

## ON THE ATTAINABLE SET FOR A SCALAR NONCONVEX CONSERVATION LAW\*

BORIS P. ANDREIANOV<sup>†</sup>, CARLOTTA DONADELLO<sup>‡</sup>, AND ANDREA MARSON<sup>§</sup>

**Abstract.** We deal with a Cauchy problem for a scalar conservation law in one space dimension. The flux function is assumed to be nonconvex, in particular, to have a single inflection point. We consider a compactly supported initial datum and regard it as a control. The main result of the paper states sufficient conditions for a function  $v$  to be attained at a fixed time  $T$  by a trajectory of the conservation law.

**Key words.** scalar conservation laws, nonconvex flux, control

**AMS subject classifications.** 35Q93, 35L65

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**1. Introduction.** The aim of this paper is to initiate the description of the set  $\mathfrak{A}_T$  of profiles  $v$  in  $BV(\mathbb{R})$  which can be attained at a fixed time  $t = T$  by the entropy admissible solution of

$$(1a) \quad \partial_t u + \partial_x f(u) = 0,$$

$$(1b) \quad u(0, x) = \begin{cases} u_c(x) & \text{if } x \in [a, b], \\ 0 & \text{if } x \notin [a, b], \end{cases}$$

where  $a < b$  are given, and  $u_c \in L^\infty(a, b)$  is regarded as a control. We consider a flux function  $f$  with a single inflection point, normalized so as to fulfill the set of conditions

- (F)    a.  $f$  is of class  $C^2$ ,  
       b.  $f(0) = f'(0) = f''(0) = 0$ ,     $uf''(u) < 0 \quad \forall u \neq 0$ ,  
       c.  $f$  has superlinear growth at  $\pm\infty$ .

A family of classical examples is given by  $f(u) = -|u|^m u$ ,  $m \geq 2$ . We stress that the conditions in (F)b are introduced here in order to make our statements and proofs involving generalized characteristics as close as possible to the original paper by Dafermos [10]. Our analysis applies to any flux function with a single inflection point satisfying (F)a and (F)c provided  $f''(u) = 0$  only at the inflection point. In section 4 we apply our results to the classical sedimentation model proposed by Kynch in [17] and we consider the flux function  $f(u) = -u(1-u)^2$  which does not satisfy (F)b.

In the existing literature we can distinguish essentially three approaches toward the study of exact controllability and characterization of the set of attainable profiles

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<sup>†</sup>Laboratoire de Mathématique et Physique Théorique, Université de Tours, Parc de Grandmont, 37200, Tours, France (boris.andreianov@lmpt.univ-tours.fr).

<sup>‡</sup>Laboratoire de Mathématique de Besançon, Université de Franche-Comté, 16 Route de Gray, 25030, Besançon, France (carlotta.donadello@univ-fcomte.fr).

<sup>§</sup>Dipartimento di Matematica, Via Trieste 63, 35121 - Padova, Italy, and Laboratorio di Matematica di Besançon, Université de Franche-Comté, 16 Route de Gray, 25030, Besançon, France (marson@math.unipd.it).

for (1a).

The pioneering paper by Ancona and Marson [3], focuses on the exact controllability for the initial boundary value problem for (1a) in the case of strictly convex flux. The authors use the theory of generalized characteristics introduced by Dafermos in [10], and describe for any given time  $T > 0$ , the set of states which are attainable in time  $T$  starting from the initial condition  $u_0 = 0$  by a control on the boundary data alone. Under similar hypotheses, Adimurthi, Ghoshal, and Gowda [1, 2] exploit the explicit representation of solutions given by the Lax–Oleinik formula to construct an explicit backward solver and give a concise characterization of the set of attainable profiles for the initial value problem and the boundary value problem in the half-space and in a strip with two boundaries. Using again the method of generalized characteristics, Corghi and Marson characterize in [8] the attainable set for a scalar balance law with strictly convex flux

$$(2) \quad u_t + f(u)_x = z(t, x) \quad \text{for } t \in [0, T], \quad \text{and } x \in \mathbb{R},$$

where the right-hand side  $z$  acts as a distributed control.

In the classical paper [15], Horsin obtains approximate controllability results for scalar conservation laws by a different approach, which relies on the return method introduced by Coron [9]. This technique is also an important ingredient in the result by Chapouly [7] concerning the attainability of  $C^1$  profiles for classical solutions of the balance law (2) on the strip  $[0, T] \times [0, 1]$ . In this paper the boundary data at  $x = 0$  and  $x = 1$  act as controls together with the source term  $z$  (distributed control) which only depends on the time variable  $t$ . A similar result, in the setting of an entropy weak solution is due to Perrollaz [19].

The return method has been applied in combination with the vanishing viscosity approach by Léautaud [18], to study the uniform controllability of scalar conservation laws. This last paper is also the only one, to our knowledge, in which the flux function  $f$  is allowed to have a finite number of inflection points. Starting from an initial condition  $u_0$  in  $L^\infty([0, 1])$  and using the boundary data at  $x = 0$  and  $x = 1$  as controls, the author proves the attainability in time  $T$  (depending on the  $L^\infty$  norm of  $u_0$ ) of constant states.

Finally, let us mention that for the viscous Burgers equation, constant states reachable by boundary controls has been studied using the Cole–Hopf transformation by Glass and Guerrero [12] and Guerrero and Imanuvilov [13].

The results in the present paper are all obtained by an extensive application of the method of generalized characteristics. This approach naturally leads to a very fine analysis of the structure of solutions which allows us to give a rather complete insight of the possible obstructions to attainability. Moreover, we succeed in providing a complete explicit backward reconstruction of the solution for a significant class of attainable profiles, which contains all basic situations. However, even if it is certainly possible to use our results as building blocks toward the analysis of more complex cases, our work fails to provide an easy to check routine to distinguish attainable states from nonattainable ones.

**1.1. Fine structure of solutions and obstacles to backward reconstruction.** We recall the definition of an entropy admissible solution to (1a)–(1b) (see [11]).

**DEFINITION 1.1.** *An entropy admissible solution to (1a)–(1b) in the time interval  $[0, T]$  is a continuous function  $u : [0, T] \rightarrow L^1_{loc}(\mathbb{R})$  which assumes the initial datum (1b), is a distributional solution to (1a) in  $]0, T[ \times \mathbb{R}$ , and satisfies an entropy*

admissibility condition (see [11]), i.e., for any entropy-entropy flux pair  $(\eta, q)$  with  $u \mapsto \eta(u)$  convex, there holds

$$(3) \quad \partial_t \eta(u) + \partial_x q(u) \leq 0 \quad \text{in } ]0, T[ \times \mathbb{R}$$

in the sense of distribution.

Throughout our paper we will always deal with solutions  $u = u(t, x)$  to (1a) such that  $u(t, \cdot)$  has finite total variation. Hence, it is not restrictive to normalize  $u(t, \cdot)$  to be left continuous for any  $t > 0$ . In general, distributional solutions to (1a) are not smooth, regardless of the possible regularity of the initial condition. The demise of classical solutions coincides with the appearance of jump discontinuities. The method of generalized characteristics has been introduced by Dafermos in [10] to investigate the fine structure of the solutions to (1a).

DEFINITION 1.2. Let  $u \in C([0, T]; L^1_{loc}(\mathbb{R}))$  be an entropy admissible solution of (1a).

- A classical characteristic associated with  $u$  is a curve  $\xi : [t_0, t_1] \rightarrow \mathbb{R}^2$  such that for some  $\mathbf{u}$  and for all  $t \in [t_0, t_1]$  there holds

$$\xi(t) = \xi(t_0) + f'(\mathbf{u})(t - t_0), \quad u(t, \xi(t)) = \mathbf{u}.$$

- A generalized characteristic associated with  $u$  is a Lipschitz continuous curve  $\xi : [t_0, t_1] \rightarrow \mathbb{R}^2$  which satisfies the differential inclusion

$$\frac{d\xi}{dt}(t) \in I[f'(u(t, \xi(t)^+)), f'(u(t, \xi(t)))] ,$$

where  $I[a, b]$  denotes the closed interval of extrema  $a$  and  $b$ .

Any classical characteristic is a generalized characteristic.

Through any point of jump discontinuity of  $u$  passes a funnel of backward generalized characteristic. The core of the results in [10, section 2], concerns the study of the special properties of minimal and maximal backward generalized characteristics through a point. To limit the length of our manuscript, we decided not to recall all of these results but rather to give precise references at any time we apply them.

We recall that the method of generalized characteristics is fully developed only for the case in which the flux function  $f$  admits at most one inflection point. A classical observation due to Hoff [14] is that, whenever the flux function  $f$  admits more than one inflection point, the interaction between two discontinuities in the solution  $u$  to (1a) can produce outgoing centered rarefaction waves at positive times, and this would make the analysis far more intricate.

Once a jump discontinuity has appeared in the solution it persists, and its location describes a Lipschitz continuous curve  $x = \varphi(t)$  in the  $(t, x)$ -plane. From the definition of distributional solution, one can check that the values of the left and right traces of the solution  $u$  on the sides of  $\varphi$  must satisfy the Rankine–Hugoniot conditions:

$$(4) \quad f(u(t, \varphi(t+))) - f(u(t, \varphi(t))) = \dot{\varphi}(t)[u(t, \varphi(t+)) - u(t, \varphi(t))].$$

Additionally, if the solution  $u$  is entropy admissible, the following inequalities are satisfied (Lax–Oleinik conditions):

$$(5) \quad f'(u(t, \varphi(t))) \geq \dot{\varphi}(t) \geq f'(u(t, \varphi(t+))), \quad x = \varphi(t).$$

This implies that the curves of discontinuity are generalized characteristics.

If the flux function  $f$  in (1a) is strictly convex, then both the inequalities in (5) are strict, while a classical lemma due to Dafermos [10, Lemma 2.2], specifies that under the assumptions (F), only the second inequality above is strict. In this framework we can distinguish two kinds of discontinuities propagating in the solution:

- genuine shocks (which we call *shocks* in the following), for which both inequalities in (5) are strict;
- left contact discontinuities, for which there holds  $\dot{\varphi}(t) = f'(u(t, \varphi(t)))$ .

As in the convex case, characteristics run into genuine shocks from both sides, while it may happen that characteristics radiate from one of the sides of a contact discontinuity.

The presence of contact discontinuities is responsible for the two major obstacles toward the formulation of a fast and easy to apply attainability test. First, recall that in the convex case, whenever the target profile  $v$  suffers from an admissible jump at  $x = \bar{x}$  joining the states  $v_L$  and  $v_R$ , there exist infinitely many ways to reconstruct backward a solution to (1a) in the triangular region delimited by the minimal and the maximal backward characteristics

$$\eta_{\bar{x}}^-(t) = \bar{x} + f'(v_L)(t - T), \quad \eta_{\bar{x}}^+(t) = \bar{x} + f'(v_R)(t - T),$$

respectively. This fact can be exploited to produce convenient backward reconstructions (see, for example, [4]), and it is related to the dissipative properties of admissible shocks.

On the contrary, since left contact discontinuities must satisfy the Rankine–Hugoniot condition together with  $\dot{\varphi}(t) = f'(v_L)$ , it is clear that given the left state of a left contact discontinuity, the corresponding right state is immediately determined from the fact that the chord joining  $(v_L, f(v_L))$  to  $(v_R, f(v_R))$  needs to have slope exactly equal to  $f'(v_L)$ . We call  $(v_L)^\sharp$  the unique possible right state connected to  $v_L$  by a left contact discontinuity. This means that there is no freedom in the backward reconstruction of the solution in an immediate right neighborhood of a contact discontinuity.

We are now ready to illustrate obstacles we have to face in our work. First of all, we give the following.

**DEFINITION 1.3.** *Let  $v = v(x)$  be a function with finite total variation normalized so that  $v$  be left continuous. We say that a line parameterized by*

$$(6) \quad [0, T] \ni t \mapsto x_0 + f'(v(x_0))(t - T),$$

*is a candidate backward characteristic.*

Indeed, if  $v$  is a candidate final profile, in order to reconstruct the initial datum  $u_c$  we need to trace backward the characteristic lines associated with the solution  $u$  to (1a)–(1b) attaining  $v$  at time  $t = T$ . Whenever  $|t - T|$  is sufficiently small, such characteristics can be parameterized by (6), but, in general, generalized characteristics in the solution  $u$  are not straight lines. In other words, a candidate characteristic is a useful geometrical object, which has not a priori a special meaning for the reconstructed solution. The first obstacle in reconstructing backward a solution from a given final profile  $v$  is the following.

- Assume that the target profile  $v$  suffers from a jump discontinuity at  $x = \bar{x}$ , joining two states  $v_L$  and  $v_R$  such that

$$f(v_R) = f(v_L) + f'(v_L)(v_R - v_L),$$

and that the candidate backward characteristics from the points in a left

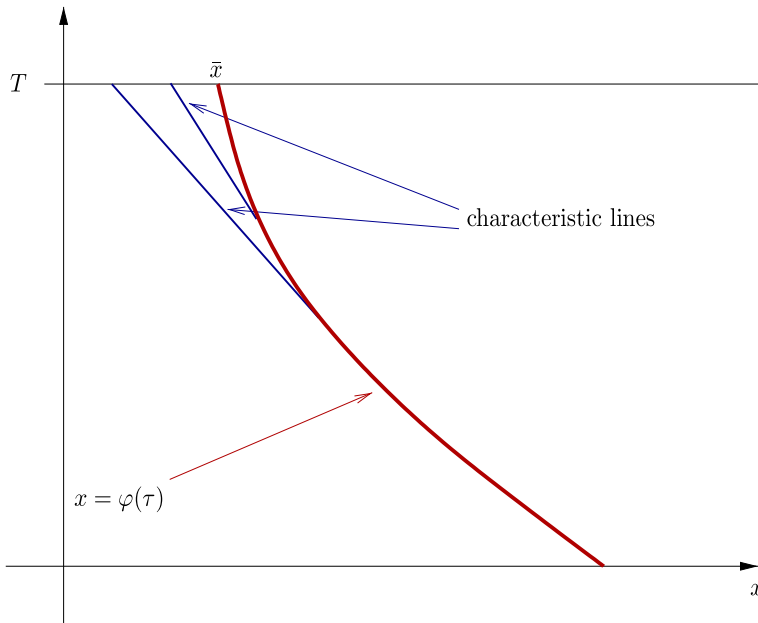


FIG. 1. The left contact discontinuity traveling along the curve  $\varphi$ .

neighborhood of  $\bar{x}$  cross one another at positive times. If  $v$  is admissible, their tangent envelope must be a convex Lipschitz curve  $t \mapsto \varphi(t)$  (see [10, Theorem 2.2]), and, in this case, our backward reconstruction contains a left contact discontinuity traveling along the curve  $\varphi$  (see Figure 1).

This geometric requirement translates into the fact that the attainability of  $v$  may depend on the values of  $v$ ,  $f'(v)$ , and (see Theorem 3.1, in particular, conditions (27)–(28))  $f''(v)$  on a whole left neighborhood of  $\bar{x}$  and not just on the values of  $v(\bar{x}^-)$  and  $v(\bar{x}^+)$ . Of course, the presence of several jumps in the target profile may ask for additional compatibility conditions.

The second difficulty we face is the following.

- Once we detect the presence of a left contact discontinuity  $\varphi$  in our backward reconstruction, we must trace back in a unique way the candidate backward characteristics from  $(t, \varphi(t))$ , that we denote by  $\xi_{\varphi(t)}$ . It might happen that the curves  $\xi_{\varphi(t)}$  cross one another at positive times and have a convex tangent envelope  $t \mapsto \varphi_2(t)$ . In this case we continue our backward reconstruction including a second left contact discontinuity in the solution. Again, the values on the left and on the right of  $\varphi_2$  are completely determined from the values of  $v$  and we have no choice in our reconstruction. We know from the results in [10] that, if  $v$  is an admissible target profile, this situation can occur a finite number of times before we reach  $t = 0$ . We call this structure a *nesting* of contact discontinuities.

Even in the most simple case, which is briefly described in section 5, the reconstruction of a solution in the presence of a nesting structure is a delicate matter, which leads us to look for solutions of a characteristic boundary value problem in which the boundary datum is not to be taken in the usual hyperbolic sense [5], but should be imposed as the value of the trace of the solution along the boundary curve; see section 5.2.

**1.2. Outline of the paper.** Notations are introduced in the last part of this introduction. Section 4 contains an example related to a model of sedimentation. A direction of perspective research is outlined in subsection 5.3. In hopes of palliating the technicality of the presentation, we give here a rather detailed summary of the results in sections 2, 3, and 5.

*Section 2: Necessary conditions.* Theorem 2.1 states necessary conditions toward attainability which are reminiscent of the classical necessary conditions for the convex flux case.<sup>1</sup> Proposition 2.2 rules out the presence of centered rarefaction waves focusing at positive time.

Proposition 2.3 is the first evidence of the strong rigid structure of solutions next to a left contact discontinuity. Basically, on the right of any left contact discontinuity curve  $t \mapsto \varphi(t)$  we can define a “triangular” region delimited by  $\varphi$ , the maximal backward characteristic from the point of jump of  $v$ ,  $t \mapsto \eta_x^+(t)$ , and the line  $t = 0$ . In this region the solution can be reconstructed in a unique way by following the backward characteristics steaming from the right side of the discontinuity. If nesting occurs, it must take place outside this region. Starting from this point we limit our attention to situations in which nesting does not occur, deferring to section 5 some examples and comments concerning nesting.

Given a continuous target profile  $v$  and a time  $T$  we define (candidate) backward characteristics lines as in (15c). Proposition 2.4 gives a necessary condition for this family of lines to have a convex tangent envelope. This implies a geometric requirement for the attainability of  $v$  because the characteristics associated with any solution to (1a) can cross only in a centered rarefaction wave at time  $t = 0$ , or they can radiate from a left contact discontinuity, which is a convex curve.

Proposition 2.5 puts together the results in Propositions 2.3 and 2.4 to formulate a necessary condition for a contact discontinuity in  $v$  to be admissible. Observe that while the admissibility of a shock discontinuity in the convex case is easily determined by the values of  $v$  on the left and on the right sides of the jump, here we need to impose conditions on the first and second derivatives of  $f'(v)$  on a whole neighborhood of the point of jump.

*Section 3: Sufficient conditions.* The main result in this section is Theorem 3.1, which collects a set of sufficient conditions for the attainability of a profile  $v$  in time  $T$  provided that the points at which  $v$  is discontinuous do not accumulate and that no nesting occurs. The first part of section 3 consists of the statement of the theorem and of some remarks illustrating the meaning of the conditions. In the hypothesis of Theorem 3.1 it is possible to determine whether the profile  $v$  is attainable by checking that each of its discontinuities is admissible. Therefore in section 3.1 we detail the study of three basic situations which will be used as building blocks in the proof of Theorem 3.1. Lemmas 3.2, 3.3, and 3.4 explain how to construct a control which allows one to reach the profile  $v$  in three basic situations. The last part of section 3 contains the proof of Theorem 3.1.

*Section 5: One example of nesting, a different point of view.* Section 5.1 contains the analysis of the simplest possible case of nesting. Some conditions for attainability are states in a very smooth framework. One explicit example illustrates our results. Section 5.2 is devoted to presenting a slightly different approach to the problem of nesting.

<sup>1</sup>The interested reader may compare the statements of our Theorems 2.1 and 3.1 with the statement of [3, Theorem 1], which gives necessary and sufficient conditions for attainability in the case of a strictly convex flux function.

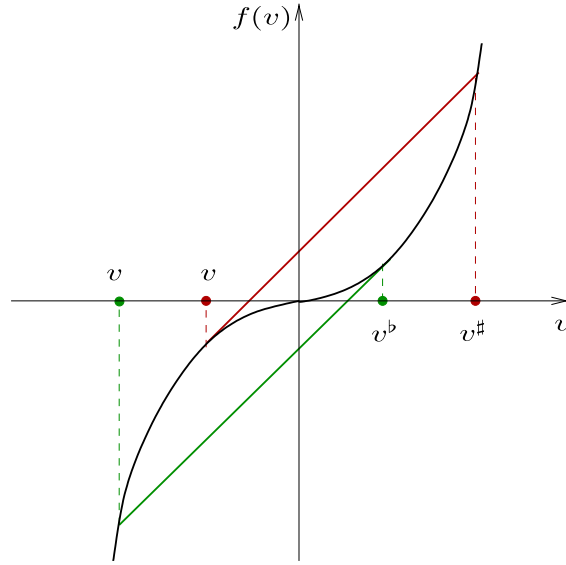


FIG. 2. The points  $v^\#$  and  $v^b$ .

**1.3. Notation.** First of all, if a solution to (1a) suffers from jump discontinuity along a Lipschitz curve  $x = \varphi(t)$ , we denote by  $u_L(t)$  and  $u_R(t)$ , respectively, the left and right limits of  $u(t, \cdot)$  at  $\varphi(t)$  whenever they exist, i.e.,

$$(7) \quad u_L(t) = \lim_{x \rightarrow \varphi(t)^-} u(t, \cdot), \quad u_R(t) = \lim_{x \rightarrow \varphi(t)^+} u(t, \cdot).$$

Let  $f$  be a flux function satisfying the set of conditions (F). Given  $v \in \mathbb{R}$ ,  $v \neq 0$ , we let  $v^\#$  be the solution to (see Figure 2)

$$(8) \quad f(v^\#) = f(v) + f'(v)(v^\# - v), \quad v^\# \neq v,$$

so that

$$(9) \quad f'(v) = \frac{f(v) - f(v^\#)}{v - v^\#}$$

holds for any  $v \neq 0$ . Symmetrically, we denote by  $v^b$  the solution to

$$(10) \quad f(v) = f(v^b) + f'(v^b)(v - v^b), \quad v^b \neq v,$$

so that

$$(11) \quad f'(v^b) = \frac{f(v) - f(v^b)}{v - v^b}$$

holds for any  $v \neq 0$ . Observe that

$$(12) \quad v = (v^\#)^b = (v^b)^\#,$$

$$(13) \quad v \cdot v^\# < 0, \quad v \cdot v^b < 0,$$

hold for any  $v \neq 0$ .

Throughout our paper we assume that the target profile  $v$  is a left continuous BV function with compact support. We define

$$(14) \quad \alpha \doteq \sup \{x \in \mathbb{R} : v(y) = 0 \ \forall y \leq x\},$$

so that  $v(\alpha) = 0$ . Moreover, if  $T$  is the fixed time at which  $v$  should be attained, we let

$$(15a) \quad \eta_x^-(\tau) \doteq x + f'(v(x))(\tau - T) = x + f'(v(x-))(\tau - T),$$

$$(15b) \quad \eta_x^+(\tau) \doteq x + f'(v(x+))(\tau - T),$$

be, respectively, the (candidate) minimal and maximal backward characteristics lines from  $(T, x)$  associated with a solution  $u$  of (1a) attaining the profile  $v$  at time  $T$ , i.e.,  $u(T, x) = v(x)$ . In case  $v$  is continuous at  $x$ , minimal and maximal backward characteristics coincide in a left neighborhood  $]T - \delta, T]$  of  $t = T$ , and therefore we can write

$$(15c) \quad \eta_x(\tau) \doteq x + f'(v(x))(\tau - T),$$

where  $\tau \in ]T - \delta, T]$ .

The notations  $D^+$  and  $D^-$  stand for the right and left upper Dini derivatives, respectively,

$$D^+g(x) = \limsup_{y \rightarrow x^+} \frac{g(y) - g(x)}{y - x}, \quad D^-g(x) = \limsup_{y \rightarrow x^-} \frac{g(y) - g(x)}{y - x}.$$

Given any two points  $a$  and  $b$  in  $\mathbb{R}$ , we use the notation  $I(a, b)$  to indicate the open interval of extrema  $a$  and  $b$ . This means that  $I(a, b) = ]a, b[$  if  $a < b$  and  $I(a, b) = ]b, a[$  if  $a > b$ . Analogously,  $I[a, b]$  denotes the closed interval of extrema  $a$  and  $b$ .

**2. Necessary conditions for attainability.** Throughout this section we assume that  $v$  is the value, at time  $t = T$ , of an entropy solution to (1a) defined on  $[0, T]$ . We analyze the necessary properties of  $v$  due to this fact, using the backward characteristics as the main tool. The conditions in the following theorem are the exact counterpart of the necessary conditions for attainability stated in [3] for the convex flux case. They differ from all other conditions stated in this work because they depend on the value (or the limit value) of the target profile at single points.

**THEOREM 2.1.** *Assume that  $v$  is a measurable, left continuous, bounded profile, with finite total variation, so that there exists  $u_c \in L^\infty(a, b)$  such that the weak entropy solution  $u = u(t, x)$  to (1a)–(1b) fulfills  $u(T, \cdot) = v$ . Let  $\alpha$  be defined as (14). Then the following properties hold.*

1. *Let  $x_0 \in \mathbb{R}$  be given and assume that  $v(x_0+) \neq 0$ . Then the candidate maximal backward characteristic from the point  $(T, x_0)$ ,  $t \mapsto \eta_{x_0}^+(t)$ , satisfies*

$$(16) \quad \eta_{x_0}^+(0) = x_0 - f'(v(x_0+))T \leq b.$$

2. *Assume that  $v(\alpha+) \neq 0$ . Then the candidate maximal backward characteristic from the point  $(T, \alpha)$ ,  $t \mapsto \eta_\alpha^+(t)$ , satisfies*

$$(17) \quad \eta_\alpha^+(0) = \alpha - f'(v(\alpha+))T \geq a.$$



3. Assume that  $v(\alpha+) = 0$ . Then there exists a sequence  $\{x_n\}_{n \in \mathbb{N}}$  such that  $x_n \downarrow \alpha$  and for any  $n \in \mathbb{N}$ , the candidate maximal backward characteristic from the point  $(T, x_n)$ ,  $t \mapsto \eta_{x_n}^+(t)$ , satisfies

$$(18) \quad \eta_{x_n}^+(0) = x_n - f'(x_n+)T > a.$$

4. Assume that at  $x = \bar{x}$  the function  $v$  suffers from a jump discontinuity with left and right states  $v_L = v(\bar{x})$  and  $v_R = v(\bar{x}+)$ , respectively. Then the following conditions on  $v_L$  and  $v_R$  hold.

- (a)  $f'(v_L) > f'(v_R)$ ;
- (b) if  $v_L \cdot v_R < 0$ , then let  $v_L^\sharp$  be as in (8) with  $v = v_L$ . Then we have

$$(19) \quad v_R \notin I(v_L, v_L^\sharp).$$

*Proof.*

1. Assume that (16) fails, and let  $y = \zeta_{x_0}^+(t)$  be the maximal generalized backward characteristic from  $(T, x_0)$ . Since  $y = \zeta_{x_0}^+(t)$  is a convex polygonal [10, Theorem 2.1], we have

$$\zeta_{x_0}^+(t) \geq x_0 + f'(v(x_0+))(t - T) = \eta_{x_0}^+(t)$$

with  $\eta_{x_0}^+(\cdot)$  being defined by (15c) (candidate backward characteristic). Since  $u(0, x) = 0$  for any  $x > b$ , in particular, we have  $u(0, x) = 0$  for all  $x \geq \zeta_{x_0}^+(0)$ , so that  $u(t, x) = 0$  for any  $x \geq \zeta_{x_0}^+(t)$  and any  $t \leq T$ . Hence  $v(x_0+) = 0$ , contrary to the assumption.

2. Assume by contradiction that (17) fails and, to fix the ideas, assume  $v(\alpha+) > 0$ . Let  $x_1 > \alpha$  be such that  $v(x) > 0$  for any  $\alpha < x < x_1$ . Then, there exists  $\delta > 0$  such that in the region

$$\{(\tau, x) : T - \delta \leq \tau \leq T, x \leq x_1 + f'(v(x_1))(\tau - T)\}$$

there holds  $u(\tau, x) \geq 0$ . Hence  $u$  is a solution to a conservation law with convex flux. Since arguing as before we get  $u(\tau, x) = 0$  for any  $x < \alpha + f'(v(\alpha+))(\tau - T)$ , it turns out that  $u(T, \cdot)$  cannot suffer from a jump discontinuity at  $x = \alpha$ .

3. Again, we argue by contradiction, and assume that there exists  $\bar{x} > \alpha$  such that

$$x - f'(v(x+))T \leq a \quad \forall \alpha < x \leq \bar{x}.$$

Let  $\eta_x(\cdot)$  be defined as at (15c). Two cases may occur.

- (a) All of the lines  $\eta_x(\cdot)$ ,  $\alpha < x \leq \bar{x}$ , focus at  $(0, a)$ , i.e.,

$$x - f'(v(x+))T = a \quad \forall \alpha < x \leq \bar{x}.$$

In this case the solution  $u$  would contain a centered rarefaction wave with 0 left state, leading to a contradiction.

- (b) If some of the lines  $\eta_x(\cdot)$ ,  $\alpha < x \leq \bar{x}$ , do not focus at  $(0, a)$ , we may assume that

$$x - f'(v(x+))T < a \quad \forall \alpha < x \leq \bar{x}$$

holds. But then, arguing as above, we can deduce that  $v(y) = 0$  in a right neighborhood of  $x = \alpha$ , contrary to definition (14).

4. The proof of such properties of admissible jump discontinuities for a weak entropy solution to (1a) is classical (e.g., see [11, Chapter 8]).  $\square$

As we limit our attention to the case in which the flux function  $f$  in (1a) has exactly one inflection point, we know that no rarefaction wave can be produced at positive time by the interaction of two or more wavefronts. Therefore, any candidate target profile  $v$  whose candidate backward characteristics focus at a single point of  $\mathbb{R}^+ \times \mathbb{R}$  is not attainable. This fact motivates the following proposition.

**PROPOSITION 2.2.** *Assume that  $v$  is an attainable profile, continuous in an interval  $I$ , with<sup>2</sup>  $D_x f'(v(x)) = c > 0$  for any  $x \in I$ . Then  $c \leq 1/T$ .*

*Proof.* If  $D_x f'(v(x)) = c$  for all  $x \in I$ , then the lines  $\eta_x$ , defined as in (15c) for  $x \in I$ , take the form

$$\eta_x(\tau) = x + c(x + a)(\tau - T),$$

where  $a$  is a constant, and intersect at  $\tau = T - 1/c$ ,  $y = -a$ . Since in any solution to (1a) no new rarefactions are created at positive times,  $\tau \leq 0$  must hold.  $\square$

*Remark 1.* The proposition above, based on the spread rate of rarefaction waves, can be compared to the classical Oleinik's estimate on the decay of positive waves. It should be noticed that Oleinik's estimate only holds in the strictly convex/concave setting, and under the assumptions of Proposition 2.2,  $f''(v(x)) \neq 0$  for  $x \in I$ . A partial generalization of Oleinik's estimate in the nonconvex case is the main result in [16], but as it involves the total variation of the initial condition, we cannot exploit it in our backward reconstruction.

Now let  $u$  be a bounded weak entropy solution to (1a) suffering from a discontinuity along a timelike curve  $x = \varphi(t)$  for  $t \in [t_0, t_1]$ , with right state  $u_R(t)$  (see (7)). The (candidate) maximal backward characteristic curve from the point  $(t, \varphi(t))$  is given by the equation

$$(20) \quad \xi_t(\tau) = \varphi(t) + f'(u_R(t))(\tau - t).$$

We stress that all through the paper, for the reader's convenience, backward characteristics originated at time  $T$  are denoted by the greek letter  $\eta$ , while backward characteristics originated at earlier times, steaming from a curve of discontinuity in the solution  $u$ , are denoted by the greek letter  $\xi$ . It is possible, of course, that some of the candidate backward characteristics  $\xi_t$ ,  $t \in [t_0, t_1]$ , cross each other. This is at the origin of the nesting phenomenon evoked in the introduction. However, the next proposition shows that the lines  $\xi_t$  do not intersect in the region between  $\varphi$  and  $\xi_{t_1}$ .

**PROPOSITION 2.3.** *Assume that  $u$  is a bounded measurable weak entropy solution to (1a) suffering from a left contact discontinuity  $x = \varphi(t)$ ,  $t \in [t_0, t_1]$ , with right state  $u_R(t)$ . Let  $\xi_t(\cdot)$  be as at (20), and assume that  $\xi_t(\tau) \leq \xi_{t_1}(\tau)$  for any  $\tau \in ]0, t]$  and  $t \in [t_0, t_1]$ . Then the function*

$$t \mapsto \xi_t(0) = \varphi(t) - f'(u_R(t))t$$

*is increasing.*

*Proof.* If  $t \mapsto \xi_t(0)$  is not increasing, then lines (20) cross each other in the time interval  $]0, T]$ . Hence, the solution  $u$  has a (left contact) discontinuity  $x = \psi(t)$  such that

$$\varphi(\tau) < \psi(\tau) < \xi_{t_1}(\tau).$$

Since the discontinuity at  $x = \psi(t)$  cannot be canceled [10, Theorem 2.3], and since

<sup>2</sup>Here  $D_x f'(v(x))$  denotes the derivative of the function  $x \mapsto f'(v(x))$ .

it cannot intersect the line  $x = \xi_{t_1}(\tau)$  due to the structure of minimal and maximal backward characteristics [10, Theorems 2.1 and 2.2], it turns out that  $x = \psi(t)$  interacts with  $\varphi(t)$  at a time  $\bar{t} \leq t_1$ . This cannot occur, otherwise  $\varphi(t)$  would not be a left contact discontinuity for  $t \geq \bar{t}$ .  $\square$

Lemma 2.4 from [10] states that any left contact discontinuity associated with a BV solution  $u$  of (1a) propagates along a strictly convex curve. This fact translates into a geometric condition on the lines  $\eta_x$ : If  $v$  is admissible and the candidate backward characteristic lines  $\eta_x$  do not reach  $t = 0$  without crossing each other, then their tangent envelope must necessarily be a strictly convex curve. The following proposition puts forward conditions on  $v$  which ensure that this desirable geometry is realized.

**PROPOSITION 2.4.** *Let  $I \subseteq \mathbb{R}$  be an interval, and  $v \in C^0(I)$  be bounded. Assume that the function  $I \ni x \mapsto f'(v(x))$  is differentiable with  $D_x f'(v(x)) > 0$  and  $D^+(D_x f'(v(x))) > 0$  for all  $x \in I$ . Then, the tangent envelope of the lines  $t \mapsto \eta_x(t)$  defined at (15c) for  $x \in I$  is a strictly convex  $C^1$  curve  $x = \varphi(\tau)$ , defined in a suitable interval  $[t_0, t_1] \subseteq ]-\infty, T]$ . Moreover, if  $D_x f'(v(x)) \geq 1/T$  for all  $x \in I$ , then the interval  $[t_0, t_1]$  is contained in  $[0, T]$ .*

*Proof.* Observe that, if  $\varphi$  exists, then for any  $\tau \in [t_0, t_1]$  there exists  $x(\tau) \in I$  such that

$$\varphi(\tau) = x(\tau) + f'(v(x(\tau)))(\tau - T) \quad \text{and} \quad \varphi'(\tau) = f'(v(x(\tau))).$$

Moreover, since we want  $\varphi$  to be strictly convex, the function  $\tau \mapsto x(\tau)$  has to be strictly increasing, and have a strictly increasing inverse, say  $I \ni x \mapsto t(x)$ . Let us compute  $t(x)$  in order to express  $\varphi$  in terms of the  $x$ -variable. For any  $x_1, x_2 \in I$  the time at which the lines  $\eta_{x_1}$  and  $\eta_{x_2}$  intersect is given by

$$t(x_1, x_2) = T - \frac{x_2 - x_1}{f'(v(x_2)) - f'(v(x_1))} < T,$$

where we have used that  $D_x f'(v(x)) > 0$  for all  $x \in I$ . It follows that

$$(21) \quad t(x_1) = \lim_{x_2 \rightarrow x_1^-} t(x_1, x_2) = T - \frac{1}{D_x f'(v(x_1))}.$$

Hence

$$(22) \quad \varphi(t(x)) = x + f'(v(x))(t(x) - T) = x - \frac{f'(v(x))}{D_x f'(v(x))}.$$

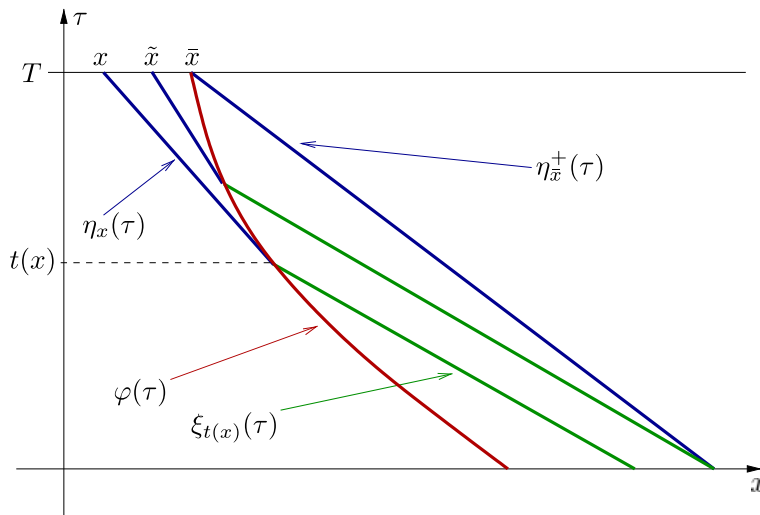
By construction, being  $D^+(D_x f'(v(x))) > 0$  for all  $x \in I$ ,  $t(x)$  turns out to be strictly increasing. Moreover, as  $\tau \mapsto x(\tau)$  is the inverse of  $x \mapsto t(x)$ , (21) implies that for a.e.  $t$

$$\begin{aligned} \varphi'(\tau) &= x'(\tau) + D_x f'(v(x(\tau)))x'(\tau)(\tau - T) + f'(v(x(\tau))) \\ &= x'(\tau)[1 + D_x f'(v(x(\tau)))(\tau - T)] + f'(v(x(\tau))) = f'(v(x(\tau))). \end{aligned}$$

Now one can easily conclude that the map  $\varphi$  at (22) fulfills the desired properties.  $\square$

*Remark 2.* Observe that, having in mind an admissible profile  $v$ , Proposition 2.4 and (21) imply that, if  $v$  suffers from a left contact discontinuity at  $x = \bar{x}$ , then

$$D_x f'(v(x)) = -\frac{1}{t(x) - T} \rightarrow +\infty \quad \text{as } x \rightarrow \bar{x}^-.$$

FIG. 3. The lines  $\eta_x$  and  $\xi_{t(x)}$ .

This behavior is forecast by the general theory on nonconvex conservation law; see [16, Theorem 3.1].

Assume that an attainable profile  $v$  suffers from a jump discontinuity at point  $x = \bar{x}$  between the states  $v(\bar{x})$  and  $v(\bar{x}^+) = (v(\bar{x}))^\sharp$ , and satisfies the hypothesis of Proposition 2.4 in a left neighborhood  $I$  of  $\bar{x}$ . In this framework we can define a curve  $\tau \mapsto \varphi(\tau)$  and conclude that any solution  $u$  of (1a) attaining the profile  $v$  at time  $T$  needs to suffer from a left contact discontinuity along the curve  $\varphi$ . It is easy and extremely important to notice that the values of  $u$  on the two sides of  $\varphi$  are completely determined as  $u(\tau, \varphi(\tau)) = v(x(\tau))$  and  $u(\tau, \varphi(\tau)^+) = (v(x(\tau)))^\sharp$ , the latter being defined as at (8) with  $v = v(x(\tau))$ . Using the inverse function  $x \mapsto t(x)$  introduced in the proof of Proposition 2.4 and writing  $v^\sharp(x)$  for  $(v(x))^\sharp$ , we can adapt the expression (20) to describe the candidate maximal backward characteristic from  $(\tau, x(\tau)) = (t(x), \varphi(t(x)))$  in terms of the profile  $v$  only (see Figure 3),

$$(23) \quad \xi_{t(x)}(\tau) = \varphi(t(x)) + f'(v^\sharp(x))(\tau - t(x)).$$

For future reference, we notice here that (21) and (22) imply

$$(24) \quad \begin{aligned} \xi_{t(x)}(0) &= \varphi(t(x)) - f'(v^\sharp(x))t(x) = x - \frac{f'(v(x))}{D_x f'(v(x))} - f'(v^\sharp(x)) \left[ T - \frac{1}{D_x f'(v(x))} \right] \\ &= x + \frac{f'(v^\sharp(x)) - f'(v(x))}{D_x f'(v(x))} - f'(v^\sharp(x))T. \end{aligned}$$

The proof of the following proposition is omitted, since it follows easily from Propositions 2.3 and 2.4.

**PROPOSITION 2.5.** *Assume that  $v$  is a measurable, left continuous, bounded attainable profile, with finite total variation. Assume that  $v$  suffers from a left contact discontinuity at  $x = \bar{x}$  with right state  $v_R$ . Assume that there exists an open interval  $I \subset ]-\infty, \bar{x}]$  such that*

1.  $\sup I = \bar{x}$ ;

- 2.  $v \in C^0(I)$ ;
- 3. the function  $I \ni x \mapsto f'(v(x))$  is differentiable and

$$D_x f'(v(x)), D^+(D_x f'(v(x))) > 0 \quad \text{for any } x \in I.$$

Moreover, let  $\eta_{\bar{x}}^+(\cdot)$  be defined as at (15b) with  $x = \bar{x}$ ,  $\xi_{t(x)}(\cdot)$  be defined as at (23), and let

$$(25) \quad \begin{aligned} \tilde{x} &= \sup \{x \in I : \xi_{t(x)}(0) \leq \eta_{\bar{x}}^+(0)\} \\ &= \sup \left\{ x \in I : x + \frac{f'(v^\sharp(x)) - f'(v(x))}{D_x f'(v(x))} - f'(v^\sharp(x))T \leq \bar{x} - f'(v_R)T \right\} \end{aligned}$$

(see Figure 3). Then, the function  $x \mapsto \xi_{t(x)}(0)$  is increasing in  $I \setminus ]\tilde{x}, \bar{x}[$ .

**Remark 3.** In the framework of Proposition 2.5, some properties of the attainable profile  $v$  in the interval  $x \in ]\tilde{x}, \bar{x}[$  will be discussed in section 5. We point out that Proposition 2.5 states that, if a second left contact appears at the left of  $x = \varphi(t)$ , then it is necessarily located at the left of  $x = \eta_{\bar{x}}^+(t)$ .

**3. Sufficient conditions for attainability.** We start by formulating the main result of this paper, which provides a partial description of the set  $\mathcal{U}_T$ . The meaning of the most technical hypotheses is explained in the three remarks following the statement.

**THEOREM 3.1.** *Let  $v \in BV(\mathbb{R})$  be a left continuous function with a nowhere dense set of points of jump discontinuities. Let  $[\alpha_0, \beta_0]$  be the support of  $v$  and let  $\alpha$  be defined as at (14). Assume that the following conditions hold.*

- 1.  $v$  fulfills conditions 1–4 of Theorem 2.1.
- 2. We consider the partition of  $[\alpha_0, \beta_0]$  into maximal subintervals in which  $v$  is continuous. For any of these maximal subintervals, say  $I$ , the following conditions are fulfilled. Define

$$(26) \quad J \doteq \{x \in I : D_x f'(v(x)) > 1/T\}.$$

Assume that either  $J$  is an empty set, or that the following conditions hold:

- (a)  $J$  is a subinterval of  $I$  and  $\sup J = \sup I = \bar{x}$  is a point of jump of  $v$ . We let  $v_L$  and  $v_R$  be the left and right states of  $v$  at  $\bar{x}$ , respectively.
- (b)  $v_L \doteq v(\bar{x})$  is different from zero.
- (c) Let  $v^\sharp(x)$  be  $(v(x))^\sharp$  as it is defined by formula (8). We have

$$(27) \quad D^+(D_x f'(v(x))) > 0 \quad \forall x \in J,$$

$$(28) \quad D^+ \left[ x + \frac{f'(v^\sharp(x)) - f'(v(x))}{D_x f'(v(x))} - f'(v^\sharp(x))T \right] \geq 0 \quad \forall x \in J.$$

- (d) Let  $(v_L^\sharp)^\sharp$  be as  $v^\sharp$  in (8) with  $v = v_L^\sharp$ . If  $v_R \in I(v_L, (v_L^\sharp)^\sharp)$ , there holds

$$(29) \quad \frac{(v_L^\sharp - v_R)[f'(v_L) - f'(v_L^\sharp)]}{f(v_R) - f(v_L) - (v_R - v_L)f'(v_L)} \left[ T - \frac{1}{D_x^- f'(v(\bar{x}))} \right] \leq T.$$

Then,  $v$  is an attainable profile, i.e., there exists  $u_c \in L^\infty(a, b)$  such that the weak entropy solution  $u = u(t, x)$  to (1a)–(1b) satisfies  $u(T, \cdot) = v$ .

*Remark 4.* If  $J$  is empty, all the candidate backward characteristics  $\eta_x$  for  $x \in I$  are classical characteristics and reach  $t = 0$  without crossing each other. The backward reconstruction of  $v$  in this case is known, as it only contains genuine shocks and rarefaction waves; see also Proposition 2.2. When  $J$  is not empty the backward reconstruction of  $v$  contains a left contact discontinuity.

Conditions 2(a) and 2(b) are actually necessary conditions. In particular, 2(b) is related to a property stated by Dafermos in [10, Theorem 2.2]. Indeed, if  $v_L = 0$ , the minimal backward characteristic from  $(t, \bar{x})$  is the straight line  $x = \bar{x}$  and no crossing of characteristics occurs. This is in contradiction with the hypothesis that  $J$  is not empty.

*Remark 5.* Condition 2(c) tells us that the family of candidate backward characteristics generated from the points  $(T, x)$ , for  $x \in J$ , admits a convex tangent envelope  $\tau \mapsto \varphi(\tau)$  and that the candidate maximal backward characteristics  $\xi_{t(x)}$  defined as at (23) do not cross at positive times. Indeed, condition (28) means that the function  $x \mapsto \xi_{t(x)}(0)$  (see (24)), is increasing for all  $x \in J$ . In other words this condition excludes the occurrence of nesting phenomena. We postpone to section 5 the analysis of some situations in which nesting takes place.

*Remark 6.* In the setting of condition 2(d), namely, if  $v_R \in I(v_L, (v_L^\sharp)^\sharp)$ , the discontinuity at  $x = \bar{x}$  in the target profile  $v$  is not a left contact but a shock. However, given the structure of the candidate backward characteristics ( $J$  is nonempty), we expect that in any solution  $u$  of (1a) attaining  $v$  at time  $T$ , such a shock originates from an interaction at  $\tau < T$  involving a left contact discontinuity. Therefore we stress that  $D_x^- f'(v(\bar{x}))$  is finite, as we are not in the same setting as Remark 2, and being that  $D_x^- f'(v(\bar{x})) \geq 1/T$ , condition (29) implies

$$\frac{(v_L^\sharp - v_R)[f'(v_L) - f'(v_L^\sharp)]}{f(v_R) - f(v_L) - (v_R - v_L)f'(v_L)} \geq 1.$$

The precise form of inequality (29) comes from the construction in the proof of Lemma 3.4 below.

**3.1. The building blocks of our construction.** This section is devoted to the detailed analysis of three basic cases in which the existence of a control function  $u_c$  can be easily proved. These cases are used as building blocks in the proof of Theorem 3.1. Indeed, under the assumptions of Theorem 3.1 the nesting effect cannot be observed; see Remark 5. Therefore, it is possible to split the support of  $v$  into subintervals  $I_k$ , show the existence of controls  $u_c^k$  which allow us to attain the truncated functions  $v|_{I_k}$ , and finally “glue” together the functions  $u_c^k$  to obtain a control  $u_c$  such that the weak entropy solution  $u = u(t, x)$  to (1a)–(1b) satisfies  $u(T, \cdot) = v$ . This argument works because under the assumptions of Theorem 3.1 the domain of dependence of  $u(T, \cdot)$  on each interval  $I_k$  is delimited by the maximal backward generalized characteristics from  $t = T$ ,  $x = \inf I_k$  and  $t = T$ ,  $x = \sup I_k$ .

For the reader’s convenience, we fix here the notation and the setting which will be used all through this section. Roughly speaking, we isolate one of the intervals  $I_k$ .

(V) Let  $v \in \text{BV}(\mathbb{R})$  satisfy all the hypotheses of Theorem 3.1 together with the following stronger assumptions. Let  $v$  be left continuous, compactly supported on  $]\alpha_0, \beta_0[$ , and suffering of a single jump discontinuity at the point  $x = \bar{x}$ , connecting the values  $v(\bar{x}) = v_L$  and  $v_R$ . For definiteness we assume that  $v_L < 0$  (the case  $v_L > 0$  is symmetric) and that  $v(x) = v_R$  for

all  $x$  in  $]\bar{x}, \beta_0[$ .<sup>3</sup> We assume, as prescribed by condition 1 of Theorem 3.1, that the jump  $(v_L, v_R)$  is admissible in the sense of condition 4 of Theorem 2.1. Using the same notation as in Theorem 3.1 we set  $I \doteq ]\alpha_0, \bar{x}[$  and  $J \doteq \{x \in I : D_x f'(v(x)) > 1/T\}$ . If  $J$  is not empty, we define

$$(30) \quad \hat{x} \doteq \inf J,$$

while  $\sup J$  has to be  $\bar{x}$  according to condition 2(a) of Theorem 3.1.

We recall that the (candidate) backward characteristic lines  $\eta_x^+, \eta_x^-$ , and  $\eta_x$  have been introduced at (15). Whenever  $J$  is nonempty, condition (27) ensures that the tangent envelope of the lines  $\tau \mapsto \eta_x(\tau)$  for  $x \in J$ , is a convex curve  $x = \varphi(\tau)$ , defined for  $\tau \in [t_0, t_1]$  with  $t_1 = t(\bar{x})$  (see (21)) given by

$$(31) \quad \begin{aligned} t_1 &= \sup \{t \leq T : \eta_x(t) = \eta_{\bar{x}}(t), x \in J, x < \bar{x}\} \\ &= \sup_{\substack{x \in I \\ x < \bar{x}}} \left\{ T - \frac{x - \bar{x}}{f'(v(x)) - f'(v_L)} \right\} = T - \frac{1}{D_x^- f'(v(\bar{x}))}, \end{aligned}$$

and  $t_0$  given by

$$(32) \quad t_0 = \max\{t \geq 0 \text{ such that } \varphi'(t) = f'(v(\hat{x}))\}.$$

The presentation articulates into three lemmas, focusing, respectively, on the following.

*Case 1.* The profile  $v$  suffers from a shock discontinuity at  $x = \bar{x}$  and the candidate backward characteristics  $\eta_x$ , for  $x \in I$  do not cross in the time interval  $]0, T]$ .

*Case 2.* The set  $J$  is not empty, therefore, the candidate backward characteristics  $\eta_x$  for  $x \in J$  cross in positive time. The state  $v_R$  does not belong to the open interval with extrema  $v_L^\sharp$  and  $(v_L^\sharp)^\sharp$ . This means that the profile  $v$  suffers either from a left contact discontinuity at  $x = \bar{x}$ , i.e.,  $v_R = (v_L)^\sharp$ , or the discontinuity of  $v$  at  $\bar{x}$  can be solved backward as an interaction occurring at time  $\tau = T$  and involving a left contact discontinuity.

*Case 3.* The set  $J$  is not empty but the state  $v_R$  lies in-between  $v_L$  and  $(v_L^\sharp)^\sharp$ . Therefore, on the one hand, the jump of  $v$  at  $\bar{x}$  can only be interpreted as a shock; on the other hand, the crossing of backward characteristics forces the presence of a left contact discontinuity in any entropy weak solution  $u$  such that  $u(T) = v$ . This construction is the most intricate.

We stress that in Cases 1, 2, and 3 we can explicitly construct a control function  $u_c$ .

LEMMA 3.2 (Case 1). *Assume that  $v$  is as in (V) and that  $J$  is empty. Then one can explicitly provide an initial condition  $u_1 \in L^\infty(\eta_{\alpha_0}^+(0), \eta_{\beta_0}^-(0))$ , and a weak entropy solution  $u$  (see (34) below), to*

$$(33) \quad \begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(0, x) = u_1(x), \end{cases} \quad 0 \leq t \leq T, \quad \eta_{\alpha_0}^+(t) < x < \eta_{\beta_0}^-(t),$$

such that  $u(T, x) = v(x)$  a.e. in  $]\alpha_0, \beta_0[$ .

*Proof.* In this case we can explicitly construct a backward solution  $u$ , hence a control  $u_1$ , in the following classical way. The shock wave joining the states  $v_L$  and

<sup>3</sup>Please notice that the constructions we perform are essentially the same under the (slightly) weaker hypothesis that the set  $J_{x > \bar{x}} \doteq \{x \in ]\bar{x}, \beta_0[: D_x f'(v(x)) > 1/T\}$  is empty.

$v_R$  propagates along the line

$$\mathfrak{s}(\tau) = \bar{x} + \sigma(v_L, v_R)(\tau - T),$$

where

$$\sigma(v_L, v_R) = \frac{f(v_R) - f(v_L)}{v_R - v_L}$$

is the Rankine–Hugoniot speed of the jump  $(v_L, v_R)$ . Then

$$(34) \quad u(\tau, y) = \begin{cases} v(x) & \text{if } \eta_{\alpha_0}^+(\tau) < y = \eta_x(\tau) \leq \eta_{\bar{x}}^-(\tau), \tau \in ]0, T[, \\ v_L & \text{if } \eta_{\bar{x}}(\tau) < y \leq \mathfrak{s}(\tau), \\ v_R & \text{if } \mathfrak{s}(\tau) < y < \eta_{\beta_0}^-(\tau), \end{cases}$$

that is obtained just following up to  $t = 0$  the backward characteristics. Please notice that in this case the backward reconstruction is far from being unique.  $\square$

Now we turn our attention to the most interesting case  $J \neq \emptyset$ .

LEMMA 3.3 (Case 2). *Assume that  $v$  is as in  $(\mathbf{V})$ ,  $J$  is not empty, and  $v_R$  does not belong to the open interval of extrema  $v_L^\sharp$  and  $(v_L^\sharp)^\sharp$ . Then one can explicitly provide an initial condition  $u_2 \in L^\infty(\eta_{\alpha_0}^+(0), \eta_{\beta_0}^-(0))$ , and a weak entropy solution  $u$  (see (35)–(36c) below) to*

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(0, x) = u_2(x), \end{cases} \quad 0 \leq t \leq T, \quad \eta_{\alpha_0}^+(t) < x < \eta_{\beta_0}^-(t),$$

such that  $u(T, x) = v(x)$  a.e. in  $]\alpha_0, \beta_0[$ .

*Proof.* As we assume that  $v_L < 0$ , either  $0 < v_L^\sharp \leq v_R$  or  $v_R \leq (v_L^\sharp)^\sharp < 0$ . If  $v_R = v_L^\sharp$  the discontinuity of  $v$  at  $x = \bar{x}$  is a left contact discontinuity. If  $v_R = (v_L^\sharp)^\sharp$  the discontinuity of  $v$  at  $x = \bar{x}$  can be solved backward as an interaction occurring at time  $T$  between the left contact discontinuities from  $v_L$  to  $v_L^\sharp$  and from  $v_L^\sharp$  to  $(v_L^\sharp)^\sharp$ . In all other cases the discontinuity of  $v$  at  $x = \bar{x}$  can be solved backward as an interaction occurring at time  $T$  between the left contact discontinuity from  $v_L$  to  $v_L^\sharp$  and a shock joining  $v_L^\sharp$  to  $v_R$ ; see Figure 4.

Once we fixed the structure of a backward resolution of the discontinuity, the construction of a control becomes routine. Indeed, assumption (28) ensures that nesting does not occur, therefore, the backward reconstruction of the solution can be done up to  $t = 0$  without the appearance of unexpected waves.

In order to keep our presentation as light as possible, we limit the discussion to the case  $v_R \notin \{v_L^\sharp, (v_L^\sharp)^\sharp\}$ , as the two other cases can easily be inferred from this one.

Call  $\mathfrak{s}$  the line along which the shock joining  $v_L^\sharp$  to  $v_R$  propagates in the backward solution,

$$\mathfrak{s}(\tau) = \bar{x} + \sigma(v_L^\sharp, v_R)(\tau - T),$$

where  $\sigma(v_L^\sharp, v_R)$  is the Rankine–Hugoniot speed of the jump  $(v_L^\sharp, v_R)$ . We fix the values of  $t_0$  and  $t_1$  as in (32) and (31). A backward solution to (1a) with datum



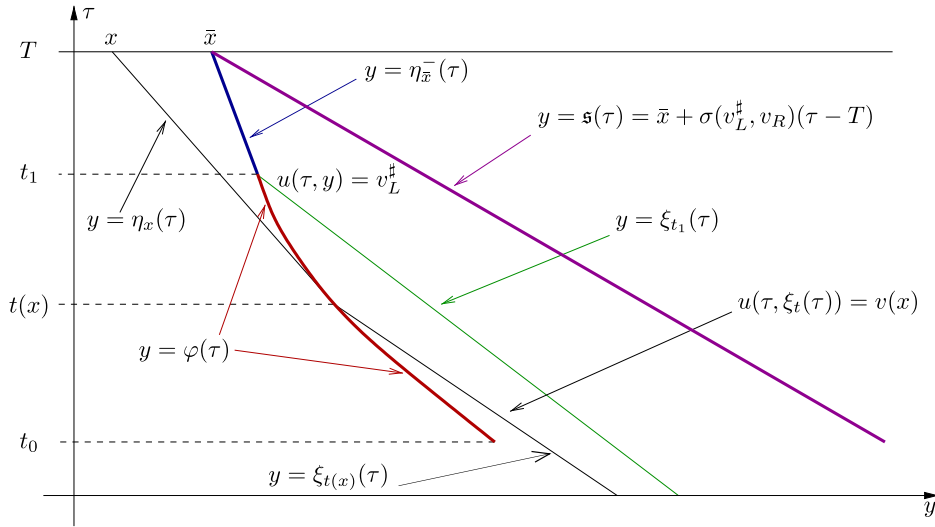


FIG. 4. The function  $u$  at (35).

$u(T, x) = v(x)$  takes the form

$$(35) \quad u(\tau, y) = \begin{cases} v(x) & \text{if } (\tau, y = \eta_x(\tau)) \in \Omega_1, \\ (v(x))^\# & \text{if } (\tau, y = \xi_{t(x)}(\tau)) \in \Omega_2, \\ v_L^\# & \text{if } (\tau, y) \in \Omega_3, \\ v_R & \text{if } y \in ]\mathfrak{s}(\tau), \eta_{\bar{x}}^-(\tau)[, \tau \in ]0, T[, \end{cases}$$

where the regions  $\Omega_i$  for  $i = 1, 2, 3$ , are defined as follows:

$$(36a) \quad \Omega_1 = \{(\tau, y) \in ]t_1, T[ \times ]\eta_{\alpha_0}^+(\tau), \eta_{\bar{x}}^-(\tau)[\} \cup \{(\tau, y) \in ]t_0, t_1[ \times ]\eta_{\alpha_0}^+(\tau), \varphi(\tau)[\} \cup \{(\tau, y) \in ]0, t_0[ \times ]\eta_{\alpha_0}^+(\tau), \eta_{\bar{x}}(\tau)[\},$$

$$(36b) \quad \Omega_2 = \{(\tau, y) \in ]t_0, t_1[ \times ]\varphi(\tau), \xi_{t_1}(\tau)[\} \cup \{(\tau, y) \in ]0, t_0[ \times ]\eta_{\bar{x}}(\tau), \xi_{t_1}(\tau)[\},$$

$$(36c) \quad \Omega_3 = \{(\tau, y) \in ]t_1, T[ \times ]\eta_{\bar{x}}^-(\tau), \mathfrak{s}(\tau)[\} \cup \{(\tau, y) \in ]0, t_1[ \times ]\xi_{t_1}(\tau), \mathfrak{s}(\tau)[\}.$$

Then  $u_2$  can be constructed, just following backward the backward characteristics up to  $t = 0$ .  $\square$

LEMMA 3.4 (Case 3). Assume that  $v$  is as in (V),  $J$  is not empty, and  $v_R \in I(v_L, (v_L^\#)^\#)$ . Then there exists an initial condition  $u_3 \in L^\infty(\eta_{\alpha_0}^+(0), \eta_{\beta_0}^-(0))$ , such that the weak entropy solution  $u$  to

$$(37) \quad \begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(0, x) = u_3(x), \end{cases} \quad 0 \leq t \leq T, \quad \eta_{\alpha_0}^+(t) < x < \eta_{\beta_0}^-(t),$$

satisfies  $u(T, x) = v(x)$  a.e. in  $]\alpha_0, \beta_0[$ .

Remark 7. Regarding Lemma 3.4, we provide at the end of the proof an explicit formulation of the initial datum  $u_3$  and of the solution  $u$  to (37) attaining  $v$  at time  $T$ .

*Proof.* Our first step in this proof is to find a wave pattern which is compatible with the information we have on the profile  $v$ . Here the jump of  $v$  at  $\bar{x}$  can only be interpreted as a shock and, since  $J \neq \emptyset$ , we can argue the presence of a left contact discontinuity  $\varphi$  in any entropy weak solution  $u$  such that  $u(T) = v$ . It is well known (see [10, Theorem 2.3]) that once a left contact discontinuity appears in a weak entropy solution of (1a), it cannot disappear but can interact with other waves. Therefore we conclude that the shock jump in  $v$  at  $x = \bar{x}$  corresponds to a shock wave  $\mathfrak{s}_3$  in  $u$  originated from the interaction between  $\varphi$  and another discontinuity  $\mathfrak{s}_0$ . The idea, roughly speaking, is the following. First, we use the values of  $v$  for  $x < \bar{x}$  to perform a partial backward reconstruction of the left contact discontinuity  $\varphi$  and, in particular, find the time  $t_1$  at which the candidate minimal backward characteristic from  $(T, \bar{x})$  is tangent to  $\varphi$ ; see (31). Second, we trace the maximal backward characteristic  $\xi_{t_1}$  from  $(t_1, \varphi(t_1))$  and we call  $x_2$  the point  $\xi_{t_1}(0)$ . We also trace the maximal backward characteristic  $\eta_{\bar{x}}^+$  from  $(T, \bar{x})$  and we call  $(t^*, x^*)$  the intersection between  $\xi_{t_1}$  and  $\eta_{\bar{x}}^+$ ,

$$(38) \quad \xi_{t_1}(t^*) = \eta_{\bar{x}}^+(t^*) = x^*.$$

The condition  $t^* \leq 0$  turns out to be necessary for the attainability of  $v$  (see below).

Finally, we construct a one parameter family of initial conditions  $(w_\gamma^0)_{\gamma \in [t^*, 0]}$  such that

1. the function  $\gamma \mapsto w_\gamma^0$  is continuous;
2. all corresponding forward solutions  $w_\gamma$  to (1a) have the desired wave pattern (incoming  $\varphi$  and  $\mathfrak{s}_0$ , interaction, outgoing  $\mathfrak{s}_3$ );
3. there exists a value  $\gamma^* \in [t^*, 0]$  such that the solution  $w_{\gamma^*}$  associated with  $w_{\gamma^*}^0$  attains exactly the profile  $v$  at time  $T$ .

The rigorous presentation of the proof is divided into several steps.

*Step 1: Partial backward reconstruction.* We consider the case  $(v_L^\sharp)^\sharp < v_R < v_L < 0$ . With exactly the same notation as in the proof of Lemma 3.3 we use the method of characteristics to reconstruct the solution backward in the regions  $\Omega_1$  and  $\Omega_2$ ; see (36a), (36b), and Figure 5.

We have

$$(39) \quad x_1 \doteq \eta_{\bar{x}}^-(t_1) = \bar{x} + f'(v_L)(t_1 - T),$$

$$(40) \quad x_2 \doteq \xi_{t_1}(0) = x_1 - f'(v_L^\sharp)t_1.$$

*Step 2: The Riemann problem with data  $(v_L^\sharp, v_R)$ .* In this step we introduce some notations and describe the self-similar solution to a Riemann problem with data  $(v_L^\sharp, v_R)$  centered at  $(t = \mathfrak{t}, x = \mathfrak{a})$ :

$$(41) \quad \begin{cases} u_t + f(u)_x = 0 & \text{for } t > \mathfrak{t} \text{ and } x \in \mathbb{R}, \\ u(\mathfrak{t}, x) = \begin{cases} v_L^\sharp & \text{if } x \leq \mathfrak{a}, \\ v_R & \text{if } x > \mathfrak{a}. \end{cases} \end{cases}$$

Observe that our assumptions on  $f$  and the choice  $v_L < 0$  imply  $0 < v_R^b < v_L^\sharp$ . Therefore, the solution to the problem (41) consists of a centered rarefaction wave with left state  $v_L^\sharp$  and right state  $v_R^b$ , followed by a left contact discontinuity joining  $v_R^b$  to  $v_R$ ,  $\tau \mapsto \mathfrak{s}_0(\tau)$ . More precisely, we have

$$(42) \quad \begin{aligned} \mathfrak{s}_0(\tau) &= \mathfrak{a} + f'(v_R^b)(\tau - \mathfrak{t}) \\ &= \mathfrak{a} + \frac{f(v_R) - f(v_R^b)}{v_R - v_R^b} (\tau - \mathfrak{t}), \end{aligned} \quad \tau \geq \mathfrak{t},$$



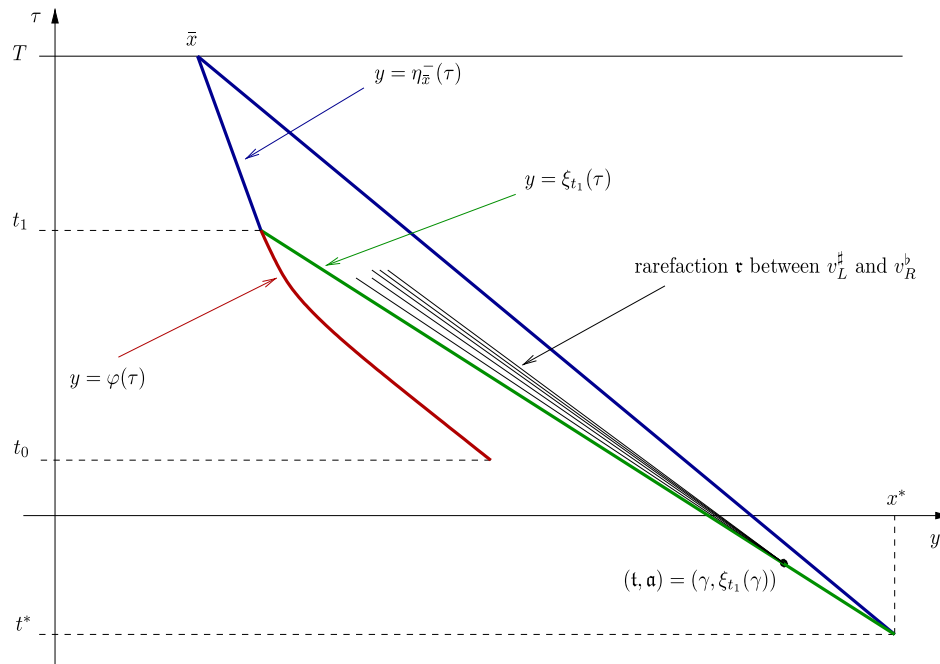


FIG. 6. The construction of a family of initial conditions described in Step 3.

*Step 4: Forward construction of  $w_\gamma$  for  $\tau > t_1$ .* For the reader's convenience, we point out that Figure 7 illustrates the procedure described in this paragraph in the case  $\gamma = 0$ . For  $\tau > t_1$  the contact discontinuity  $\varphi$  interacts with the rarefaction fan  $\mathbf{r}$ . The result of this interaction is a contact discontinuity with left state  $\mathbf{v}^b(\tau, \mathbf{s}_1(\tau))$  and right state  $\mathbf{r}(\tau, \mathbf{s}_1(\tau))$ , traveling along the curve  $\tau \mapsto \mathbf{s}_1(\tau)$ , defined as the unique solution of the Cauchy problem

$$(44) \quad \begin{cases} \mathbf{s}'_1 = f'(\mathbf{v}^b(\tau, \mathbf{s}_1)), \\ \mathbf{s}_1(t_1) = x_1, \end{cases} \quad \tau \geq t_1.$$

Observe that, thanks to (11), we have

$$\mathbf{s}'_1(\tau) = \frac{f(\mathbf{v}^b(\tau, \mathbf{s}_1(\tau))) - f(\mathbf{r}(\tau, \mathbf{s}_1(\tau)))}{\mathbf{v}^b(\tau, \mathbf{s}_1(\tau)) - \mathbf{r}(\tau, \mathbf{s}_1(\tau))}.$$

Geometrical considerations show that the curve  $\mathbf{s}_1$  approaches  $\mathbf{s}_0$  until  $\mathbf{r}$  attains the value  $v_R^b$ . Call  $t_3$  the time at which the interaction takes place, i.e.,  $\mathbf{s}_1(t_3) = \mathbf{s}_0(t_3)$ . Let

$$(45) \quad x_3 \doteq \mathbf{s}_1(t_3) = \mathbf{s}_0(t_3) = \xi_{t_1}(\gamma) + f'(v_R^b)t_3,$$

and observe that, by construction,

$$\eta_{\bar{x}}^-(\tau) < \mathbf{s}_1(\tau) < \mathbf{s}_0(\tau) \quad \forall t_1 < \tau < t_3.$$

For future reference, we call  $\Omega_4$  the region in the  $t$ - $x$  plane delimited by  $\xi_{t_1}$ ,  $\mathfrak{s}_0$ , and  $\mathfrak{s}_1$ ,

$$(46) \quad \Omega_4 = \{(\tau, y) \in \mathbb{R}^2 : \tau \in ]\gamma, t_1], \xi_{t_1}(\tau) < y \leq \mathfrak{s}_0(\tau)\} \\ \cup \{(\tau, y) \in \mathbb{R}^2 : \tau \in ]t_1, t_3], \mathfrak{s}_1(\tau) < y \leq \mathfrak{s}_0(\tau)\} .$$

The interaction between  $\mathfrak{s}_0$  and  $\mathfrak{s}_1$ , taking place at  $(t_3, x_3)$ , generates a single outgoing shock discontinuity traveling along the curve  $\tau \mapsto \mathfrak{s}_3(\tau)$ . We use again the method of characteristics to define precisely the definition of  $\mathfrak{s}_3$ . For any  $(\tau, y)$  such that

- either  $\tau \in [t_1, t_3]$  and  $\eta_{\bar{x}}^-(\tau) \leq y \leq \mathfrak{s}_1(\tau)$ ,
- or  $\tau > t_3$  and  $\eta_{\bar{x}}^-(\tau) \leq y < \mathfrak{s}_0(\tau) = x_3 + f'(v_R^b)(\tau - t_3)$ ,

let  $s = s(\tau, y) \in [t_1, t_3]$  be such that

$$y = \mathfrak{s}_1(s) + \mathfrak{s}'_1(s)(\tau - s) ,$$

and define

$$\mathfrak{q}(\tau, y) = \mathfrak{r}^b(s(\tau, y), \mathfrak{s}_1(s(\tau, y))) .$$

In such a way  $\mathfrak{q} = \mathfrak{q}(\tau, y)$  is a Lipschitz continuous solution to (1a) within its domain of definition. Then, we let  $\tau \mapsto \mathfrak{s}_3(\tau)$  be the solution to the Cauchy problem

$$(47) \quad \begin{cases} \mathfrak{s}'_3(\tau) = \frac{f(v_R) - f(\mathfrak{q}(\tau, \mathfrak{s}_3(\tau)))}{v_R - \mathfrak{q}(\tau, \mathfrak{s}_3(\tau))} , \\ \mathfrak{s}_3(t_3) = x_3 , \end{cases}$$

so that  $y = \mathfrak{s}_3(\tau)$  is a shock curve joining the left state  $\mathfrak{q}(\tau, \mathfrak{s}_3(\tau))$  to the right state  $v_R$ .

The solution to (47) is defined starting from  $\tau = t_3$  until the point  $(\tau, \mathfrak{s}_3(\tau))$  reaches the line  $\eta_{\bar{x}}^-$ . This happens in finite time since the slope of  $\eta_{\bar{x}}^-$  is larger than  $\mathfrak{s}'_3(\tau)$  for all  $\tau > t_3$ . We call  $\bar{t}$  the time at which the crossing takes place. This allows for the following definition

$$(48) \quad \Omega_5 = \{(\tau, y) \in \mathbb{R}^2 : \tau \in ]t_1, t_3], \eta_{\bar{x}}^-(\tau) < y < \mathfrak{s}_1(\tau)\} \\ \cup \{(\tau, y) \in \mathbb{R}^2 : \tau \in ]t_3, \bar{t}], \eta_{\bar{x}}^-(\tau) < y < \mathfrak{s}_3(\tau)\} .$$

The solution of the Cauchy problem (1a)–(43),  $w_\gamma$ , is the piecewise Lipschitz continuous function defined as follows for  $\tau \in ]0, \bar{t}]$  (see Figure 7 for the case  $\gamma = 0$ ),

$$(49) \quad w_\gamma(\tau, y) = \begin{cases} v(x) & \text{if } (\tau, y = \eta_x(\tau)) \in \Omega_1 , \\ (v(x))^\# & \text{if } (\tau, y = \xi_{t(x)}(\tau)) \in \Omega_2 , \\ \mathfrak{r}(\tau, y) & \text{if } (\tau, y) \in \Omega_4 , \\ \mathfrak{q}(\tau, y) & \text{if } (\tau, y) \in \Omega_5 , \\ v^R & \text{if } y \in ]\mathfrak{s}_0(\tau), \eta_{\beta_0}^-(\tau)] \text{ and } \tau \leq t_3 , \\ & \text{or } y \in ]\mathfrak{s}_3(\tau), \eta_{\beta_0}^-(\tau)] \text{ and } \tau \in ]t_3, \bar{t}] . \end{cases}$$

The lemma is proved if we can show that there exists a value of  $\gamma$  for which  $\bar{t}$  is exactly equal to  $T$ . Observe that, as we pointed out in Remark 5, the hypotheses of the lemma exclude the presence of a second left contact discontinuity in the region  $\Omega_2$ ,

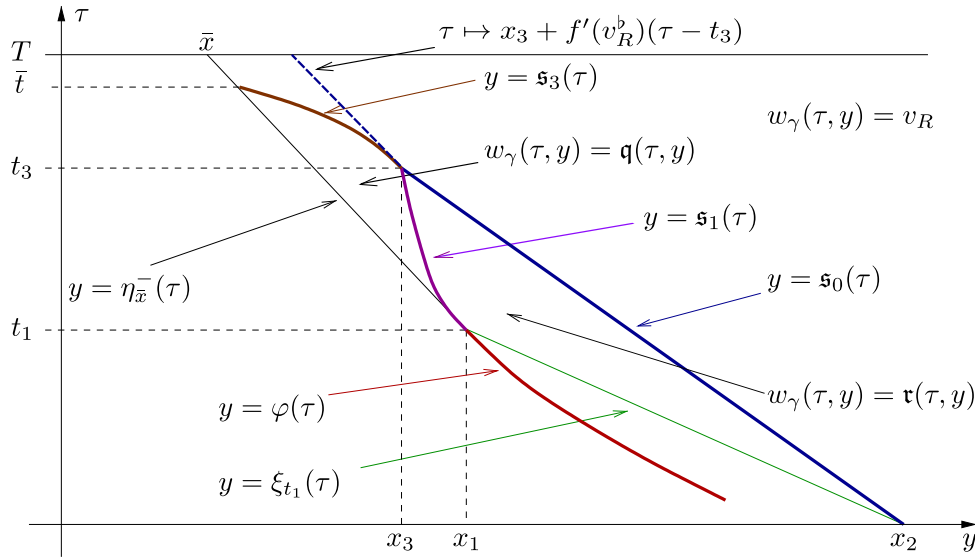


FIG. 7. The case  $(v_L^\#)^\# < v_R < v_L < 0$ .

as the backward characteristics issued from the curve  $\phi$  reach  $t = 0$  before intersecting each other. For this reason we are not allowed to consider initial conditions suffering from a jump discontinuity at a point  $y \in ]\eta_{\bar{x}}(0), x_2[$ . Therefore, when we consider  $\gamma = 0$  we obtain the solution of (1a) with the desired wave pattern (incoming  $\varphi$  and  $s_0$ , interaction, outgoing  $s_3$ ) in which the curves  $s_3$  and  $\eta_{\bar{x}}$  intersect earlier than for any larger value of  $\gamma$ .

*Remark 8.* We remark that a necessary condition toward the attainability of the profile  $v$  is  $t^* < 0$ , i.e.,

$$(50) \quad x_2 < \eta_{\bar{x}}^+(0) = \bar{x} - f'(v_R)T.$$

Indeed, assume (50) does not hold (see Figure 8). Since  $f'(v_R^b) > f'(v_R)$ , we get

$$(51) \quad x_3 > \bar{x} + f'(v_R)(t_3 - T).$$

Notice that  $s_3'(\tau) > f'(v_R)$  for any  $\tau$  due to the admissibility conditions for shock discontinuities, and that  $v_R$  is the right state of  $s_3$ . Due to (51) we have  $s_3(T) > \bar{x}$ . Hence,  $s_3$  and  $\eta_{\bar{x}}$  do not interact in the time interval  $[t_3, T]$ , contrary to the fact that  $\bar{t} \leq T$ . A similar argument shows that if  $\gamma = t^*$ , then, for sure,  $\bar{t} > T$ .

The next step in this proof shows that if our original target profile satisfies the hypothesis (29) of Theorem 3.1, then in  $w_{\gamma=0}$ ,  $s_3(\bar{t}) = \eta_{\bar{x}}(\bar{t})$  at time  $\bar{t} < T$ .

*Step 5:* Under assumption (29), in  $w_{\gamma=0}$  there holds  $\bar{t} < T$ . Now we use conservation in the region  $\Omega_5$ , defined at (48), in order to determine  $\bar{t}$ . We have

$$\iint_{\Omega_5} \partial_t u + \partial_x f(u) \, dxdt = 0,$$

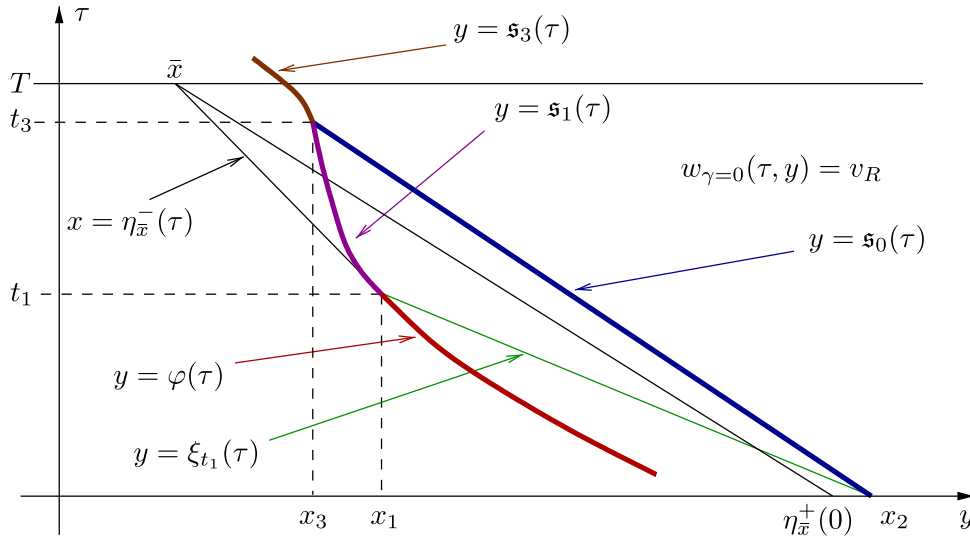


FIG. 8. Remark 8.

so that, by the divergence theorem,

$$(52a) \quad 0 = \int_{t_3}^{\bar{t}} -q(t, s_3(t))s_3'(t) + f(q(t, s_3(t))) dt$$

$$(52b) \quad + \int_{t_1}^{t_3} -q(t, s_1(t))s_1'(t) + f(q(t, s_1(t))) dt$$

$$(52c) \quad + \int_{t_1}^{\bar{t}} q(t, \eta_x^-(t))(\eta_x^-)'(t) - f(q(t, \eta_x^-(t))) dt.$$

Let us compute the three integrals in (52). Regarding (52a), since Rankine–Hugoniot conditions hold along  $x = s_3(t)$ , we get

$$-q(t, s_3(t))s_3'(t) + f(q(t, s_3(t))) = -v_R s_3'(t) + f(v_R) \quad \forall t_3 < t < \bar{t},$$

and, hence,

$$\begin{aligned} & \int_{t_3}^{\bar{t}} -q(t, s_3(t))s_3'(t) + f(q(t, s_3(t))) dt \\ &= -v_R (s_3(\bar{t}) - x_3) + f(v_R)(\bar{t} - t_3) \\ &= -v_R (\bar{x} - x_3) + v_R f'(v_L)T - f(v_R)t_3 + [f(v_R) - v_R f'(v_L)]\bar{t}. \end{aligned}$$

Now we compute (52b). Since Rankine–Hugoniot conditions hold along  $s_1$ , we get

$$-q(t, s_1(t))s_1'(t) + f(q(t, s_1(t))) = -r(t, s_1(t+))s_1'(t) + f(r(t, s_1(t+))) \quad \forall t \in ]t_1, t_3[.$$

Using, again, conservation in the region  $\Omega_4$  (see (46)), we get

$$\begin{aligned} & \int_{t_1}^{t_3} -\mathbf{q}(t, \mathbf{s}_1(t))\mathbf{s}'_1(t) + f(\mathbf{q}(t, \mathbf{s}_1(t))) dt \\ &= \int_{t_1}^{t_3} -\mathbf{r}(t, \mathbf{s}_1(t+))\mathbf{s}'_1(t) + f(\mathbf{r}(t, \mathbf{s}_1(t+))) dt \\ &= \int_{\gamma}^{t_3} -\mathbf{r}(t, \mathbf{s}_0(t))\mathbf{s}'_0(t) + f(\mathbf{r}(t, \mathbf{s}_0(t))) dt \\ & \quad + \int_{\gamma}^{t_1} \mathbf{r}(t, \xi_{t_1}(t+))\xi'_{t_1}(t) - f(\mathbf{r}(t, \xi_{t_1}(t+))) dt, \end{aligned}$$

and being that

$$\begin{aligned} \int_{\gamma}^{t_3} -\mathbf{r}(t, \mathbf{s}_0(t))\mathbf{s}'_0(t) + f(\mathbf{r}(t, \mathbf{s}_0(t))) dt &= \int_{\gamma}^{t_3} -v_R^b \mathbf{s}'_0(t) + f(v_R^b) dt \\ &= [-v_R^b f'(v_R^b) + f(v_R^b)](t_3 - \gamma), \\ \int_{\gamma}^{t_1} \mathbf{r}(t, \xi'_{t_1}(t+))\xi'_{t_1}(t) - f(\mathbf{r}(t, \xi'_{t_1}(t+))) dt &= \int_{\gamma}^{t_1} v_L^\sharp \xi'_{t_1}(t) - f(v_L^\sharp) dt \\ &= [v_L^\sharp f'(v_L^\sharp) - f(v_L^\sharp)](t_1 - \gamma), \end{aligned}$$

and using Rankine–Hugoniot conditions along  $\mathbf{s}_0$ , we obtain

$$\begin{aligned} & \int_{t_1}^{t_3} -\mathbf{q}(t, \mathbf{s}_1(t))\mathbf{s}'_1(t) + f(\mathbf{q}(t, \mathbf{s}_1(t))) dt \\ &= [-v_R f'(v_R^b) + f(v_R)](t_3 - \gamma) + [v_L^\sharp f'(v_L^\sharp) - f(v_L^\sharp)](t_1 - \gamma). \end{aligned}$$

Regarding (52c), observe that  $\mathbf{q}(t, \eta_{\bar{x}}^-(t)) = v_L$ , so that we obtain

$$\begin{aligned} \int_{t_1}^{\bar{t}} \mathbf{q}(t, \eta_{\bar{x}}^-(t))(\eta_{\bar{x}}^-)'(t) - f(\mathbf{q}(t, \eta_{\bar{x}}^-(t))) dt &= \int_{t_1}^{\bar{t}} v_L (\eta_{\bar{x}}^-)'(t) - f(v_L) dt \\ &= [v_L f'(v_L) - f(v_L)](\bar{t} - t_1). \end{aligned}$$

From (52), and observing that by construction of  $x_1$ ,  $\xi_{t_1}(\gamma)$ , and  $x_3$ ,

$$x_3 - \bar{x} = \gamma f'(v_L^\sharp) - f'(v_L)T + f'(v_R^b)t_3 + [f'(v_L) - f'(v_L^\sharp)]t_1,$$

we obtain

$$\begin{aligned} (53) \quad & [f(v_R) - f(v_L) - (v_R - v_L)f'(v_L)]\bar{t} \\ & + [f(v_L) - f(v_L^\sharp) + v_R(f'(v_L) - f'(v_L^\sharp)) + v_L^\sharp f'(v_L^\sharp) - v_L f'(v_L)]t_1 \\ & + [v_R f'(v_L^\sharp) + v_R f'(v_R^b) - v_L^\sharp f'(v_L^\sharp) + f(v_L^\sharp) - f(v_R)]\gamma = 0. \end{aligned}$$

Since

$$f(v_R) - f(v_L) - (v_R - v_L)f'(v_L) \neq 0$$

due to the fact that  $(v_L^\sharp)^\sharp < v_R < v_L$ , and being that

$$f(v_L) - f(v_L^\sharp) = f'(v_L)(v_L - v_L^\sharp),$$



we recover

$$(54) \quad \bar{t} = -\frac{(v_R - v_L^\sharp)[f'(v_L) - f'(v_L^\sharp)]}{f(v_R) - f(v_L) - (v_R - v_L)f'(v_L)}t_1 + \frac{v_R f'(v_L^\sharp) + v_R f'(v_R^\flat) - v_L^\sharp f'(v_L^\sharp) + f(v_L^\sharp) - f(v_R)}{f(v_R) - f(v_L) - (v_R - v_L)f'(v_L)}\gamma.$$

Using the expression of  $t_1$  at (31) and due to (29), we get  $\bar{t} \leq T$  if  $\gamma = 0$ . As  $v_R < 0 < v_R^\flat < v_L^\sharp$  the assumption (F)b ensures that the coefficient in front of  $\gamma$  is negative. Then it is possible to find  $\gamma = \gamma^* \in [t^*, 0]$  in order to get  $\bar{t} = T$ . It follows that  $u_3 = w_\gamma^0$  with  $\gamma = \gamma^*$  in (43), and  $u = w_\gamma$  with  $\gamma = \gamma^*$  in (49) are, respectively, the explicit formulation of an initial datum  $u_3$  and a solution  $u$  to (37) attaining  $v$  at time  $T$ .  $\square$

**3.2. Proof of Theorem 3.1.** Assume that  $v$  satisfies all conditions in the statement of Theorem 3.1. We consider the partition of  $[\alpha_0, \beta_0]$  into maximal subintervals in which  $v$  is continuous, namely,  $[\alpha_0, \beta_0] = \cup_{n \geq 1} I_n$ ,  $I_n = ]x_n, x_{n+1}[$ . Define  $J_n = \{x \in I_n : D_x f'(v(x)) > 1/T\}$ . Consider now the function

$$(55) \quad v_n(x) = \begin{cases} v(x_n^+) & \text{if } \alpha_0 \leq x \leq x_n, \\ v(x) & \text{if } x \in I_n, \\ v(x_{n+1}^+) & \text{if } x_{n+1} < x \leq \beta_0, \end{cases}$$

and observe that  $v_n$  satisfies the hypotheses of one of the three lemmas, Lemmas 3.2, 3.3, or 3.4. Therefore, there exists a control  $u_c^n$  which we can use as initial condition in (1) to attain the profile  $v_n$  at time  $T$ . Under the hypotheses of Theorem 3.1, in particular, in the absence of nesting structure, we are sure that the control function  $u_c^n$  is constant outside the interval  $Y_n = ]\eta_{x_n^+}(0), \eta_{x_{n+1}^+}(0)[$ . Therefore, the function

$$(56) \quad u_c(y) = \sum_{n \geq 0} u_c^n(y) \mathbb{1}_{Y_n}(y)$$

can be used as initial condition in (1) to attain the profile  $v$  at time  $T$ .  $\square$

**4. Application to Kynch’s sedimentation model.** In this section we give an explicit example on the application of Lemma 3.4 to a real life problem. We consider the classical model for sedimentation proposed by Kynch in [17]; see also [6] for an introduction to the model from an historical point of view and a short account of the related literature.

The model describes the sedimentation of the solid part of a suspension in a cylindrical batch of height  $L$ . The unknown function in the equation is the local solid fraction of the suspension,  $u$ , which varies between 0 and a maximal value  $u_{\max}$ . For technical reasons (presence of a suitable mixing device), the only relevant space dimension in the problem is the height above the bottom of the batch,  $x \in [0, L]$ . The *ansatz* used by Kynch is that the velocity of sedimentation,  $V$ , at any level  $x$  depends on the value of the local solid fraction  $u$ , so that the flux function is given by  $f(u) = V(u)u$ . If we write  $t$  for the time variable, the model takes the form of a scalar conservation law in one space dimension,

$$(57) \quad \partial_t u + \partial_x (V(u)u) = 0.$$

Given the parameterization of the batch it is natural to impose that  $f$  is negative for  $u \in [0, u_{\max}]$  and attains zero for  $u = 0$  and  $u = u_{\max}$ . One can find many different constitutive equations for  $V$  in the existing literature. In this example, to allow for explicit computations, we take  $u_{\max} = 1$  and we consider  $V(u) = -(1 - u)^2$ , which leads to the flux function  $f(u) = -u(1 - u)^2$ , so that (57) becomes

$$(58) \quad \partial_t u - \partial_x (u(1 - u)^2) = 0.$$

Such a flux function suffers from an inflection point at  $u = 2/3$ , but it does not satisfy the conditions we fixed in (F)b in order to simplify our presentation. Nevertheless all the results in the previous sections apply to this case. In analogy with the notation introduced in (8)–(10), for any  $v$  we have

$$(59) \quad v^\# = 2(1 - v), \quad (v^\#)^\# = 2(2v - 1), \quad v^\flat = 1 - \frac{v}{2}.$$

Since the volume fraction of solid  $u$  varies in the interval  $[0, 1]$ , we need  $v \in [1/2, 3/4]$  in order to have  $v^\#$  and  $(v^\#)^\#$  in the same interval  $[0, 1]$ .

Fix  $T = 2$ , and consider the final profile

$$(60) \quad v_2(x) = \begin{cases} 1/2 & \text{if } x \leq 1/2, \\ \frac{4\sqrt{7} - \sqrt{4 + 3\sqrt{15 - 28x}}}{6\sqrt{7}} & \text{if } 1/2 < x \leq \frac{59}{112}, \\ \frac{224 - 13\sqrt{154}}{84} & \text{if } x > \frac{59}{112}. \end{cases}$$

We illustrate how the backward reconstruction exploited in the proof of Lemma 3.4, allows us to construct an initial datum  $u_0$  such that the solution  $u = u(t, x)$  to (58) with  $u(0, \cdot) = u_0$  fulfills  $u(2, \cdot) = v_2$ .

The target profile suffers from a jump discontinuity at  $\bar{x} = 59/112$ . In the following we call

$$v_L \doteq v(59/112) = \frac{56 - \sqrt{154}}{84}, \quad v_R \doteq v(59/112+) = \frac{224 - 13\sqrt{154}}{336}.$$

A straightforward computation (see (59)), gives us

$$v_L^\# = \frac{28 + \sqrt{154}}{42}, \quad (v_L^\#)^\# = \frac{14 - \sqrt{154}}{21},$$

so we can check that  $v_R \in [(v_L^\#)^\#, v_L]$ . For future use we also compute

$$v_R^\flat = \frac{448 + 13\sqrt{154}}{672}.$$

Observe that, being that  $D_x f'(v(x)) = (2\sqrt{15 - 28x})^{-1}$ , the interval  $J$  at (26) is nonempty,  $\sup J = \bar{x}$ , and  $\inf J = \hat{x} = 1/2$ , and  $t_1 = 1$ ; see (31). The convex envelope of the lines  $\eta_x$  defined at (15c) for  $x \in J = ]1/2, 59/112]$  is

$$\varphi(t) = \frac{1}{112}t^2 + \frac{1}{4}t, \quad t \leq 1 \doteq t_1,$$

Moreover, being that  $(v_L^\#)^\# < v_R < v_L$  and

$$-\frac{(v_R - v_L^\#)[f'(v_L) - f'(v_L^\#)]}{f(v_R) - f(v_L) - (v_R - v_L)f'(v_L)}t_1 = \frac{16}{9} < 2 = T,$$

the assumption (29) is satisfied. Hence, all hypotheses of Lemma 3.4 are fulfilled. Plugging  $\bar{t} = T = 2$  and  $t_1 = 1$  into (54), we easily find the value  $\gamma = \gamma^*$  which allows us to construct the initial condition

$$\begin{aligned}
 (61) \quad \gamma^* &= - \left[ \frac{f(v_R) - f(v_L) - (v_R - v_L)f'(v_L)}{v_R f'(v_L^\sharp) + v_R f'(v_R^\flat) - v_L^\sharp f'(v_L^\sharp) + f(v_L^\sharp) - f(v_R)} \right] \\
 &\quad \cdot \left[ 2 + \frac{(v_R - v_L^\sharp)[f'(v_L) - f'(v_L^\sharp)]}{f(v_R) - f(v_L) - (v_R - v_L)f'(v_L)} \right] \\
 &= - \frac{6823432 + 5667200\sqrt{154}}{391268487} \approx -0.197183.
 \end{aligned}$$

From (39) and (40) we get

$$x_1 = \frac{29}{112}, \quad x_2 = \frac{3}{16}.$$

With the choice (61) of  $\gamma = \gamma^*$  and using (42), we get

$$\begin{aligned}
 \mathbf{a} &= x_1 + f'(v_L^\sharp)(\gamma^* - 1) = \frac{1166007253 - 6476800\sqrt{154}}{6260295792} \approx 0.173415, \\
 \mathfrak{s}_0(0) &= \mathbf{a} - f'(v_R^\flat)\gamma^* = \frac{4734091037 + 32282800\sqrt{154}}{25041183168} \approx 0.205051.
 \end{aligned}$$

Hence, from (43), we get that an initial datum by means of which we get  $v_2$  at time  $T = 2$  is given by

$$u_0(x) = \begin{cases} 1/2 & \text{if } x \leq 0, \\ \frac{2\sqrt{7} + \sqrt{4 + 3\sqrt{1 - 4x}}}{3\sqrt{7}} & \text{if } 0 < x \leq 3/16, \\ \frac{2}{3} + \sqrt{\frac{\gamma^* + 3x - 3\mathbf{a}}{\gamma^*}} & \text{if } 3/16 < x \leq \mathfrak{s}_0(0), \\ \frac{224 - 13\sqrt{154}}{336} & \text{if } x > \mathfrak{s}_0(0). \end{cases}$$

**5. More complex structures.** This section consists of two parts. First, in section 5.1 we present a simple case of nesting in a very smooth framework. We state some conditions for attainability and we provide an explicit example. This presentation is, of course, far from being exhaustive but it is sufficient to illustrate the main features of the problem.

In sections 5.2 and 5.3 we present some ideas toward the construction of a recursive procedure which could be used to simplify the problem in more general situations, and that leads to a prospective research direction.

**5.1. Two nested contact discontinuities.** Let  $v$  be an attainable profile suffering from a jump discontinuity at  $x = \bar{x}$ , connecting the states  $v_L = v(\bar{x}^-)$ , and  $v_R = v(\bar{x}^+) = v_L^\sharp$ . We assume that tracing back the candidate backward characteristics  $\eta_x$ , one finds that there exist  $x_1$  and  $x_2$  such that

- for all  $x \in J_{\bar{x}}^L = ]x_1, \bar{x}]$  the lines  $\eta_x$  graze from the left contact discontinuity  $\tau \mapsto \varphi_1(\tau)$ ;
- for all  $x \in J_{\bar{x}}^R = ]\bar{x}, x_2]$  the lines  $\eta_x$  graze from the left contact discontinuity  $\tau \mapsto \varphi_2(\tau)$ .

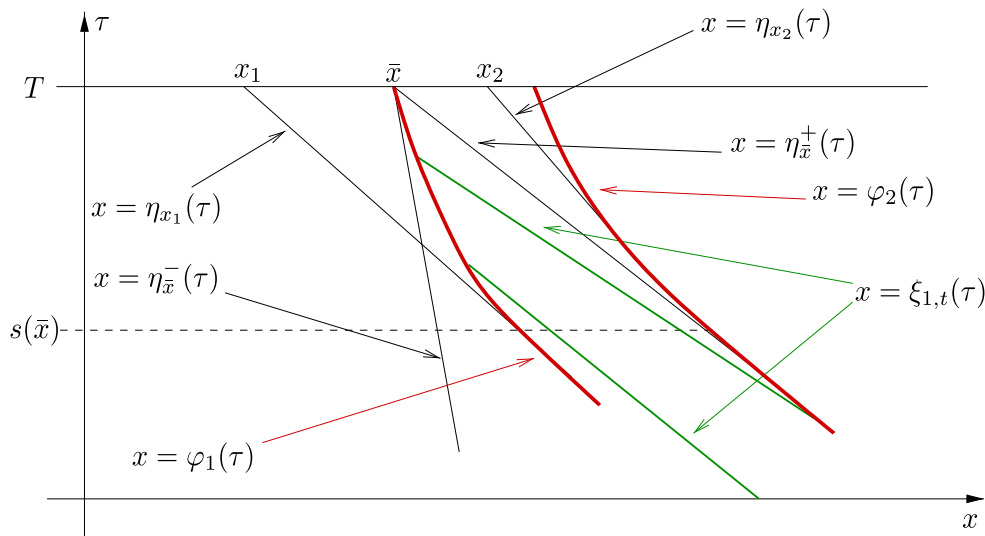


FIG. 9. Configuration with two consecutive contact discontinuities.

It is well known that, once a left contact discontinuity appears in the solution, it persists until it interacts with a second left contact or with a shock; see [10, Theorem 2.3]. This means that the admissible profile  $v$  must suffer from a second jump discontinuity, taking place at  $x = x_2$ . The proposition below, whose proof is straightforward in the light of the analysis in the preceding sections, formalizes these necessary conditions.

**PROPOSITION 5.1.** *In the hypothesis above there exists  $\bar{x} \in \mathbb{R}$ , such that*

- $\bar{x} > \bar{x}$  and  $v(\bar{x}^-) \neq v(\bar{x}^+)$ ;
- $(v(\bar{x}^-), v(\bar{x}^+))$  is an admissible jump in the sense of condition 4 of Theorem 2.1;
- the set  $J_{\bar{x}} \doteq \{\bar{x} < x \leq \bar{x} : D_x f'(v(x)) > 1/T\}$ , analogous to the set  $J$  defined in (26), is exactly  $]\bar{x}, \bar{x}]$  and for all  $x \in J_{\bar{x}}$  there holds  $D^+(D_x f'(v(x))) > 0$ .

Regarding the lines  $\tau \mapsto \xi_{t(x)}(\tau)$  defined as in (23) as the candidate backward characteristics from  $(t(x), \varphi_1(t(x)))$ , two configurations are possible. If they do not cross each other in positive time, it means that the left contact discontinuities  $\varphi_1$  and  $\varphi_2$  are essentially disconnected one from the other. Otherwise, we conclude that the lines  $\xi_{t(x)}$  graze from  $\varphi_2$  and nesting occurs (see Figure 9). This second case is the one we are interested in. Without surprise, we have that in a configuration like the one in Figure 9, not only the jump in  $\bar{x}$  must be admissible in the sense of Theorem 2.1, but there is also a compatibility condition among the values of  $v$  in a left and in a right neighborhood of  $\bar{x}$ .

**THEOREM 5.2.** *Let  $v$  be a piecewise  $C^2$  attainable profile at time  $T$ , suffering from a jump at  $x = \bar{x}$  with left and right states  $v_L$  and  $v_R = v_L^\sharp$ , respectively. Assume that in a neighborhood of  $(T, \bar{x})$  the configuration described above and illustrated in Figure 9 occurs. Then*

$$(62) \quad \lim_{x \rightarrow \bar{x}^+} D_x f'(v(x)) = \lim_{x \rightarrow \bar{x}^-} \frac{[D_x f'(v(x))]^2 D_x f'(v^\sharp(x))}{D_{xx}^2 f'(v(x)) [f'(v_L) - f'(v_R)]}.$$

*Proof.* Let  $s(\bar{x}) \in ]0, T[$  be such that  $\eta_{\bar{x}}^+(s(\bar{x})) = \varphi_2(s(\bar{x}))$ . Due to (21) it turns out that

$$(63) \quad s(\bar{x}) = T - \lim_{x \rightarrow \bar{x}^+} \frac{1}{D_x f'(v(x))}.$$

Now we compute  $s(\bar{x})$  using the fact that there exists  $t_0 < s(\bar{x})$  such that for  $\tau \in ]t_0, s(\bar{x})]$  the curve  $x = \varphi_2(\tau)$  is the convex envelope of the maximal characteristics grazing from  $x = \varphi_1(\tau)$ . Let  $I$  be a suitable left neighborhood of  $\bar{x}$  such that  $x = \varphi_1(\tau)$  is the tangent envelope of the backward characteristics grazing from  $(T, x)$ ,  $x \in I$ , so that (22) holds with  $\varphi = \varphi_1$ . Hence, we can write the maximal backward characteristics from  $\varphi_1$  as

$$(64) \quad \xi_{t(x)}(\tau) = \varphi_1(t(x)) + f'(v^\sharp(x))(\tau - t(x)).$$

Such a line intersects  $\eta_{\bar{x}}^+$  at time

$$s(\bar{x}, x) = - \frac{\varphi_1(t(x)) - \bar{x} - (f'(v^\sharp(x))t(x) - f'(v_R)T)}{f'(v^\sharp(x)) - f'(v_R)}.$$

Letting  $x \rightarrow \bar{x}^-$  we get

$$s(\bar{x}) = T + \lim_{x \rightarrow \bar{x}^-} \frac{D_{xx}^2 f'(v(x)) [f'(v_R) - f'(v_L)]}{[D_x f'(v(x))]^2 D_x f'(v^\sharp(x))},$$

that together with (63) gives (62) □

In the spirit of Proposition 2.4 and having in mind the configuration at Figure 9, one can easily formulate conditions on the values of  $v$  on  $J_{\bar{x}}^L$  in order that the tangent envelope of the the family of lines  $\{\xi_{t(x)}\}_{x \in J_{\bar{x}}^L}$  at (64), turns out to be a convex, Lipschitz continuous curve.

**5.1.1. An example.** Consider again the Kynch model for sedimentation (58), and let

$$(65) \quad v_3(x) = \begin{cases} 1/2 & \text{if } x \leq \frac{3}{4}, \\ \frac{1}{6} \left[ 4 - \sqrt{\frac{5}{14} + \frac{3}{14} \sqrt{93 - 112x}} \right] & \text{if } \frac{3}{4} < x \leq \frac{93}{112}, \\ \frac{14 + \sqrt{10 + 3\sqrt{169 - 196x}}}{21} & \text{if } \frac{93}{112} < x \leq \frac{165}{196}, \\ \frac{2}{7} & \text{if } x > \frac{165}{196}, \end{cases}$$

be a candidate final profile at time  $T = 3$  for a solution to (58). In order to reconstruct an initial datum for (58) from  $v_3$ , we trace backward the lines  $\{\eta_x\}_{x \in ]3/4, 165/196]}$  (see Figure 9, where  $\bar{x} = 93/112$ ),

$$\eta_x(t) = x + f'(v_3(x))(t-3) = \begin{cases} x + \frac{17 - \sqrt{93 - 112x}}{56}(t-3) & \text{if } 3/4 < x \leq \frac{93}{112}, \\ x + \frac{13 - \sqrt{169 - 196x}}{49}(t-3) & \text{if } \frac{93}{112} < x \leq \frac{165}{196}. \end{cases}$$

The tangent envelope of  $\eta_x$  for  $x \in ]3/4, 93/111]$  can be easily computed by means of (21)–(22), and it turns out to be the convex curve

$$\varphi_1(t) = \frac{1}{112}t^2 + \frac{1}{4}t, \quad 0 \leq t \leq 3.$$

Such a curve is a candidate left contact discontinuity in the (candidate) solution  $u = u(t, x)$  to (58) attaining  $v_3$  at time  $T = 3$ . The left and right states of  $x = \varphi_1(t)$ , say  $u_1(t)$  and  $u_1^\sharp(t)$ , respectively, must fulfill

$$f'(u_1(t)) = \varphi_1'(t) = \frac{f(u_1^\sharp(t)) - f(u_1(t))}{u_1^\sharp(t) - u_1(t)},$$

and hence we obtain that

$$u_1(t) = \frac{1}{6} \left( 4 - \sqrt{\frac{14-3t}{14}} \right), \quad u_1^\sharp(t) = \frac{1}{3} \left( 2 + \sqrt{\frac{14-3t}{14}} \right)$$

hold. Now, trace backward the lines  $\{\xi_{1,t}\}_{t \in [0,3]}$  (see Figure 9),

$$\xi_{1,t}(\tau) \doteq \varphi_1(t) + f'(u_1^\sharp(t))(\tau - t) = -\frac{1}{16}t^2 + \frac{1}{4}t + \frac{1}{14}t\tau,$$

that are the candidate backward maximal characteristics from  $(t, \varphi_1(t))$  in the candidate solution  $u = u(t, x)$  that we would like to reconstruct. It turns out that

$$\xi_{1,3}(7/4) = \lim_{x \rightarrow 93/112+} \eta_x(7/4),$$

where  $7/4 = s(\bar{x})$  in Figure 9. Hence we can consider the tangent envelope of the family of lines

$$\{\eta_x : x \in ]93/112, 165/196]\} \cup \{\xi_{1,t} : t \in [2, 3]\}.$$

With an easy computation we get the convex curve

$$\varphi_2(t) = \left( \frac{1}{7}t + \frac{1}{2} \right)^2, \quad t \in [0, 2],$$

which is another candidate left contact discontinuity in the solution  $u$  (see Figure 9 again). The (candidate) forward left characteristics grazing from  $(t, \varphi_2(t))$  for  $t \in [0, 7/4]$  are exactly the (candidate) backward maximal characteristics from  $(t, \varphi_1(t))$  for  $t \in [2, 3]$  (nesting configuration). Now, observe that

$$\frac{f(v_3(165/196+)) - f(v_3(165/196))}{v_3(165/196+) - v_3(165/196)} = \frac{11}{49} = \varphi_2'(2).$$

Hence, we can redefine  $\varphi_2(\cdot)$  as

$$\varphi_2(t) = \begin{cases} \left( \frac{1}{7}t + \frac{1}{2} \right)^2 & \text{if } t \in [0, 2], \\ \frac{11}{49}t + \frac{33}{196} & \text{if } t > 2, \end{cases}$$

and it turns out to be a (candidate) left contact discontinuity with left and right states

$$u_2(t) = \begin{cases} \frac{14 + 2\sqrt{2(14 - 3t)}}{21} & \text{if } 0 \leq t < 2, \\ \frac{6}{7} & \text{if } t \geq 2, \end{cases}$$

$$u_2^\#(t) = \begin{cases} \frac{14 - 4\sqrt{2(14 - 3t)}}{21} & \text{if } 0 \leq t < 2, \\ \frac{2}{7} & \text{if } t \geq 2, \end{cases}$$

respectively. The lines  $\{\xi_{1,t}\}_{t \in [0,2[}$  do not intersect each other in the time interval  $[0, 2[$ , and the same happens for the family of lines

$$\xi_{2,t}(\tau) = \varphi_2(t) + f'(u_2^\#(t))(\tau - t) = \begin{cases} \left(\frac{1}{7}t + \frac{1}{2}\right)^2 + \frac{8t - 21}{49}(\tau - t) & \text{if } t \in [0, 2[, \\ \frac{33}{196} + \frac{16}{49}t - \frac{5}{49}\tau & \text{if } t \geq 2, \end{cases}$$

that are the candidate backward maximal characteristics from  $(t, \varphi_2(t))$ ,  $t \in [0, 3]$ . It follows that we can reconstruct the initial datum  $u_0$  which is driven to  $v_3$  by means of (58) by using the method of characteristics, and thus obtain

$$u_0(x) = \begin{cases} 1/2 & \text{if } x \leq 0, \\ \frac{2\sqrt{7} + \sqrt{4 + 3\sqrt{1 - 4x}}}{3\sqrt{7}} & \text{if } 0 < x \leq 1/4, \\ \frac{14 - 2\sqrt{16 + 3\sqrt{23 - 28x}}}{21} & \text{if } 1/4 < x \leq 23/28, \\ 2/7 & \text{if } x > 23/28. \end{cases}$$

**5.2. A recursive procedure.** In this section we introduce a recursive procedure which might help in the investigation of the cases in which a nesting of contact discontinuities is detected. This different approach does not lead to a complete answer to the problem of characterization of the set of attainable states, but allows us to formulate one additional necessary condition for attainability.

The main idea in the recursive procedure is the following. Let  $v$  be a candidate admissible profile suffering from a jump discontinuity at  $x = \bar{x}$  between the states  $v_L$  and  $v_R$ . We do not make any assumption on the values of  $v$  for  $x > \bar{x}$ . Assume that backward characteristics from a left neighborhood  $J$  of  $\bar{x}$  do intersect, and that their tangent envelope is a convex curve  $x = \varphi_1(\tau)$ , defined in a left neighborhood of  $t_1 \leq T$ , with  $t_1$  given by (31), namely,

$$t_1 = T - \frac{1}{D_{\bar{x}} f'(v(\bar{x}))}.$$

The profile  $v$  is not supposed to satisfy condition 2(c) of Theorem 3.1, therefore, it might happen that the maximal backward characteristics from the points  $(t, \varphi_1(t))$ , which we call  $\xi_{1,t}$ , cross each other before reaching  $t = 0$ .

We are interested in the case in which the lines  $\xi_{1,t}$  admit a convex tangent envelope  $\tau \mapsto \varphi_2(\tau)$ , along which any possible backward reconstruction of  $v$  experiences





At time  $t_2$  we are now able to construct a candidate final profile  $w_2$  defined in a left neighborhood  $J_2$  of  $\varphi_2(t_2)$  using the same method exploited to construct  $w_1$  and, hence, considering the lines

$$\xi_{2,t}(\tau) = \varphi_2(t) + f'(u_{2,R}(t))(\tau - t),$$

where  $u_{2,R}(t)$  is the state at the right of  $\varphi_2$  chosen so that, if  $\varphi_2'(t) = f'(u_{2,L}(t))$ , then  $u_{2,R}(t) = (u_{2,L}(t))^\sharp$ . Then  $w_2$  is defined as

$$w_2(\xi_{2,t}(t_2)) = u_{2,R}(t).$$

3. Now it is clear how we can proceed. Once we reconstruct the convex curve  $x = \varphi_n(\tau)$ , that turns out to be a left contact discontinuity with left state  $u_{n,L}(t)$  and right state  $u_{n,R}(t) = (u_{n,L}(t))^\sharp$ , we can define the lines

$$\xi_{n,t}(\tau) = \varphi_n(t) + f'(u_{n,R}(t))(\tau - t),$$

and then  $w_n$  as

$$w_n(\xi_{n,t}(t_n)) = u_{n,R}(t).$$

If the lines  $\xi_{n,t}$  do intersect in  $]0, T]$ , then their tangent envelope is a convex curve  $x = \varphi_{n+1}(\tau)$  starting at time

$$t_{n+1} = t_n - \frac{1}{D_x^- f'(w_n(x))|_{x=\varphi_n(t_n)}}.$$

Since in a solution  $u$  to (1a) the maximal backward characteristic starting from a point  $(T, \bar{x})$  with  $u(T, \bar{x}+) \neq 0$  is a polygonal line with a finite number of nodes [10, Theorem 2.1] (see Figure 11), we can hope to obtain a control  $u_c$  in (1b) only if there exists  $N \in \mathbb{N}$  such that

$$(66) \quad t_N < 0 \quad \text{and} \quad \beta \doteq \varphi_1(t_1) + \sum_{i=1}^N (\varphi_{i+1}(t_{i+1}) - \varphi_i(t_i)) < b.$$

Using the expression for  $t_i$ , such conditions can be rewritten as

$$(67) \quad \frac{1}{D_x f'(v(\bar{x}))} + \sum_{n=1}^N \frac{1}{D_x^- f'(w_n(x))|_{x=\varphi_n(t_n)}} > T$$

and

$$(68) \quad \bar{x} + \frac{f'(v(\bar{x}-))}{D_x^- f'(v(\bar{x}))} + \sum_{n=1}^N \frac{f'(w_n(\varphi_n(t_n)-))}{D_x^- f'(w_n(x))|_{x=\varphi_n(t_n)}} < b.$$

Notice that the recursive procedure above does not allow for a complete treatment of the problem. This is related to the fact that, even at the first iteration of the procedure, we would not be able to define the profile  $w_1$  on the right of  $\varphi_1(t_1)$ . Indeed, even if condition (66) is fulfilled, we would not be able to decide whether the final profile is attainable or not; see the proof of Lemma 3.4 for an example in this direction.

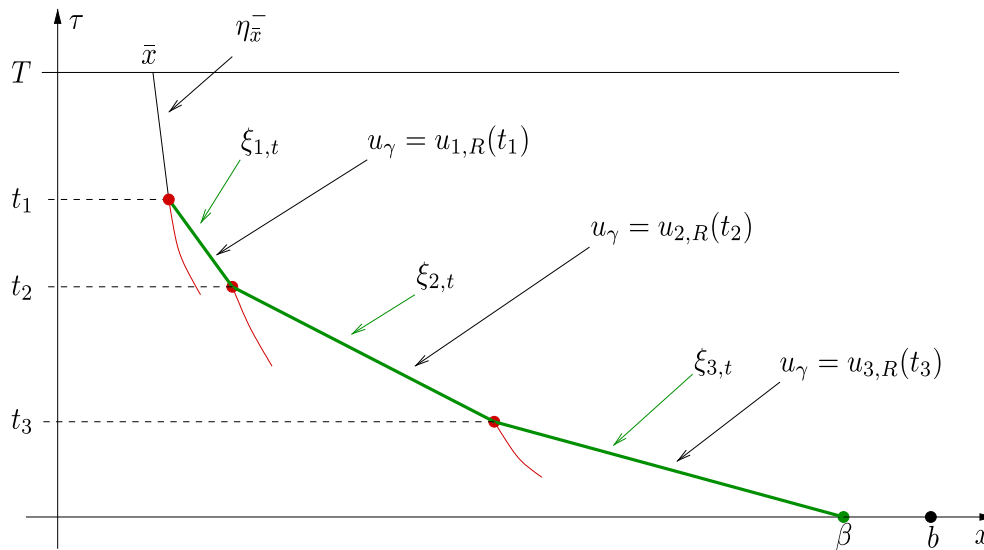


FIG. 11. Condition (66).

**5.3. A prospective research direction.** Starting from the recursive procedure introduced above, it should be possible to obtain a numerical algorithm testing attainability of profiles associated with simple cases of nesting.

**DEFINITION 5.3.** For any  $T > 0$ , we call  $\mathcal{A}_T$  the set of all profiles  $v$  verifying the assumptions of Theorem 3.1 for attainability at  $t = T$ .

Obviously, if  $T_1 < T_2$ , then  $\mathcal{A}_{T_2} \subset \mathcal{A}_{T_1}$ . Assume that  $v$  is a profile attainable at time  $T$  associated with a nested structure (see Figure 10). In particular, this means that  $v$  does not belong to  $\mathcal{A}_T$ . We start the backward reconstruction of the profile  $v$ , as described in section 3, up to the first time at which nesting occurs,  $t_2 \in ]0, T[$  (see Figures 10 and 11). This means that the solution  $u_1$  that we obtain from  $v$  by backward reconstruction contains a left contact discontinuity traveling along the curve  $x = \varphi_1(t)$ . Moreover, the right backward characteristics  $\tau \mapsto \xi_t(\tau)$  issued from the points  $(t, \varphi_1(t))$  cross each other before  $t = 0$ , and their tangent envelope is a new left contact  $x = \varphi_2(t)$  (see Figure 10). We call  $t_2$  the largest time at which this interaction takes place, so that  $v \in \mathcal{A}_{T-t_2}$ . Therefore the tangent envelope of the curves  $\xi_t$  must be a second contact discontinuity  $x = \varphi_2(t)$  (see Figure 10). Assume now that

**(NH)** no nesting occurs in the portion of the plane  $\{(t, x) : t \in [0, T], x \leq \varphi_1(t)\}$ .

Then we can use the idea of the recursive procedure to construct a profile  $v_1$ , at time  $t = T$ , defined piecewise as

1.  $v_1(\xi_{1,t(x)}(T)) = v^\sharp(x)$  for  $x \leq \xi_{1,t_1}(T)$ ;
2.  $v_1(x) = v^\sharp(\bar{x})$  for  $x \in ]\xi_{1,t_1}(T), \bar{x}]$ ;
3.  $v_1(x) = v(x)$  for all  $x > \bar{x}$ .

In such a way, if we apply the procedure of the backward reconstruction of section 3 starting from the profile  $v_1$ , we obtain a solution, call it  $u_2$ , which does not contain any longer the left contact discontinuity  $x = \varphi_1(t)$ . Moreover, for such a backward reconstruction,  $x = \varphi_2(t)$  is the leftmost left contact discontinuity obtained as the tangent envelope of the backward characteristics from  $(t, T)$ ,  $x < \bar{x}$ . The solution  $u_2$

is defined up to  $t_3 < t_2$  (see Figure 11), where a nesting configuration may well appear again. Observe that in such a way  $v_1$  belongs to  $\mathcal{A}_{T-t_3}$  and, moreover,  $v$  belongs to  $\mathcal{A}_{T-t_3}$  too. Indeed, we can piece together  $u_1$  and  $u_2$  defining

$$u(t, x) = \begin{cases} u_1(t, x) & \text{if } x < \varphi_1(t) \text{ or } t \geq t_2, \\ u_2(t, x) & \text{if } t_3 \leq t < t_2 \text{ and } x \geq \varphi_1(t), \end{cases}$$

and obtain a solution  $u$  to (1a) in  $[t_3, T] \times \mathbb{R}$  such that  $u(t, \cdot) = v$ . We can iterate inductively this procedure, and, if the analogous part of the **(NH)** hypothesis holds at each step, we obtain a decreasing sequence of times  $t_k$ , and a sequence of profiles  $v_k$  belonging to  $\mathcal{A}_{T-t_{k+2}}$ .

The explicit verification of the set of properties imposed by this construction (basically, one should check the assumptions of Theorem 3.1 at each step, plus a condition similar to **(NH)**) seems extremely expensive in practice. Nonetheless, these assumptions illustrate well the nature of the backward resolution of the nonconvex conservation law (1a), where solutions should be constructed by piecing together patches of backward and forward solutions, as in the proof of Lemma 3.4. Numerical experiments with this version of the recursive procedure will be considered elsewhere.

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