CLIFFORD ALGEBRAS AND LITTLEWOOD-RICHARDSON COEFFICIENTS

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ABSTRACT. We show how to use Clifford algebra techniques to describe the de Rham cohomology ring of equal rank compact symmetric spaces G/K . In particular, for $G/K = U(n)/U(k) \times U(n-k)$, we obtain a new way of multiplying Schur polynomials, i.e., computing the Littlewood-Richardson coefficients. The corresponding multiplication on the Clifford algebra side is, in a convenient basis given by projections of the spin module, simply the componentwise multiplication of vectors in \mathbb{C}^N , also known as the Hadamard product.

1. INTRODUCTION

In this paper we consider the classical compact symmetric space $G/K =$ $U(n)/U(k) \times U(n-k)$. It is well known that this space is diffeomorphic to the complex Grassmannian $\mathrm{Gr}(k,\mathbb{C}^n)$ of k-dimensional subspaces of \mathbb{C}^n . The de Rham cohomology $H(G/K)$ (with complex coefficients) is very well understood through Schubert calculus. One of the well known results about $H(G/K)$ (due to E.Cartan and de Rham) is that it is isomorphic to $({\bf \Lambda}\mathfrak{p}^*)^{\mathfrak{k}},$ where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the decomposition of the complexified Lie algebra of G into eigenspaces of the (differentiated) involution defining K . By classical results of Borel, H. Cartan and others,

$$
\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}/I_+ \cong (\bigwedge \mathfrak{p}^*)^{\mathfrak{k}}.\tag{1.1}
$$

Here t is a Cartan subalgebra of $\mathfrak k$ and $\mathfrak g$, $W_{\mathfrak k}$ is the Weyl group of $(\mathfrak k, \mathfrak t)$, and I_+ is the ideal in $\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}$ generated by the polynomials without constant term invariant under the Weyl group $W_{\mathfrak{g}}$ of $(\mathfrak{g}, \mathfrak{t})$.

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Furthermore, there is a basis for $({\wedge \mathfrak{p}}^*)^{\mathfrak{k}}$ given by the images under the isomorphism [\(1.1\)](#page-0-0) of Schur polynomials $s_{\lambda} \in \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}$ associated to partitions λ with Young diagrams inside the $k \times (n - k)$ box. For a definition of s_{λ} see e.g. [\[Mac,](#page-9-0) Chapter 1].

The multiplication of Schur polynomials is described by the Littlewoood-Richardson rule [\[LR\]](#page-9-1):

$$
s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu,
$$

where $c_{\lambda\mu}^{\nu}$ are the Littlewood-Richardson (LR) coefficients. The LR coefficients also appear in other places. For example, let V_{λ} , V_{μ} and V_{ν} be finite-dimensional representations of $GL_m(\mathbb{C})$ with highest weights given by partitions λ, μ, ν . Then one has

$$
\dim \mathrm{Hom}_{GL_m(\mathbb{C})}(V_{\nu}, V_{\lambda} \otimes V_{\mu}) = c_{\lambda\mu}^{\nu}.
$$

The LR coefficients also appear in branching laws as well as in Schubert calculus.

The multiplication of the basis of Schur polynomials in $H(\mathrm{Gr}(k, \mathbb{C}^n))$ is again given by the Littlewood-Richardson rule, but we replace $c^{\nu}_{\lambda\mu}$ by zero if ν does not fit in the $k \times (n - k)$ box; for example, see [\[Les,](#page-9-2) [Bor,](#page-9-3) [BGG\]](#page-9-4) and [\[Sta,](#page-9-5) Theorem 3.1].

In the following, we denote the subspace of $\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}$ spanned by the above Schur polynomials by $\mathcal{P}_{k\times(n-k)}$.

In [\[CNP\]](#page-9-6) and [\[CGKP\]](#page-9-7) we study a filtered deformation of $({\wedge \mathfrak{p}}^*)^{\mathfrak{k}} \cong ({\wedge \mathfrak{p}})^{\mathfrak{k}},$ given as the \mathfrak{k} -invariants $Cl(\mathfrak{p})^{\mathfrak{k}}$ in the Clifford algebra $Cl(\mathfrak{p})$ with respect to the trace form B. Recall that $Cl(p)$ is defined as the quotient of the tensor algebra $T(\mathfrak{p})$ by the ideal generated by

$$
X \otimes Y + Y \otimes X - 2B(X, Y), \qquad X, Y \in \mathfrak{p}.
$$

The grading by degree of $T(\mathfrak{p})$ induces a filtration of $Cl(\mathfrak{p})$, and the associated graded algebra of the filtered algebra $Cl(p)$ is Λp . Passing to ℓ invariants, we conclude that $Cl(p)^{\ell}$ is a filtered algebra with the associated graded algebra equal to $(\bigwedge \mathfrak{p})^{\mathfrak{k}}$.

The main advantage of $Cl(p)^{\ell}$ over $({\wedge p})^{\ell}$ is that the algebra structure is much simpler. Namely, $Cl(\mathfrak{p})$ is the endomorphism algebra of the spin module S, which becomes a \mathfrak{k} -module through the map $\alpha: U(\mathfrak{k}) \to \mathrm{Cl}(\mathfrak{p})$ defined on $X \in \mathfrak{k}$ by

$$
\alpha(X) = \frac{1}{4} \sum_{i} [X, b_i] d_i,
$$
\n(1.2)

where b_i is any basis of $\mathfrak p$ and d_i is the dual basis with respect to B. One knows that the \mathfrak{k} -module S is multiplicity free, with $|W_{\mathfrak{g}}|/|W_{\mathfrak{k}}| = \binom{n}{k}$ $\binom{n}{k}$ components. In fact, these components have highest weights $\sigma \rho - \rho_{\text{t}}$, where ρ and ρ_{t} are the half sums of compatible positive root systems for $(\mathfrak{g}, \mathfrak{t})$ respectively $(\mathfrak{k}, \mathfrak{t})$, and σ runs over the set

$$
W^{1} = \{ \sigma \in W_{\mathfrak{g}} \, \big| \, \sigma \rho \text{ is } \mathfrak{k}\text{-dominant} \}. \tag{1.3}
$$

It now follows from Schur's lemma that $\text{Cl}(\mathfrak{p})^{\mathfrak{k}} = \text{End}_{\mathfrak{k}} S$ is spanned by the orthogonal projections to these components. In this way the algebra $Cl(p)^{\ell}$ becomes isomorphic to $\mathbb{C}^{n \choose k}$ with componentwise multiplication, also called the Hadamard product.

The following filtered analogue of the Cartan–Borel theorem is proved in [\[CNP,](#page-9-6) Corollary 3.15]; we give a slightly more direct argument in Sec-tion [2.](#page-3-0) A version of this theorem when $\mathfrak g$ and $\mathfrak k$ are not of equal rank will appear in [\[CGKP\]](#page-9-7).

Theorem 1.4. The algebra homomorphism α of [\(1.2\)](#page-1-0) restricted to $U(\mathfrak{k})^{\mathfrak{k}} \cong$ $\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}$ induces a filtered algebra isomorphism

$$
\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}/I_{\rho}\cong \mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}},
$$

where I_{ρ} is the ideal in $\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}$ generated by the $W_{\mathfrak{g}}$ -invariant polynomials vanishing at ρ.

The Cartan–Borel theorem [\(1.1\)](#page-0-0) can be deduced from Theorem [1.4](#page-2-0) by passing to associated graded algebras. In particular, the isomorphism [\(1.1\)](#page-0-0) is induced by $gr \alpha$.

Through the above identifications, (images of) Schur polynomials define a basis for $\text{Cl}(\mathfrak{p})^{\mathfrak{k}}$, and since the multiplication in $\text{Cl}(\mathfrak{p})^{\mathfrak{k}}$ is equal to the multiplication in $(\bigwedge \mathfrak{p}^*)^{\mathfrak{k}}$ modulo lower order terms, we can recover the Littlewood-Richardson coefficients from the multiplication in $Cl(p)^{\ell}$, which is simply the Hadamard product in the basis of projections. Following Kostant, we view $\Lambda \mathfrak{p}$ and $Cl(\mathfrak{p})$ as the same space with two multiplications, Λ respectively \bullet . In particular, this space is graded, and we denote by $a_{[d]}$ the dth graded component of a in this space. (We point out that while \land is compatible with this grading, • is not.)

The main result of this paper is

Theorem 1.5. Given Schur polynomials s_{λ} and s_{μ} of degree $l(\lambda)$ respectively $l(\mu)$, we can calculate the wedge product of their images in $H^*(\mathop{\rm Gr}(k, \mathbb C^n)) \cong$ $\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}/I_+$ as

$$
\operatorname{gr} \alpha(s_{\lambda}) \wedge \operatorname{gr} \alpha(s_{\mu}) = (\alpha(s_{\lambda}) \bullet_H \alpha(s_{\mu}))_{[2l(\lambda)+2l(\mu)]},
$$

This describes a novel way to calculate the Littlewood-Richardson coefficients using the Hadamard product (componentwise multiplication) \bullet_H , a linear isomorphism from the space $\mathcal{P}_{k\times(n-k)}$ of Schur polynomials to $\text{Cl}(\mathfrak{p})^{\mathfrak{k}}$ induced by α , and the inverse of this isomorphism. Let M be the matrix sending the basis $s_{\lambda} \in \mathcal{P}_{n \times (n-k)}$ to $\alpha(s_{\lambda}) \in \text{Span}\{ \text{pr}_{\sigma}\},$ and $P_{[d]}$ be the projection to degree d polynomials. Then the multiplication of Schur polynomials s_{λ} and s_{μ} in the cohomology ring $H(\mathrm{Gr}(k, \mathbb{C}^n)) \cong \bigwedge(\mathfrak{p})^{\mathfrak{k}}$ is

$$
s_{\lambda} \cdot s_{\mu} = P_{[l(\lambda)+l(\mu)]} M^{-1}(M(s_{\lambda}) \bullet_H M(s_{\mu})).
$$

The above is illustrated by Algorithm [1](#page-3-1) and examples in Section [3.](#page-6-0) Furthermore, the linear isomorphism corresponding to M is easily described as evaluation of Schur polynomials at points $\sigma \rho$ for $\sigma \in W^1$, see [\(2.1\)](#page-4-0).

To obtain the full list of LR coefficients for a given λ and μ , one can consider the case of $n = 2(l(\lambda) + l(\mu))$, $k = l(\lambda) + l(\mu)$. But in order to compute $c^{\nu}_{\lambda\mu}$ for given λ , μ , and ν it is enough to consider, for example, the minimal box $\mathcal{P}_{k\times(n-k)}$ such that $\lambda, \mu, \nu \in \mathcal{P}_{k\times(n-k)}$.

Require: $\lambda, \mu \in \mathcal{P}_{k \times (n-k)}$ **Ensure:** the list of $c_{\lambda\mu}^{\nu}$ for $\nu \in \mathcal{P}_{k \times (n-k)}$ $Poly = \{s_{\nu} : \nu \in \mathcal{P}_{k \times (n-k)}\}, N = {n \choose k}$ $_{k}^{n}$, Pts = {}. $E_{\lambda} \leftarrow$ the coordinate vector of s_{λ} in the basis Poly $E_{\mu} \leftarrow$ the coordinate vector of s_{μ} in the basis Poly $\rho \leftarrow {\frac{n+1}{2}, \frac{n+1}{2} - 1, \cdots, \frac{n+1}{2} - (n-1), \frac{n+1}{2} - (n-1)}$ \triangleright Calculate ρ for σ a k shuffle of n do Pts \leftarrow Pts $\cup \sigma(\rho)$ \triangleright Calculate the points $\{\sigma\rho : \sigma \in W^1\}$ end for for $i = 1, \dots, N$ do for $j = 1, \dots, N$ do $M[i, j] \leftarrow \text{Poly}[i](\text{Pts}[j]) \Rightarrow M \text{ is the basis change } \{s_{\lambda}\}\text{ to }\{\text{pr}_{\sigma}\}\$ end for end for $X_{\lambda} \leftarrow M \cdot E_{\lambda}, X_{\mu} \leftarrow M \cdot E_{\mu} \triangleright$ Coordinates of s_{λ}, s_{μ} in the basis of $\{pr_{\sigma}\}\$ for $i = 1, \dots, N$ do $W[i] \leftarrow X_{\lambda}[i] \times X_{\mu}[i]$ \triangleright Hadamard product end for $Z \leftarrow M^{-1} \cdot W$ \triangleright Back to the basis of $\{s_{\lambda}\}\$ for $i = 1, \dots, N$ do if deg Poly $[i] \neq \deg s_\lambda + \deg s_\mu$ then $Z[i] \leftarrow 0$ \triangleright Removing lower order terms end if end for **return** Z \triangleright The list of $c^{\nu}_{\lambda\mu}$

In Section [2](#page-3-0) we discuss the ℓ -decomposition of the spin module S and prove Theorems [1.4](#page-2-0) and [1.5.](#page-2-1) In Section [3](#page-6-0) we compute a couple of examples, calculated using [\[Wol,](#page-9-8) Mathematica].

2. DESCRIPTION OF $Cl(p)^{\ell}$ for equal rank symmetric pairs

In this section G and K are as in the introduction, but (G, K) could equally well be any equal rank compact symmetric pair with G and K connected (for disconnected K , see [\[CNP,](#page-9-6) Section 3]). Since G and K are connected it is enough to work with Lie algebras $\mathfrak g$ and $\mathfrak k$. In this more general situation, the form B is replaced by any nondegenerate invariant symmetric bilinear form on \mathfrak{g} , and \mathfrak{p} is still the -1 eigenspace of the involution defining $\mathfrak k$ (or equivalently, the orthogonal of $\mathfrak k$ with respect to B).

Let us first review some facts about the spin module S for $Cl(\mathfrak{p})$. Since $\dim \mathfrak{p}$ is even, the Clifford algebra $Cl(\mathfrak{p})$ has only one simple module S, and moreover $\text{Cl}(\mathfrak{p}) \cong \text{End }S$. To construct S, write

$$
\mathfrak{p}=\mathfrak{p}^+\oplus \mathfrak{p}^-,
$$

where \mathfrak{p}^+ and \mathfrak{p}^- are maximal isotropic subspaces of \mathfrak{p} in duality under B. Then one can take $S = \Lambda p^+$, with elements of p^+ acting by wedging, and elements of \mathfrak{p}^- by contracting. As mentioned in the introduction, S becomes a ℓ -module through the map α of [\(1.2\)](#page-1-0). Furthermore, the ℓ -module S is multiplicity free and decomposes as

$$
S = \bigoplus_{\sigma \in W^1} E_{\sigma \rho - \rho_{\mathfrak{k}}}
$$

,

where ρ , ρ_{ℓ} and W^1 are as in the introduction, see [\(1.3\)](#page-1-1), and $E_{\sigma \rho - \rho_{\ell}}$ denotes the irreducible finite-dimensional \mathfrak{k} -module with highest weight $\sigma \rho - \rho_{\mathfrak{k}}$.

Now $Cl(p) \cong End S$ implies

$$
\operatorname{Cl}(\mathfrak{p})^{\mathfrak{k}} = \operatorname{End}_{\mathfrak{k}} S,
$$

and by Schur's Lemma this is the algebra of projections $pr_{\sigma}: S \to E_{\sigma \rho - \rho_t}$, which we denote by $Pr(S)$. This is a very simple commutative algebra, isomorphic to $\mathbb{C}^{|W^1|}$ with coordinatewise multiplication (Hadamard product), by identifying pr_{σ} with $(0, \ldots, 0, 1, 0, \ldots, 0)$, with 1 in the place corresponding to σ .

The restriction of α to the center $U(\mathfrak{k})^{\mathfrak{k}}$ of $U(\mathfrak{k})$ has a very simple description in terms of the Harish-Chandra isomorphism $U(\mathfrak{k})^{\tilde{\mathfrak{k}}}\cong \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}$. Namely, under the Harish-Chandra isomorphism, the infinitesimal character of $E_{\sigma\rho-\rho_{\mathfrak{k}}}$ corresponds to evaluation on $\sigma\rho$. Therefore, the restriction of α to $U(\mathfrak{k})^{\mathfrak{k}}$ corresponds to the algebra morphism, $ev_{\rho}: \mathbb{C}[\mathfrak{k}^*]^{W_{\mathfrak{k}}} \to Cl(\mathfrak{p})^{\mathfrak{k}} = \Pr(S)$ given by

$$
\operatorname{ev}_{\rho}(p) = \sum_{\sigma \in W^1} p(\sigma \rho) \operatorname{pr}_{\sigma}.
$$
 (2.1)

Lemma 2.2. The map ev_{ρ} is surjective.

Proof. We need to prove that for any choice of scalars a_{σ} , $\sigma \in W^1$, there is $P \in \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}$ such that $P(\sigma \rho) = a_{\sigma}$ for all $\sigma \in W^1$. The polynomial P is similar to the Lagrange interpolation polynomial, but we have to ensure it is W_{ℓ} -invariant.

It is enough to find the polynomials $P_{\sigma} \in \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}, \sigma \in W^1$, such that $P_{\sigma}(\tau \rho) = \delta_{\sigma \tau}$ for any $\tau \in W^1$.

We define

$$
Q_{\sigma}(\gamma) = \prod_{\substack{w \in W_{\mathfrak{k}} \\ w \neq id \\ \tau \in W^1}} \frac{||\gamma - w\tau\rho||^2}{||\sigma\rho - w\tau\rho||^2} \prod_{\tau \neq \sigma} \frac{||\gamma - \tau\rho||^2}{||\sigma\rho - \tau\rho||^2}, \qquad \gamma \in \mathfrak{t}^*.
$$

Then

$$
Q_{\sigma}(\sigma \rho) = 1,
$$

\n
$$
Q_{\sigma}(w\tau \rho) = 0 \quad \text{ if } w \in W_{\mathfrak{k}} \setminus {\text{id}} \text{ or } \sigma \neq \tau.
$$

So $Q_{\sigma}(\tau \rho) = \delta_{\sigma \tau}$ as required, but the Q_{σ} are not necessarily $W_{\mathfrak{k}}$ -invariant. So we set

$$
P_{\sigma} = \frac{1}{|W_{\mathfrak{k}}|} \sum_{w \in W_{\mathfrak{k}}} w Q_{\sigma}
$$

and obtain the polynomials we were looking for. \Box

The kernel of ev_ρ clearly contains the ideal I_p generated by $\{p \in \mathbb{C}[{\mathfrak{t}}^*]^{W_{\mathfrak{g}}}$: $p(\rho) = 0$. To see that in fact ker ev_{$\rho = I_\rho$}, we prove

Lemma 2.3. The ideal I_{ρ} is of codimension $|W^1|$ in $\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}$.

Proof. Let J be the ideal of $\mathbb{C}[\mathfrak{t}^*]$ generated by $\{p \in \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{g}}}: p(0) = 0\},\$ then $\mathbb{C}[\mathfrak{t}^*]/J$ is the coinvariant algebra associated to $W_{\mathfrak{g}}$ and is isomorphic as a $W_{\mathfrak{a}}$ -module to $\mathbb{C}[W_{\mathfrak{a}}]$, see [\[Hum,](#page-9-9) Section 3.6], hence has dimension $|W_{\mathfrak{a}}|$. Therefore J has codimension $|W_{\mathfrak{g}}|$. The associated graded ideal of I_{ρ} is I_{+} . Applying $W_{\mathfrak{k}}$ invariance to the exact sequence of $W_{\mathfrak{g}}$ -modules

$$
0 \to J \to \mathbb{C}[\mathfrak{t}^*] \to \mathbb{C}[\mathfrak{t}^*]/J \cong \mathbb{C}[W_{\mathfrak{g}}] \to 0
$$

we obtain the exact sequence

$$
0 \to J^{W_{\mathfrak{k}}} \cong I_+ \to \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}} \to (\mathbb{C}[\mathfrak{t}^*]/J)^{W_{\mathfrak{k}}} \cong \mathbb{C}[W_{\mathfrak{g}}]^{W_{\mathfrak{k}}} \to 0.
$$

We can conclude that $I_+ = J^{W_{\mathfrak{k}}}$ is of codimension $\dim \mathbb{C}[W_{\mathfrak{g}}]^{W_{\mathfrak{k}}} = |W^1|$ and hence, since the functor gr preserves codimension, so is I_{ρ} . \Box

Proof of Theorem [1.4.](#page-2-0) We have seen that the map $\alpha: U(\mathfrak{k})^{\mathfrak{k}} \to \mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}$ becomes ev_{ρ} under identifications $U(\mathfrak{k})^{\mathfrak{k}} = \mathbb{C}[\mathfrak{k}^*]^{W_{\mathfrak{k}}}$ and $Cl(\mathfrak{p})^{\mathfrak{k}} = Pr(S)$. By Lemma [2.2,](#page-4-1) ev_{ρ} is onto, and by Lemma [2.3,](#page-5-0) its kernel is exactly I_{ρ} . \square

The above discussion can be summarized by the following commutative diagram, with bottom row exact:

$$
U(\mathfrak{k})^{\mathfrak{k}} \xrightarrow{\alpha} \mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}
$$

$$
\downarrow \text{hc} \qquad \downarrow \cong
$$

$$
0 \longrightarrow I_{\rho} \longrightarrow \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}} \xrightarrow{\text{ev}_{\rho}} \mathrm{Pr}(S) \longrightarrow 0
$$

Remark 2.4. The above proof of Theorem [1.4](#page-2-0) works equally well for an unequal rank symmetric pair $(\mathfrak{g},\mathfrak{k})$. In this case, the statement is that α (or ev_{ρ}) maps $U(\mathfrak{k})^{\mathfrak{k}} \cong \mathbb{C}[\mathfrak{k}^*]^{W_{\mathfrak{k}}}$ onto $Pr(S)$, which is however no longer all of $\text{Cl}(\mathfrak{p})^{\mathfrak{k}}$, and that the kernel of α is generated by the polynomials invariant under the Weyl group of the pair (g, t) . Here t is a Cartan subalgebra of \mathfrak{k} , which is no longer a Cartan subalgebra of \mathfrak{g} .

In the more general case when \mathfrak{k} is a reductive quadratic subalgebra of \mathfrak{g} which is not necessarily symmetric, the above proof still shows that α maps $U(\mathfrak{k})^{\mathfrak{k}}$ onto $Pr(S)$, but it is more difficult to describe its kernel.

Proof of Theorem [1.5.](#page-2-1) Recall that we view $Cl(\mathfrak{p})$ and $\bigwedge \mathfrak{p}$ as the same space with two multiplications, • and \wedge . Note that for elements a and b of this space such that a (resp. b) is in the kth (resp. lth) filtered piece but not the $(k-1)$ st (resp. $(l-1)$ st) filtered piece, we have

$$
a_{[k]} \wedge b_{[l]} = (a \bullet b)_{[k+l]}.
$$

Also, gr $\alpha(s_\lambda) = \alpha(s_\lambda)_{[2l(\lambda)]}$, hence

$$
\operatorname{gr} \alpha(s_{\lambda}) \wedge \operatorname{gr} \alpha(s_{\mu}) = \alpha(s_{\lambda})_{[2l(\lambda)]} \wedge \alpha(s_{\mu})_{[2l(\mu)]} = (\alpha(s_{\lambda}) \bullet \alpha(s_{\mu}))_{[2l(\lambda) + 2l(\mu)]}.
$$

3. Examples

We highlight the application of Theorem [1.5](#page-2-1) to the calculation of Littlewood-Richardson coefficients below with a couple of examples.

Example 3.1. For $\mathfrak{g} = \mathfrak{gl}_4$ and $\mathfrak{k} = \mathfrak{gl}_2 \oplus \mathfrak{gl}_2$, the set $\sigma \rho$ for $\sigma \in W^1$ is

$$
\begin{aligned}\n\left(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right), & \left(\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}\right), & \left(\frac{3}{2}, -\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}\right), \\
\left(\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}\right), & \left(\frac{1}{2}, -\frac{3}{2}, \frac{3}{2}, -\frac{1}{2}\right), & \left(-\frac{1}{2}, -\frac{3}{2}, \frac{3}{2}, \frac{1}{2}\right)\n\end{aligned}
$$

Let $\{\text{pr}_i : i = 1, \ldots, 6\} = \{\text{pr}_\sigma : \sigma \in W^1\}$ be the basis of orthogonal projections in $Pr(S)$. By (2.1) ,

$$
\alpha(s_{\lambda}) = \operatorname{ev}_{\rho}(s_{\lambda}) = \sum_{\sigma \in W^{1}} s_{\lambda}(\sigma \rho) \operatorname{pr}_{\sigma}, \quad s_{\lambda} \in \mathcal{P}_{k \times (n-k)}.
$$

For simplicity of notation, for the remainder of these examples we will replace $\alpha(s_\lambda)$ by s_λ . Expressing s_λ as a vector in $\mathbb{C}^6 \cong \Pr(S)$

$$
s_{(0,0)} = (1,1,1,1,1,1), \t s_{(1,0)} = (2,1,0,0,-1,-2),
$$

\n
$$
s_{(2,0)} = (\frac{13}{4}, \frac{7}{4}, \frac{9}{4}, \frac{1}{4}, \frac{7}{4}, \frac{13}{4}), \t s_{(1,1)} = (\frac{3}{4} - \frac{3}{4}, -\frac{9}{4}, -\frac{1}{4}, -\frac{3}{4}, \frac{3}{4}),
$$

\n
$$
s_{(2,1)} = (\frac{3}{2}, -\frac{3}{4}, 0, 0, \frac{3}{4}, -\frac{3}{2}), \t s_{(2,2)} = (\frac{9}{16}, \frac{9}{16}, \frac{81}{16}, \frac{1}{16}, \frac{9}{16}, \frac{9}{16}).
$$

The change of basis from s_{λ} to pr_{i} and its inverse are given by the matrices

$$
M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 & -1 & -2 \\ \frac{13}{4} & \frac{7}{4} & \frac{9}{4} & \frac{1}{4} & \frac{7}{4} & \frac{13}{4} \\ \frac{3}{2} & \frac{-3}{4} & 0 & 0 & \frac{3}{4} & \frac{-3}{2} \\ \frac{9}{16} & \frac{9}{16} & \frac{81}{16} & \frac{1}{16} & \frac{9}{16} & \frac{9}{16} \end{bmatrix}, M^{-1} = \begin{bmatrix} \frac{3}{64} & \frac{1}{8} & \frac{1}{16} & \frac{13}{48} & \frac{1}{6} & \frac{1}{12} \\ \frac{-3}{16} & \frac{1}{4} & \frac{1}{4} & \frac{7}{12} & \frac{-1}{3} & \frac{-1}{3} \\ \frac{1}{16} & 0 & \frac{-1}{16} & \frac{1}{16} & 0 & \frac{1}{4} \\ \frac{-3}{16} & \frac{-1}{4} & \frac{1}{4} & \frac{-7}{12} & \frac{-1}{3} & \frac{-1}{3} \\ \frac{-3}{16} & \frac{-1}{4} & \frac{1}{4} & \frac{-7}{12} & \frac{1}{3} & \frac{-1}{3} \\ \frac{3}{64} & \frac{-1}{8} & \frac{1}{16} & \frac{13}{48} & \frac{-1}{6} & \frac{1}{12} \end{bmatrix}.
$$

The multiplication \bullet in Cl(p)^t is given by expressing s_{λ} and s_{μ} as vectors in $\mathbb{C}^6 = \text{Span}_{\mathbb{C}}(\text{pr}_i : i = 1, \ldots, 6)$ using M, then multiplying with the $8\,$ KIERAN CALVERT, KARMEN GRIZELJ, ANDREY KRUTOV, AND PAVLE PANDŽIĆ

Hadamard product. The resulting vector is then expressed as a combination of s_λ by applying M^{-1} . Explicitly $s_\lambda \bullet s_\mu = M^{-1}(M(s_\lambda) \bullet_H M(s_\mu)),$ where \bullet _H is the Hadamard (component-wise) product. This gives the new multiplication below: (The terms vanishing in the associated graded algebra $\bigwedge(\mathfrak{p})^{\mathfrak{k}}$, equivalently those vanishing under the projection $P_{[l(\lambda)+l(\mu)]}$, are underlined.)

$$
s_{(1,0)} \bullet s_{(1,0)} = s_{(1,1)} + s_{(2,0)},
$$
\n
$$
s_{(1,0)} \bullet s_{(1,1)} = s_{(2,1)},
$$
\n
$$
s_{(1,0)} \bullet s_{(1,1)} = s_{(2,1)},
$$
\n
$$
s_{(1,0)} \bullet s_{(2,1)} = \frac{9}{16} s_{(1,0)},
$$
\n
$$
s_{(1,0)} \bullet s_{(2,2)} = \frac{9}{16} s_{(1,0)},
$$
\n
$$
s_{(2,0)} \bullet s_{(2,0)} = \frac{5}{2} s_{(1,1)} + \frac{5}{2} s_{(2,0)} + s_{(2,2)},
$$
\n
$$
s_{(1,1)} \bullet s_{(2,0)} = \frac{9}{16} s_{(0,0)} + \frac{5}{2} s_{(1,1)},
$$
\n
$$
s_{(2,0)} \bullet s_{(2,1)} = \frac{9}{16} s_{(1,0)} + \frac{5}{2} s_{(2,1)},
$$
\n
$$
s_{(2,0)} \bullet s_{(2,1)} = \frac{9}{16} s_{(1,1)} + \frac{5}{2} s_{(2,2)},
$$
\n
$$
s_{(1,1)} \bullet s_{(1,1)} = s_{(2,2)},
$$
\n
$$
s_{(1,1)} \bullet s_{(2,1)} = \frac{9}{16} s_{(1,0)},
$$
\n
$$
s_{(1,1)} \bullet s_{(2,2)} = \frac{9}{16} s_{(2,0)} - \frac{5}{2} s_{(2,2)},
$$
\n
$$
s_{(2,1)} \bullet s_{(2,1)} = \frac{9}{16} s_{(1,1)} + \frac{9}{16} s_{(2,0)},
$$
\n
$$
s_{(2,1)} \bullet s_{(2,2)} = \frac{9}{16} s_{(2,1)},
$$
\n
$$
s_{(2,2)} \bullet s_{(2,2)} = \frac{81}{256} s_{(0,0)} + \frac{45}{32} s_{(1,1)}
$$
\n
$$
s_{(2,2)} \bullet s_{(2,2)} = \frac{45}{32} s_{(2,0)} + \frac{
$$

Example 3.2. In this example $\mathfrak{g} = \mathfrak{gl}_5$ and $\mathfrak{k} = \mathfrak{gl}_3 \oplus \mathfrak{gl}_2$. Then $|W^1| = 10$. The change of bases matrices are:

.

Again the new multiplication is calculated by expressing s_λ and s_μ as vectors in \mathbb{C}^{10} , performing the Hadamard product then applying the change of basis from $\{pr_i\}$ to $\{s_\lambda\}$. (Again the terms vanishing in the associated graded algebra $\Lambda(\mathfrak{p})^{\mathfrak{k}}$ are underlined.)

$$
\begin{aligned} &s_{(1,1)}\bullet s_{(1,1)}=s_{(2,2)},\\ &s_{(1,1)}\bullet s_{(3,0)}=\frac{5s_{(2,1)}}{4s_{(1,1)}+5s_{(2,2)}},\\ &s_{(1,1)}\bullet s_{(3,1)}=\frac{4s_{(1,1)}+5s_{(2,2)}}{4s_{(2,1)}},\\ &s_{(1,1)}\bullet s_{(3,2)}=\frac{4s_{(2,1)}}{4s_{(2,1)}},\\ &s_{(1,1)}\bullet s_{(3,3)}=\frac{4s_{(3,1)}-5s_{(3,3)}}{4s_{(3,1)}-5s_{(3,3)}},\\ &s_{(1,1)}\bullet s_{(3,3)}=\frac{4s_{(3,1)}-5s_{(3,3)}}{8s_{(3,0)}-5s_{(3,2)}},\\ &s_{(2,2)}\bullet s_{(3,0)}=\frac{5s_{(2,2)}+5s_{(3,1)}}{5s_{(2,2)}+5s_{(3,1)}},\\ &s_{(3,0)}\bullet s_{(3,1)}=\frac{25s_{(2,1)}+5s_{(3,2)}}{20s_{(1,1)}+25s_{(2,2)}+5s_{(3,3)}},\\ &s_{(2,2)}\bullet s_{(3,0)}=\frac{5s_{(3,2)}}{20s_{(2,1)}},\\ &s_{(2,1)}\bullet s_{(2,1)}=s_{(3,3)}+\frac{4s_{(1,1)}+5s_{(2,2)}}{s_{(2,1)}+5s_{(3,1)}}\\ &s_{(2,1)}\bullet s_{(3,1)}=\frac{4s_{(2,1)}+5s_{(3,2)}}{4s_{(2,2)}+4s_{(3,1)}},\\ &s_{(2,1)}\bullet s_{(3,2)}=\frac{4s_{(2,2)}+4s_{(3,1)}}{4s_{(3,2)}+4s_{(3,1)}},\\ &s_{(2,1)}\bullet s_{(3,1)}=\frac{20s_{(1,1)}+25s_{(2,2)}+4s_{(3,1)}}{8s_{(3,1)}+25s_{(2,2)}+4s_{(3,1)}},\\ \end{aligned}
$$

$$
s_{(2,2)} \bullet s_{(3,1)} = 4s_{(2,2)} + 5s_{(3,3)},
$$

\n
$$
s_{(3,1)} \bullet s_{(3,2)} = \frac{20s_{(2,1)} + 4s_{(3,2)}},
$$

\n
$$
s_{(3,1)} \bullet s_{(3,3)} = \frac{20s_{(3,1)} - 21s_{(3,3)}},
$$

\n
$$
s_{(2,2)} \bullet s_{(2,2)} = \frac{4s_{(3,1)} - 5s_{(3,3)}},
$$

\n
$$
s_{(2,2)} \bullet s_{(3,2)} = \frac{4s_{(3,2)}}{16s_{(1,1)} + 20s_{(2,2)} - 20s_{(3,1)} + 25s_{(3,3)}},
$$

\n
$$
s_{(3,2)} \bullet s_{(3,2)} = \frac{16s_{(1,1)} + 20s_{(2,2)} + 4s_{(3,3)}}{16s_{(2,1)} + 20s_{(2,2)} + 25s_{(3,3)}},
$$

\n
$$
s_{(3,3)} \bullet s_{(3,3)} = \frac{16s_{(2,1)}}{-80s_{(1,1)} - 84s_{(2,2)} + 100s_{(3,1)} - 105s_{(3,3)}}.
$$

In both examples, as well as in general, deleting the underlined terms gives the Littlewood-Richardson coefficients.

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