_ib_j)\right) = (e_1, e_2)$, where $b_ib_j \in \Lambda$; so $\left(\mu_D^P(b_ib_j), \mu_D^N(b_ib_j)\right) = (e_1, e_2), \forall b_ib_j \in \Lambda$. As Δ^* is a cycle, each vertex in Δ has exactly two incident edges. Again, as Δ is regular, all the vertices have equal degrees. $\therefore d(b_i) = \left(d^P(b_i), d^N(b_i)\right) = (2e_1, 2e_2), \forall b_i \in \Gamma$.

Again we have, Δ^* is a cycle and $|\Gamma| = n$. So, Δ has exactly *n* edges.

Therefore,

$$\begin{aligned}
FI^P_{BF}(\Delta) &= \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \left[\left\{\mu^P_C(b_i)\, d^P(b_i)\right\}^2 + \left\{\mu^P_C(b_j)\, d^P(b_j)\right\}^2\right] \\
&= \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \left[\{v_{1i} 2e_1\}^2 + \{v_{1j} 2e_1\}^2\right] \\
&= 4e_1^2 \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \left[{v_{1i}}^2 + {v_{1j}}^2\right] \\
&= 4e_1^2\left[({v_{11}}^2 + {v_{12}}^2) + ({v_{12}}^2 + {v_{13}}^2) + ({v_{13}}^2 + {v_{14}}^2) + \ldots\ldots + ({v_{1n}}^2 + {v_{11}}^2)\right] \\
&= 8e_1^2\left[{v_{11}}^2 + {v_{12}}^2 + {v_{13}}^2 + \ldots\ldots + {v_{1n}}^2\right]
\end{aligned}$$

and

$$\begin{aligned}
FI^N_{BF}(\Delta) &= \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \left[\left\{\mu^N_C(b_i)\, d^N(b_i)\right\}^2 + \left\{\mu^N_C(b_j)\, d^N(b_j)\right\}^2\right] \\
&= \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \left[\{v_{2i} 2e_2\}^2 + \{v_{2j} 2e_2\}^2\right] \\
&= 4e_2^2 \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \left[{v_{2i}}^2 + {v_{2j}}^2\right] \\
&= 4e_2^2\left[({v_{21}}^2 + {v_{22}}^2) + ({v_{22}}^2 + {v_{23}}^2) + ({v_{23}}^2 + {v_{24}}^2) + \ldots\ldots + ({v_{2n}}^2 + {v_{21}}^2)\right] \\
&= 8e_2^2\left[{v_{21}}^2 + {v_{22}}^2 + {v_{23}}^2 + \ldots\ldots + {v_{2n}}^2\right].
\end{aligned}$$

$$\text{So,}\quad FI_{BF}(\Delta) = 8(e_1^2\left[{v_{11}}^2 + {v_{12}}^2 + {v_{13}}^2 + \ldots\ldots + {v_{1n}}^2\right], e_2^2\left[{v_{21}}^2 + {v_{22}}^2 + {v_{23}}^2 + \ldots\ldots + {v_{2n}}^2\right]).$$

□

Theorem 11 *Let* $\Delta = (\Gamma, C, D)$ *be a strong BFG of the graph* $\Delta^* = (\Gamma, \Lambda)$ *having n vertices, s.t.* Δ^* *is a cycle. Then,* $FI_{BF}(\Delta) = 8n\left(u_1^4, u_2^4\right)$*, where* $(u_1, u_2) = \left(\mu^P_C(b_i), \mu^N_C(b_i)\right), \forall b_i \in \Gamma$.

Proof Let, $\Gamma = \{b_1, b_2, b_3, ..., b_n\}$. Since, Δ is a strong *BFG*, and $(u_1, u_2) = (\mu^P_C(b_i), \mu^N_C(b_i)), \forall b_i \in \Gamma$, so, $\mu^P_D(b_ib_j) = \min\{\mu^P_C(b_i), \mu^P_C(b_j)\} = u_1$ and $\mu^N_D(b_ib_j) = \max\left\{\mu^N_C(b_i), \mu^N_C(b_j)\right\} = u_2, \forall b_ib_j \in \Lambda$. Every vertex has exactly two incident edges, as Δ^* is a cycle. Then, $d^P(b_i) = 2u_1$ and $d^N(b_i) = 2u_2, \forall b_i \in \Gamma$. Again, as $|\Gamma| = n$ and Δ^* is a cycle, so there are exactly *n* edges in Δ. Then,

$$\begin{aligned}
FI^P_{BF}(\Delta) &= \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \left[\left\{\mu^P_C(b_i)\, d^P(b_i)\right\}^2 + \left\{\mu^P_C(b_j)\, d^P(b_j)\right\}^2\right] \\
&= \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \left[(2u_1^2)^2 + (2u_1^2)^2\right] \\
&= \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \left[8u_1^4\right] \\
&= 8nu_1^4
\end{aligned}$$

and

$$\begin{aligned}
FI^N_{BF}(\Delta) &= \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \left[\left\{\mu^N_C(b_i)\, d^N(b_i)\right\}^2 + \left\{\mu^N_C(b_j)\, d^N(b_j)\right\}^2\right] \\
&= \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \left[(2u_2^2)^2 + (2u_2^2)^2\right] \\
&= \sum_{b_ib_j \in \Lambda: 1 \le i \ne j \le n} \left[8u_2^4\right] \\
&= 8nu_2^4.
\end{aligned}$$

Therefore,

$$FI_{BF}(\Delta) = 8n\left(u_1^4, u_2^4\right), \quad \text{where} \quad (u_1, u_2) = \left(\mu^P_C(b_i), \mu^N_C(b_i)\right), \forall b_i \in \Gamma.$$

□

Theorem 12 *Let,* $\Delta_1 = (\Gamma_1, C_1, D_1)$ *and* $\Delta_2 = (\Gamma_2, C_2, D_2)$ *are two isomorphic bipolar fuzzy graphs of the graphs* $\Delta_1^*(\Gamma_1, \Lambda_1)$ *and* $\Delta_2^*(\Gamma_2, \Lambda_2)$*, respectively. Then,* $FI_{BF}^P(\Delta_1) = FI_{BF}^P(\Delta_2)$ *and* $FI_{BF}^N(\Delta_1) = FI_{BF}^N(\Delta_2)$.

Proof As, Δ_1 and Δ_2 are isomorphic, so $\exists$ a bijective mapping $\psi : \Gamma_1 \to \Gamma_2$, s.t. $\mu_{C_1}^P(b_i) = \mu_{C_2}^P(\psi(b_i))$, $\mu_{C_1}^N(b_i) = \mu_{C_2}^N(\psi(b_i)), \forall b_i \in \Gamma_1$; and $\mu_{D_1}^P(b_ib_j) = \mu_{D_2}^P(\psi(b_i)\psi(b_j))$, $\mu_{D_1}^N(b_ib_j) = \mu_{D_2}^N(\psi(b_i)\psi(b_j)), \forall b_ib_j \in \Lambda_1$.

Now,

$$\begin{aligned} d_{\Delta_1}^P(b_i) &= \sum_{b_j : b_i \neq b_j, b_ib_j \in \Lambda_1} \mu_{D_1}^P(b_ib_j) \\ &= \sum_{b_j : b_i \neq b_j, b_ib_j \in \Lambda_1} \mu_{D_2}^P(\psi(b_i)\psi(b_j)) \\ &= \sum_{\psi(b_j) : \psi(b_i) \neq \psi(b_j), \psi(b_i)\psi(b_j) \in \Lambda_2} \mu_{D_2}^P(\psi(b_i)\psi(b_j)) \\ &= d_{\Delta_2}^P(\psi(b_i)). \end{aligned}$$

Therefore,

$$\begin{aligned} &FI_{BF}^P(\Delta_1) \\ &= \sum_{b_ib_j \in \Lambda_1 : 1 \le i \neq j \le n} \left[\left\{ \mu_{C_1}^P(b_i)\, d_{\Delta_1}^P(b_i) \right\}^2 + \left\{ \mu_{C_1}^P(b_j)\, d_{\Delta_1}^P(b_j) \right\}^2 \right] \\ &= \sum_{b_ib_j \in \Lambda_1 : 1 \le i \neq j \le n} \left[\left\{ \mu_{C_2}^P(\psi(b_i))\, d_{\Delta_2}^P(\psi(b_i)) \right\}^2 + \left\{ \mu_{C_2}^P(\psi(b_j))\, d_{\Delta_2}^P(\psi(b_j)) \right\}^2 \right] \\ &= \sum_{\psi(b_i)\psi(b_j) \in \Lambda_2 : 1 \le i \neq j \le n} \left[\left\{ \mu_{C_2}^P(\psi(b_i))\, d_{\Delta_2}^P(\psi(b_i)) \right\}^2 + \left\{ \mu_{C_2}^P(\psi(b_j))\, d_{\Delta_2}^P(\psi(b_j)) \right\}^2 \right] \\ &= FI_{BF}^P(\Delta_2). \end{aligned}$$

So, $FI_{BF}^P(\Delta_1) = FI_{BF}^P(\Delta_2)$.

Similarly, $FI_{BF}^N(\Delta_1) = FI_{BF}^N(\Delta_2)$.

$$\therefore FI_{BF}^P(\Delta_1) = FI_{BF}^P(\Delta_2) \quad \text{and} \quad FI_{BF}^N(\Delta_1) = FI_{BF}^N(\Delta_2).$$

□

Theorems on relationships of the forgotten index with some other topological indices in bipolar fuzzy graphs

Several topological indices have been developed over the decades. The establishment of relationships among them is very important. By finding the relationships among different topological indices, one can describe the characteristics of one topological index with the help of another topological index, compare different topological indices and discuss their effectiveness, and apply one topological index in the application area of the other topological index and see how it behaves there. In this subsection, we have proved some theorems regarding the relationships of different topological indices with the Forgotten Index in *BFG*s.

Theorem 13 *Let* $\Delta = (\Gamma, C, D)$ *be a BFG of the graph* $\Delta^* = (\Gamma, \Lambda)$ *s.t.* $|\Gamma| = n$*, and* $|\Lambda| = m$*. Then,*

$$\begin{aligned} &\frac{1}{m}\{SO_{BF}^P(\Delta)\}^2 \le FI_{BF}^P(\Delta) \le \{SO_{BF}^P(\Delta)\}^2, \quad \text{and} \\ &\frac{1}{m}\{SO_{BF}^N(\Delta)\}^2 \le FI_{BF}^N(\Delta) \le \{SO_{BF}^N(\Delta)\}^2. \end{aligned}$$

Proof We have,

$$\begin{aligned} FI_{BF}^P(\Delta) &= \sum_{b_ib_j \in \Lambda : 1 \le i \neq j \le n} \left[\left\{ \mu_C^P(b_i)\, d^P(b_i) \right\}^2 + \left\{ \mu_C^P(b_j)\, d^P(b_j) \right\}^2 \right] \\ &= \sum_{b_ib_j \in \Lambda : 1 \le i \neq j \le n} \left[\sqrt{ \left\{ \mu_C^P(b_i)\, d^P(b_i) \right\}^2 + \left\{ \mu_C^P(b_j)\, d^P(b_j) \right\}^2 } \right]^2 \\ &\le \left[\sum_{b_ib_j \in \Lambda : 1 \le i \neq j \le n} \sqrt{ \left\{ \mu_C^P(b_i)\, d^P(b_i) \right\}^2 + \left\{ \mu_C^P(b_j)\, d^P(b_j) \right\}^2 } \right]^2 \\ &= \{SO_{BF}^P(\Delta)\}^2. \end{aligned}$$

$\therefore FI_{BF}^P(\Delta) \le \{SO_{BF}^P(\Delta)\}^2$.

Similarly, we can prove that, $FI^N_{BF}(\Delta) \leq \{SO^N_{BF}(\Delta)\}^2$.

From the Cauchy-Schwartz inequality, we get

$$\left(\sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n} 1^2\right)\left(\sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n} \left\{\sqrt{\left[\left\{\mu_C^P(b_i)\,d^P(b_i)\right\}^2+\left\{\mu_C^P(b_j)\,d^P(b_j)\right\}^2\right]}\right\}^2\right)$$
$$\geq \left(\sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n} (1)\left(\sqrt{\left[\left\{\mu_C^P(b_i)\,d^P(b_i)\right\}^2+\left\{\mu_C^P(b_j)\,d^P(b_j)\right\}^2\right]}\right)\right)^2$$
$$\Rightarrow m\left(\sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n} \left[\left\{\mu_C^P(b_i)\,d^P(b_i)\right\}^2+\left\{\mu_C^P(b_j)\,d^P(b_j)\right\}^2\right]\right)$$
$$\geq \left(\sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n} \sqrt{\left[\left\{\mu_C^P(b_i)\,d^P(b_i)\right\}^2+\left\{\mu_C^P(b_j)\,d^P(b_j)\right\}^2\right]}\right)^2$$
$$\Rightarrow mFI^P_{BF}(\Delta) \geq \{SO^P_{BF}(\Delta)\}^2$$
$$\Rightarrow FI^P_{BF}(\Delta) \geq \frac{1}{m}\{SO^P_{BF}(\Delta)\}^2.$$

Similarly, it can be proved that, $FI^N_{BF}(\Delta) \geq \frac{1}{m}\{SO^N_{BF}(\Delta)\}^2$.

Therefore,

$$\frac{1}{m}\{SO^P_{BF}(\Delta)\}^2 \leq FI^P_{BF}(\Delta) \leq \{SO^P_{BF}(\Delta)\}^2, \quad \text{and}$$
$$\frac{1}{m}\{SO^N_{BF}(\Delta)\}^2 \leq FI^N_{BF}(\Delta) \leq \{SO^N_{BF}(\Delta)\}^2.$$

□

Theorem 14 *Let $\Delta = (\Gamma, C, D)$ be a BFG of the graph $\Delta^* = (\Gamma, \Lambda)$ s.t. $|\Gamma| = n$, and $|\Lambda| = m$. Then,*

$$\frac{1}{2m}\{FZI^P_{BF}(\Delta)\}^2 \leq FI^P_{BF}(\Delta) \leq \{FZI^P_{BF}(\Delta)\}^2, \quad \text{and}$$
$$\frac{1}{2m}\{FZI^N_{BF}(\Delta)\}^2 \leq FI^N_{BF}(\Delta) \leq \{FZI^N_{BF}(\Delta)\}^2.$$

Proof We have, $0 \leq \mu^P_C(b_i) \leq 1, 0 \leq \mu^P_C(b_j) \leq 1, -1 \leq \mu^N_C(b_i) \leq 0, -1 \leq \mu^N_C(b_j) \leq 0, \forall b_i, b_j \in \Gamma$, and $0 \leq \mu^P_D(b_ib_j) \leq 1, -1 \leq \mu^N_D(b_ib_j) \leq 0, \forall b_ib_j \in \Lambda$.

$$\text{So,} \quad d^P(b_i) = \sum_{b_j:b_i\neq b_j, b_ib_j\in\Lambda} \mu^P_D(b_ib_j) \geq 0, \quad \text{and} \quad d^N(b_i) = \sum_{b_j:b_i\neq b_j, b_ib_j\in\Lambda} \mu^N_D(b_ib_j) \leq 0.$$

Therefore, $\mu^P_C(b_i)d^P(b_i) \geq 0$, $\mu^P_C(b_j)d^P(b_j) \geq 0$, $\mu^N_C(b_i)d^N(b_i) \geq 0$, and $\mu^N_C(b_j)d^N(b_j) \geq 0$.

Now, for any two non-negative real numbers *f* and *r*, we have. $f^2 + r^2 \leq (f + r)^2$.

Putting $f = \mu^P_C(b_i)d^P(b_i)$ and $r = \mu^P_C(b_j)d^P(b_j)$, we get,

$$\left[\left\{\mu_C^P(b_i)\,d^P(b_i)\right\}^2+\left\{\mu_C^P(b_j)\,d^P(b_j)\right\}^2\right]$$
$$\leq \left[\left\{\mu_C^P(b_i)\,d^P(b_i)\right\}+\left\{\mu_C^P(b_j)\,d^P(b_j)\right\}\right]^2$$
$$\Rightarrow \sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n} \left[\left\{\mu_C^P(b_i)\,d^P(b_i)\right\}^2+\left\{\mu_C^P(b_j)\,d^P(b_j)\right\}^2\right]$$
$$\leq \sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n} \left[\left\{\mu_C^P(b_i)\,d^P(b_i)\right\}+\left\{\mu_C^P(b_j)\,d^P(b_j)\right\}\right]^2$$
$$\leq \left[\sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n} \left\{\mu_C^P(b_i)\,d^P(b_i)+\mu_C^P(b_j)\,d^P(b_j)\right\}\right]^2$$
$$\Rightarrow FI^P_{BF}(\Delta) \leq \{FZI^P_{BF}(\Delta)\}^2.$$

Similarly, we can prove that, $FI^N_{BF}(\Delta) \leq \{FZI^N_{BF}(\Delta)\}^2$.

Again, for any two non-negative real numbers *h* and *t*, we have, $(h-t)^2 \geq 0 \Rightarrow h^2 + t^2 \geq 2ht \Rightarrow 2(h^2 + t^2) \geq (h + t)^2 \Rightarrow \sqrt{2}\sqrt{h^2 + t^2} \geq (h + t)$.

Putting $h = \mu^P_C(b_i)d^P(b_i)$ and $t = \mu^P_C(b_j)d^P(b_j)$, we get,

$$\begin{aligned}
&\sqrt{2}\sqrt{\left\{\mu_C^P(b_i)\,d^P(b_i)\right\}^2+\left\{\mu_C^P(b_j)\,d^P(b_j)\right\}^2}\\
&\geq\left\{\mu_C^P(b_i)\,d^P(b_i)+\mu_C^P(b_j)\,d^P(b_j)\right\}\\
&\Rightarrow\sqrt{2}\sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n}\sqrt{\left\{\mu_C^P(b_i)\,d^P(b_i)\right\}^2+\left\{\mu_C^P(b_j)\,d^P(b_j)\right\}^2}\\
&\geq\sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n}\left\{\mu_C^P(b_i)\,d^P(b_i)+\mu_C^P(b_j)\,d^P(b_j)\right\}\\
&\Rightarrow\sqrt{2}SO_{BF}^P(\Delta)\geq FZI_{BF}^P(\Delta)\\
&\Rightarrow\sqrt{2}\sqrt{m}\sqrt{FI_{BF}^P(\Delta)}\geq\sqrt{2}SO_{BF}^P(\Delta)\geq FZI_{BF}^P(\Delta)\quad\text{[From theorem 13]}\\
&\Rightarrow\sqrt{2}\sqrt{m}\sqrt{FI_{BF}^P(\Delta)}\geq FZI_{BF}^P(\Delta)\\
&\Rightarrow FI_{BF}^P(\Delta)\geq\frac{1}{2m}\{FZI_{BF}^P(\Delta)\}^2.
\end{aligned}$$

Similarly, it can be proved that, $FI_{BF}^N(\Delta)\geq\frac{1}{2m}\{FZI_{BF}^N(\Delta)\}^2$.

Therefore,

$$\begin{aligned}
&\frac{1}{2m}\{FZI_{BF}^P(\Delta)\}^2\leq FI_{BF}^P(\Delta)\leq\{FZI_{BF}^P(\Delta)\}^2,\quad\text{and}\\
&\frac{1}{2m}\{FZI_{BF}^N(\Delta)\}^2\leq FI_{BF}^N(\Delta)\leq\{FZI_{BF}^N(\Delta)\}^2.
\end{aligned}$$

□

Theorem 15 *Let* $\Delta=(\Gamma,C,D)$ *be a BFG of the graph* $\Delta^*=(\Gamma,\Lambda)$ *s.t.* $|\Gamma|=n$. *Then,*

$$FI_{BF}^P(\Delta)\geq 2SZI_{BF}^P(\Delta),\quad\text{and}\quad FI_{BF}^N(\Delta)\geq 2SZI_{BF}^N(\Delta).$$

Proof For any two non-negative real numbers *s* and *t*, by applying A.M. $\geq$ G.M. on s^2 and t^2, we get, $\frac{s^2+t^2}{2}\geq st\Rightarrow s^2+t^2\geq 2st$.

Putting $s=\mu_C^P(b_i)d^P(b_i)$ and $t=\mu_C^P(b_j)d^P(b_j)$, we get,

$$\begin{aligned}
&\left[\left\{\mu_C^P(b_i)\,d^P(b_i)\right\}^2+\left\{\mu_C^P(b_j)\,d^P(b_j)\right\}^2\right]\geq 2\left[\left\{\mu_C^P(b_i)\,d^P(b_i)\right\}\left\{\mu_C^P(b_j)\,d^P(b_j)\right\}\right]\\
&\Rightarrow\sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n}\left[\left\{\mu_C^P(b_i)\,d^P(b_i)\right\}^2+\left\{\mu_C^P(b_j)\,d^P(b_j)\right\}^2\right]\\
&\geq 2\sum_{b_ib_j\in\Lambda:1\leq i\neq j\leq n}\left[\left\{\mu_C^P(b_i)\,d^P(b_i)\right\}\left\{\mu_C^P(b_j)\,d^P(b_j)\right\}\right]\\
&\Rightarrow FI_{BF}^P(\Delta)\geq 2SZI_{BF}^P(\Delta).
\end{aligned}$$

Similarly, it can be proved that, $FI_{BF}^N(\Delta)\geq 2SZI_{BF}^N(\Delta)$.

$$\therefore FI_{BF}^P(\Delta)\geq 2SZI_{BF}^P(\Delta),\quad\text{and}\quad FI_{BF}^N(\Delta)\geq 2SZI_{BF}^N(\Delta).$$

□

Theorem 16 *Let* $\Delta=(\Gamma,C,D)$ *be a BFG of the graph* $\Delta^*=(\Gamma,\Lambda)$ *s.t.* $|\Gamma|=n$. *Then,*

$$FI_{BF}^P(\Delta)\leq HZI_{BF}^P(\Delta),\quad\text{and}\quad FI_{BF}^N(\Delta)\leq HZI_{BF}^N(\Delta).$$

Proof We have, $0\leq\mu_C^P(b_i)\leq 1, 0\leq\mu_C^P(b_j)\leq 1, -1\leq\mu_C^N(b_i)\leq 0, -1\leq\mu_C^N(b_j)\leq 0, \forall b_i,b_j\in\Gamma$, and $0\leq\mu_D^P(b_ib_j)\leq 1, -1\leq\mu_D^N(b_ib_j)\leq 0, \forall b_ib_j\in\Lambda$.

$$\text{So,}\quad d^P(b_i)=\sum_{b_j:b_i\neq b_j,b_ib_j\in\Lambda}\mu_D^P(b_ib_j)\geq 0,\quad\text{and}\quad d^N(b_i)=\sum_{b_j:b_i\neq b_j,b_ib_j\in\Lambda}\mu_D^N(b_ib_j)\leq 0.$$

Therefore, $\mu_C^P(b_i)d^P(b_i)\geq 0$, $\mu_C^P(b_j)d^P(b_j)\geq 0$, $\mu_C^N(b_i)d^N(b_i)\geq 0$, and $\mu_C^N(b_j)d^N(b_j)\geq 0$.

Now, for any two non-negative real numbers *s* and *t*, we have $s^2+t^2\leq(s+t)^2$.
Putting $s=\mu_C^P(b_i)d^P(b_i)$ and $t=\mu_C^P(b_j)d^P(b_j)$, we get,

$$
\begin{aligned}
&\left[\left\{\mu_C^P(b_i)\, d^P(b_i)\right\}^2 + \left\{\mu_C^P(b_j)\, d^P(b_j)\right\}^2\right] \le \left[\left\{\mu_C^P(b_i)\, d^P(b_i)\right\} + \left\{\mu_C^P(b_j)\, d^P(b_j)\right\}\right]^2 \\
&\Rightarrow \sum_{b_ib_j\in\Lambda:1\le i\ne j\le n} \left[\left\{\mu_C^P(b_i)\, d^P(b_i)\right\}^2 + \left\{\mu_C^P(b_j)\, d^P(b_j)\right\}^2\right] \\
&\le \sum_{b_ib_j\in\Lambda:1\le i\ne j\le n} \left[\left\{\mu_C^P(b_i)\, d^P(b_i)\right\} + \left\{\mu_C^P(b_j)\, d^P(b_j)\right\}\right]^2 \\
&\Rightarrow FI_{BF}^P(\Delta) \le HZI_{BF}^P(\Delta).
\end{aligned}
$$

Similarly, $FI_{BF}^N(\Delta) \le HZI_{BF}^N(\Delta)$.

$$\therefore FI_{BF}^P(\Delta) \le HZI_{BF}^P(\Delta), \quad \text{and} \quad FI_{BF}^N(\Delta) \le HZI_{BF}^N(\Delta).$$

□

Theorem 17 *Let $\Delta = (\Gamma, C, D)$ be a BFG of the graph $\Delta^* = (\Gamma, \Lambda)$ s.t. $|\Gamma| = n$, and $|\Lambda| = m$. Then,*

$$FI_{BF}^P(\Delta) \ge \frac{2}{m}\{RRI_{BF}^P(\Delta)\}^2, \quad \text{and} \quad FI_{BF}^N(\Delta) \ge \frac{2}{m}\{RRI_{BF}^N(\Delta)\}^2.$$

Proof We have, $0 \le \mu_C^P(b_i) \le 1, 0 \le \mu_C^P(b_j) \le 1, -1 \le \mu_C^N(b_i) \le 0, -1 \le \mu_C^N(b_j) \le 0, \forall b_i, b_j \in \Gamma$, and $0 \le \mu_D^P(b_ib_j) \le 1, -1 \le \mu_D^N(b_ib_j) \le 0, \forall b_ib_j \in \Lambda$.

$$\text{So,} \quad d^P(b_i) = \sum_{b_j:b_i\ne b_j, b_ib_j\in\Lambda} \mu_D^P(b_ib_j) \ge 0, \quad \text{and} \quad d^N(b_i) = \sum_{b_j:b_i\ne b_j, b_ib_j\in\Lambda} \mu_D^N(b_ib_j) \le 0.$$

Therefore, $\mu_C^P(b_i)d^P(b_i) \ge 0$, $\mu_C^P(b_j)d^P(b_j) \ge 0$, $\mu_C^N(b_i)d^N(b_i) \ge 0$, and $\mu_C^N(b_j)d^N(b_j) \ge 0$.

Now, for any two non-negative real numbers *s* and *t*, applying A.M. ≥ G.M., we get, $\frac{s+t}{2} \ge \sqrt{st}$.
Putting $s = \mu_C^P(b_i)d^P(b_i)$ and $t = \mu_C^P(b_j)d^P(b_j)$, we get,

$$
\begin{aligned}
&\left[\left\{\mu_C^P(b_i)\, d^P(b_i)\right\} + \left\{\mu_C^P(b_j)\, d^P(b_j)\right\}\right] \ge 2\sqrt{\left\{\mu_C^P(b_i)\, d^P(b_i)\right\}\left\{\mu_C^P(b_j)\, d^P(b_j)\right\}} \\
&\Rightarrow \sum_{b_ib_j\in\Lambda:1\le i\ne j\le n} \left[\left\{\mu_C^P(b_i)\, d^P(b_i)\right\} + \left\{\mu_C^P(b_j)\, d^P(b_j)\right\}\right] \\
&\ge 2 \sum_{b_ib_j\in\Lambda:1\le i\ne j\le n} \sqrt{\left\{\mu_C^P(b_i)\, d^P(b_i)\right\}\left\{\mu_C^P(b_j)\, d^P(b_j)\right\}} \\
&\Rightarrow FZI_{BF}^P(\Delta) \ge 2RRI_{BF}^P(\Delta) \\
&\Rightarrow \sqrt{2m}\sqrt{FI_{BF}^P(\Delta)} \ge FZI_{BF}^P(\Delta) \ge 2RRI_{BF}^P(\Delta) \quad [\text{From theorem 14}] \\
&\Rightarrow \sqrt{2m}\sqrt{FI_{BF}^P(\Delta)} \ge 2RRI_{BF}^P(\Delta) \\
&\Rightarrow FI_{BF}^P(\Delta) \ge \frac{2}{m}\{RRI_{BF}^P(\Delta)\}^2.
\end{aligned}
$$

Similarly, we can prove that, $FI_{BF}^N(\Delta) \ge \frac{2}{m}\{RRI_{BF}^N(\Delta)\}^2$.

$$\therefore FI_{BF}^P(\Delta) \ge \frac{2}{m}\{RRI_{BF}^P(\Delta)\}^2, \quad \text{and} \quad FI_{BF}^N(\Delta) \ge \frac{2}{m}\{RRI_{BF}^N(\Delta)\}^2.$$

□

The relationships of the Forgotten Index with some other topological indices in a *BFG* Δ are shown in Table 3.

Applications

Here, we have discussed a couple of real-life applications of the Forgotten Index for bipolar fuzzy graphs. This section is divided into two subsections. We have applied the Forgotten Index to matrimonial websites and gene regulatory networks in the Subsections "Application of the forgotten index in matrimonial websites" and "Application of the forgotten index in gene regulatory networks", respectively.

Application of the forgotten index in matrimonial websites

Matrimonial websites offer numerous benefits for individuals seeking life partners. Firstly, they provide a vast pool of potential matches, allowing users to connect with individuals they might not have encountered otherwise. These platforms also offer advanced search filters, enabling users to specify their preferences based on factors such as religion, ethnicity, education, and profession, thereby increasing the likelihood of finding compatible partners. Additionally, matrimonial websites facilitate communication and interaction between interested parties through messaging features, virtual chats, and video calls, fostering a comfortable and convenient environment for getting to know each other. Furthermore, these platforms often employ stringent security measures to ensure the safety and privacy of users' personal information. Overall, matrimonial websites streamline the process of finding a life partner, offering a convenient and effective solution for individuals seeking meaningful relationships.

When a person opens an account on a matrimonial website, they are asked to fill out a form where all the details about that person, like country, language, religion, hobby, education, occupation, etc., are asked; apart from that, the preferred details about the potential life partner that the person is looking for are also asked. Matrimonial websites suggest profiles that match the given criteria based on the preferences given by that person about the potential life partner.

For our convenience and better understanding, we assume there are profiles of ten women on a matrimonial website. Now let us consider a man, Sumit, who is looking for a compatible partner and opens an account on that matrimonial website. Here, the matchings and mismatchings of Sumit and these ten women are represented through a bipolar fuzzy graph having ten vertices representing those women, where the positive and negative membership degrees of the vertices represent, respectively, the matching and mismatching factors of a particular woman with Sumit. One thing to keep in mind is that while determining the membership values, every factor should not be treated equally. Factors like country, religion, language, occupation, etc., should be given more weight, and factors like hobbies, favorite sports, etc., should be given less weight. Similarly, the positive and negative degrees of membership of the edges represent the common matching and mismatching factors of two adjacent vertices. Let the names of those ten women be Sujata, Afsana, Puja, Mina, Deepti, Nafisa, Ranjita, Victoria, Bithika, and Ipsita, and these women are depicted through the vertices *S*, *A*, *P*, *M*, *D*, *N*, *R*, *V*, *B* and *I*, respectively. The bipolar fuzzy graph Δ, as mentioned above, is demonstrated in Fig. 3.

Positive and negative degrees of membership in a *BFG* have the opposite sign. Thus, although the positive and negative portions of the Forgotten Index have the same sign, their effects are opposite. Therefore, we must calculate the difference among the positive and negative portions of the Forgotten Indices of the vertex deleted bipolar fuzzy subgraphs of the *BFG* Δ, as shown in Fig. 3.

The positive and negative parts of the Forgotten Indices of the vertex deleted bipolar fuzzy subgraphs of the *BFG* given in Fig. 3 are calculated and are shown in Table 4.

From Table 4, it is clear that $FI_{BF}^{P}(\Delta - V) - FI_{BF}^{N}(\Delta - V) > FI_{BF}^{P}(\Delta - N) - FI_{BF}^{N}(\Delta - N) > FI_{BF}^{P}(\Delta - A) - FI_{BF}^{N}(\Delta - A) > FI_{BF}^{P}(\Delta - D) - FI_{BF}^{N}(\Delta - D) > FI_{BF}^{P}(\Delta - S) - FI_{BF}^{N}(\Delta - S) > FI_{BF}^{P}(\Delta - B) - FI_{BF}^{N}(\Delta - B) > FI_{BF}^{P}(\Delta - R) - FI_{BF}^{N}(\Delta - R) > FI_{BF}^{P}(\Delta - M) - FI_{BF}^{N}(\Delta - M) > FI_{BF}^{P}(\Delta - P) - FI_{BF}^{N}(\Delta - P) > FI_{BF}^{P}(\Delta - I) - FI_{BF}^{N}(\Delta - I)$, and $FI_{BF}^{P}(\Delta - V) - FI_{BF}^{N}(\Delta - V)$ is the highest and $FI_{BF}^{P}(\Delta - I) - FI_{BF}^{N}(\Delta - I)$ is the lowest. So, when we consider the bipolar fuzzy graph given in Fig. 3, the order of compatibility is $I > P > M > R > B > S > D > A > N > V$, and *I*(Ipsita) and *V*(Victoria), respectively, have the highest and the least compatibility with Sumit.

Thus, using the Forgotten Index, matrimonial websites can suggest compatibility preferences to the persons seeking life partners so that they can contact each other and start a conversation.

Application of the forgotten index in gene regulatory networks

Gene Regulatory Networks (GRNs) describe the interactions between genes and their regulatory elements. These networks are crucial for understanding cellular processes and can be modeled mathematically to predict gene behavior under various conditions. These networks are usually represented through directed networks, where the direction between transcription factors and target nodes is indicated through directed edges. Some edges representing regulatory interactions can be bidirectional. Here, we assume that each gene can be in one of two states: active (on) or inactive (off); regulatory interactions can be either activating or inhibiting, and the state of a gene is influenced by the states of its regulators. In a gene regulatory network, genes (Γ) and regulatory interactions (Λ) are represented by the nodes and the edges, respectively, of the network. A gene regulatory network can be represented by a bipolar fuzzy graph. Here, for simplicity and convenience, we have considered an undirected network. Such a network is demonstrated by a *BFG* Δ_1 in Fig. 4. The membership values of the vertices and the edges of the gene regulatory network, demonstrated in Fig. 4, are provided in Table 5.

Relationships of the forgotten index with	Relationships
Sombor Index	$\frac{1}{m}\{SO_{BF}^{P}(\Delta)\}^2 \leq FI_{BF}^{P}(\Delta) \leq \{SO_{BF}^{P}(\Delta)\}^2$, and $\frac{1}{m}\{SO_{BF}^{N}(\Delta)\}^2 \leq FI_{BF}^{N}(\Delta) \leq \{SO_{BF}^{N}(\Delta)\}^2$.
First Zagreb Index	$\frac{1}{2m}\{FZI_{BF}^{P}(\Delta)\}^2 \leq FI_{BF}^{P}(\Delta) \leq \{FZI_{BF}^{P}(\Delta)\}^2$, and $\frac{1}{2m}\{FZI_{BF}^{N}(\Delta)\}^2 \leq FI_{BF}^{N}(\Delta) \leq \{FZI_{BF}^{N}(\Delta)\}^2$.
Second Zagreb Index	$FI_{BF}^{P}(\Delta) \geq 2SZI_{BF}^{P}(\Delta)$, and $FI_{BF}^{N}(\Delta) \geq 2SZI_{BF}^{N}(\Delta)$.
Hyper Zagreb Index	$FI_{BF}^{P}(\Delta) \leq HZI_{BF}^{P}(\Delta)$, and $FI_{BF}^{N}(\Delta) \leq HZI_{BF}^{N}(\Delta)$.
Reciprocal Randic Index	$FI_{BF}^{P}(\Delta) \geq \frac{2}{m}\{RRI_{BF}^{P}(\Delta)\}^2$, and $FI_{BF}^{N}(\Delta) \geq \frac{2}{m}\{RRI_{BF}^{N}(\Delta)\}^2$.

Table 3. The relationships of the forgotten index with some other topological indices.

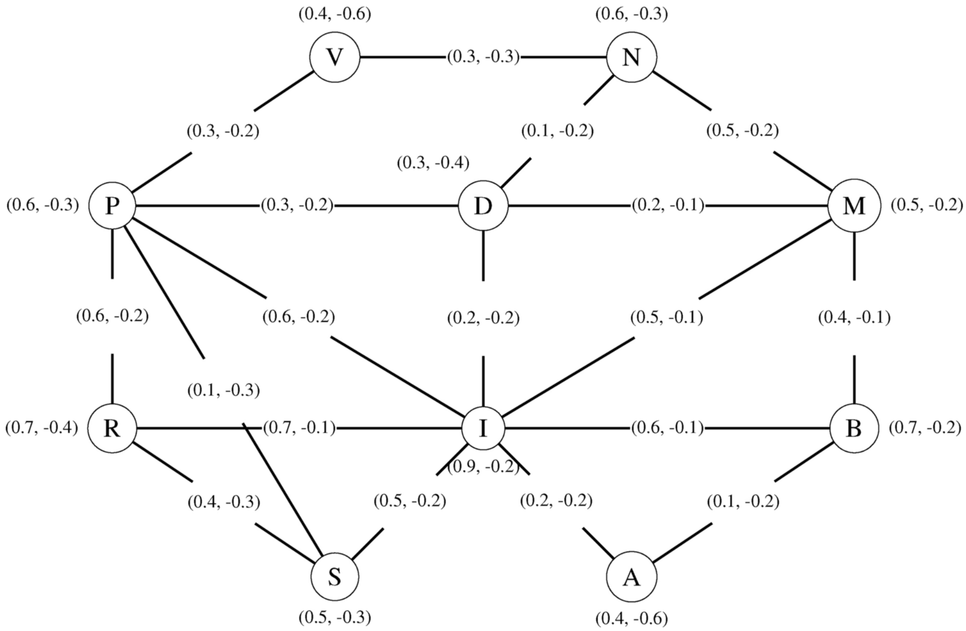

Fig. 3. A $BFG\Delta$ representing the ten women in a matrimonial website.

Bipolar fuzzy subgraphs	FI_{BF}^{P}	FI_{BF}^{N}	$FI_{BF}^{P} - FI_{BF}^{N}$
$\Delta - V$	13.20	3.21	9.99
$\Delta - N$	12.66	3.28	9.38
$\Delta - P$	10.64	2.85	7.79
$\Delta - D$	12.10	3.21	8.89
$\Delta - M$	11.49	3.62	7.87
$\Delta - R$	11.35	3.41	7.94
$\Delta - I$	06.48	2.89	3.59
$\Delta - B$	11.55	3.55	8.00
$\Delta - S$	11.83	3.55	8.28
$\Delta - A$	12.62	3.31	9.31

Table 4. Positive and negative parts of the Forgotten Indices of the vertex deleted bipolar fuzzy subgraphs and their differences.

In a gene regulatory network, the positive membership degree of a vertex indicates the extent to which a gene is actively involved in promoting a biological process, while the negative membership degree reflects the extent to which the gene inhibits or suppresses that process. The positive membership degree of an edge indicates the strength of an activating interaction between two genes, while the negative membership degree reflects the strength of an inhibitory interaction between them. The Forgotten Index for the graph Δ_1 is calculated and is given by $F_{BF}(\Delta_1) = (F_{BF}^{P}(\Delta_1), F_{BF}^{N}(\Delta_1)) = (24.9294, 00.1837)$. The Forgotten Index's positive and negative values indicate the network's positive and negative regulatory influences. In a gene regulatory network, if the value of the positive part of the Forgotten Index is greater than the negative part, it implies increased gene activity and activation of gene expression, which can lead to cell growth, response to stimuli, etc. If the negative part of the Forgotten Index is more than the positive, it implies lesser gene activity preventing growth, response to external signals, etc. If the positive and negative part values are equal, it means stable gene activity. It is crucial for normal cellular functions and responses. We have calculated the difference between the network's positive and negative values of the F-index. The higher the value, the more prominent the positive or activating regulatory influence is than the negative or inhibiting influence in the network. Clearly, $F_{BF}^{P}(\Delta_1) - F_{BF}^{N}(\Delta_1) = 24.7457$, which is significantly greater than 0. So, the overall positive or activating regulatory influences of Δ_1 are significantly more prominent than the negative or inhibiting regulatory influences.

Now, we are to find out the genes that have the most influence on characterizing the networks's higher positive or activating nature. We have calculated the Forgotten Index for all the vertex-deleted subgraphs of Δ_1 , and computed the differences between the positive and negative parts of the Forgotten Indices. The values are provided in Table 6.

From Table 6, it is clear that $FI^{P}_{BF}(\Delta_1-\Gamma_{20})-FI^{N}_{BF}(\Delta_1-\Gamma_{20}) > FI^{P}_{BF}(\Delta_1-\Gamma_{19})-FI^{N}_{BF}(\Delta_1-\Gamma_{19}) > FI^{P}_{BF}(\Delta_1-\Gamma_{5})-FI^{N}_{BF}(\Delta_1-\Gamma_{5}) > FI^{P}_{BF}(\Delta_1-\Gamma_{14})-FI^{N}_{BF}(\Delta_1-\Gamma_{14}) > FI^{P}_{BF}(\Delta_1-\Gamma_{4})-FI^{N}_{BF}(\Delta_1-\Gamma_{4}) > FI^{P}_{BF}(\Delta_1-\Gamma_{7})-FI^{N}_{BF}(\Delta_1-\Gamma_{7}) > FI^{P}_{BF}(\Delta_1-\Gamma_{2})-FI^{N}_{BF}(\Delta_1-\Gamma_{2}) > FI^{P}_{BF}(\Delta_1-\Gamma_{1})-FI^{N}_{BF}(\Delta_1-\Gamma_{1}) > FI^{P}_{BF}(\Delta_1-\Gamma_{3})-FI^{N}_{BF}(\Delta_1-\Gamma_{3}) > FI^{P}_{BF}(\Delta_1-\Gamma_{10})-FI^{N}_{BF}(\Delta_1-\Gamma_{10}) > FI^{P}_{BF}(\Delta_1-\Gamma_{8})-FI^{N}_{BF}(\Delta_1-\Gamma_{8}) > FI^{P}_{BF}(\Delta_1-\Gamma_{18})-FI^{N}_{BF}(\Delta_1-\Gamma_{18}) > FI^{P}_{BF}(\Delta_1-\Gamma_{15})-FI^{N}_{BF}(\Delta_1-\Gamma_{15}) > FI^{P}_{BF}(\Delta_1-\Gamma_{9})-FI^{N}_{BF}(\Delta_1-\Gamma_{9}) > FI^{P}_{BF}(\Delta_1-\Gamma_{13})-FI^{N}_{BF}(\Delta_1-\Gamma_{13}) > FI^{P}_{BF}(\Delta_1-\Gamma_{11})-FI^{N}_{BF}(\Delta_1-\Gamma_{11}) > FI^{P}_{BF}(\Delta_1-\Gamma_{12})-FI^{N}_{BF}(\Delta_1-\Gamma_{12}) > FI^{P}_{BF}(\Delta_1-\Gamma_{6})-FI^{N}_{BF}(\Delta_1-\Gamma_{6}) > FI^{P}_{BF}(\Delta_1-\Gamma_{17})-FI^{N}_{BF}(\Delta_1-\Gamma_{17}) > FI^{P}_{BF}(\Delta_1-\Gamma_{16})-FI^{N}_{BF}(\Delta_1-\Gamma_{16})$, and $FI^{P}_{BF}(\Delta_1-\Gamma_{20})-FI^{N}_{BF}(\Delta_1-\Gamma_{20})$ is the highest and $FI^{P}_{BF}(\Delta_1-\Gamma_{16})-FI^{N}_{BF}(\Delta_1-\Gamma_{16})$ is the lowest. So, when we consider the bipolar fuzzy graph given in Fig. 4, the order of influence of genes is $\Gamma_{16} > \Gamma_{17} > \Gamma_6 > \Gamma_{12} > \Gamma_{11} > \Gamma_{13} > \Gamma_9 > \Gamma_{15} > \Gamma_{18} > \Gamma_8 > \Gamma_{10} > \Gamma_3 > \Gamma_1 > \Gamma_2 > \Gamma_7 > \Gamma_4 > \Gamma_{14} > \Gamma_5 > \Gamma_{19} > \Gamma_{20}$, and Γ_{16} and Γ_{20} have the highest and the least influence, respectively, in characterizing the positive or activating nature of the gene regulatory network Δ_1, given in Fig. 4.

Thus, using the Forgotten Index, we can find out the activating or inhabiting nature of a gene regulatory network and compare the overall influence of a gene in characterizing the nature of the network.

Conclusion and future works

In this research article, we have developed the Forgotten Index for bipolar fuzzy graphs. We have proved several theorems on the F-index in bipolar fuzzy graphs. Then, we discussed the application of the Forgotten Index to matrimonial websites and gene regulatory networks. If the nature of a network's vertices and edges is not self-contradictory, then bipolar fuzzy graphs cannot represent those networks. Data collection in a bipolar fuzzy environment is challenging. As we have already seen that topological indices have a vast real-life application area, we would like to apply the Forgotten Index to stock markets to find more secure stocks or mutual funds and different lottery platforms to assess the most trustworthy lottery companies. We would also like to introduce the Forgotten Index for m-polar fuzzy graphs and discuss its properties. In the future, we would also like to use other topological indices introduced in this article for applications in real-life scenarios.

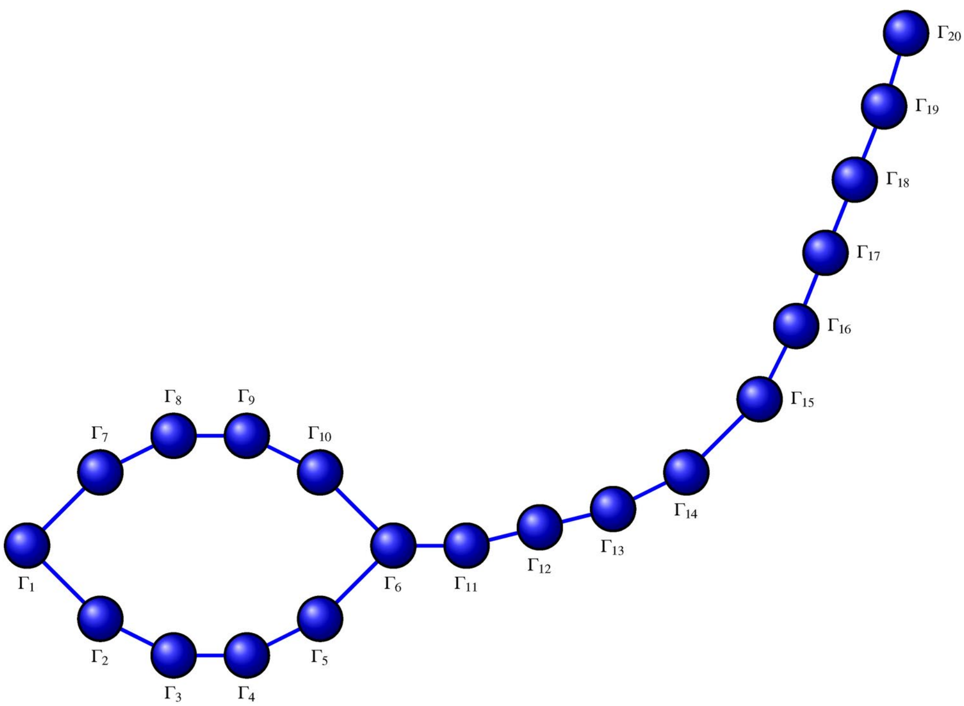

Fig. 4. A gene regulatory network consisting of twenty genes.

Gene regulatory network Δ_1			
Vertex	**Vertex membership values**	**Edge**	**Edge membership values**
Γ_1	$(0.8, -0.1)$	(Γ_1, Γ_2)	$(0.5, -0.1)$
Γ_2	$(0.6, -0.2)$	(Γ_2, Γ_3)	$(0.6, -0.1)$
Γ_3	$(0.7, -0.3)$	(Γ_3, Γ_4)	$(0.5, -0.1)$
Γ_4	$(0.9, -0.1)$	(Γ_4, Γ_5)	$(0.4, -0.1)$
Γ_5	$(0.5, -0.4)$	(Γ_5, Γ_6)	$(0.4, -0.2)$
Γ_6	$(0.6, -0.3)$	(Γ_1, Γ_7)	$(0.5, -0.1)$
Γ_7	$(0.7, -0.2)$	(Γ_7, Γ_8)	$(0.4, -0.1)$
Γ_8	$(0.8, -0.1)$	(Γ_8, Γ_9)	$(0.6, -0.1)$
Γ_9	$(0.8, -0.2)$	(Γ_9, Γ_{10})	$(0.5, -0.2)$
Γ_{10}	$(0.6, -0.3)$	(Γ_{10}, Γ_6)	$(0.6, -0.1)$
Γ_{11}	$(0.7, -0.2)$	(Γ_6, Γ_{11})	$(0.5, -0.1)$
Γ_{12}	$(0.8, -0.1)$	$(\Gamma_{11}, \Gamma_{12})$	$(0.6, -0.1)$
Γ_{13}	$(0.7, -0.2)$	$(\Gamma_{12}, \Gamma_{13})$	$(0.7, -0.1)$
Γ_{14}	$(0.6, -0.3)$	$(\Gamma_{13}, \Gamma_{14})$	$(0.5, -0.2)$
Γ_{15}	$(0.7, -0.2)$	$(\Gamma_{14}, \Gamma_{15})$	$(0.4, -0.2)$
Γ_{16}	$(0.8, -0.1)$	$(\Gamma_{15}, \Gamma_{16})$	$(0.7, -0.1)$
Γ_{17}	$(0.9, -0.1)$	$(\Gamma_{16}, \Gamma_{17})$	$(0.6, -0.1)$
Γ_{18}	$(0.6, -0.3)$	$(\Gamma_{17}, \Gamma_{18})$	$(0.6, -0.1)$
Γ_{19}	$(0.7, -0.2)$	$(\Gamma_{18}, \Gamma_{19})$	$(0.5, -0.2)$
Γ_{20}	$(0.8, -0.1)$	$(\Gamma_{19}, \Gamma_{20})$	$(0.5, -0.1)$

Table 5. Membership values of the vertices and the edges of the graph Δ_1.

Bipolar fuzzy subgraphs	FI_{BF}^P	FI_{BF}^N	$FI_{BF}^P - FI_{BF}^N$
$\Delta_1 - \Gamma_1$	22.8169	00.1797	22.6372
$\Delta_1 - \Gamma_2$	22.8253	00.1765	22.6488
$\Delta_1 - \Gamma_3$	22.6519	00.1745	22.4774
$\Delta_1 - \Gamma_4$	22.8643	00.1649	22.6994
$\Delta_1 - \Gamma_5$	23.1433	00.1401	23.0032
$\Delta_1 - \Gamma_6$	21.3109	00.1164	21.1945
$\Delta_1 - \Gamma_7$	22.8556	00.1797	22.6759
$\Delta_1 - \Gamma_8$	22.4781	00.1777	22.3004
$\Delta_1 - \Gamma_9$	22.3050	00.1680	22.1370
$\Delta_1 - \Gamma_{10}$	22.4738	00.1495	22.3243
$\Delta_1 - \Gamma_{11}$	21.8520	00.1657	21.6863
$\Delta_1 - \Gamma_{12}$	21.4677	00.1777	21.2900
$\Delta_1 - \Gamma_{13}$	22.1450	00.1617	21.9833
$\Delta_1 - \Gamma_{14}$	23.0477	00.1477	22.9000
$\Delta_1 - \Gamma_{15}$	22.3704	00.1617	22.2087
$\Delta_1 - \Gamma_{16}$	21.0069	00.1789	20.8280
$\Delta_1 - \Gamma_{17}$	21.0794	00.1744	20.9050
$\Delta_1 - \Gamma_{18}$	22.4018	00.1635	22.2383
$\Delta_1 - \Gamma_{19}$	23.3538	00.1683	23.1855
$\Delta_1 - \Gamma_{20}$	24.2794	00.1800	24.0994

Table 6. Positive and negative parts of the Forgotten Indices of the vertex deleted bipolar fuzzy subgraphs of the graph Δ_1 and their differences.

Data availability
All data generated or analysed during this study are included in this published article.

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Author contributions
Review of literature has been done by S.I.A., K.D., A.K. and T.A. Study conception and design was performed by S.I.A., S.S. and J.G.L. S.I.A., S.S. and K.D. has done mathematical calculations and proofs. Tables and Figures have been developed by S.I.A., A.K., J.G.L. and T.A. Interpretation of results and area of applications have been demonstrated by all the authors. J.G.L. funded this research work. Draft version of the manuscript has been written by S.I.A. Revised and final version of the manuscript has been prepared by S.I.A. S.S. and K.D. supervised the work. All authors reviewed the results and approved the final version of the manuscript.

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Competing interests
The authors declare no competing interests.

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