

Space-time deep neural network  
approximations for high-dimensional partial  
differential equations

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# Space-time deep neural network approximations for high-dimensional partial differential equations

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## Abstract

It is one of the most challenging issues in applied mathematics to approximately solve high-dimensional partial differential equations (PDEs) and most of the numerical approximation methods for PDEs in the scientific literature suffer from the so-called curse of dimensionality in the sense that the number of computational operations employed in the corresponding approximation scheme to obtain an approximation precision  $\varepsilon > 0$  grows exponentially in the PDE dimension and/or the reciprocal of  $\varepsilon$ . Recently, certain deep learning based approximation methods for PDEs have been proposed and various numerical simulations for such methods suggest that deep neural network (DNN) approximations might have the capacity to indeed overcome the curse of dimensionality in the sense that the number of real parameters used to describe the approximating DNNs grows at most polynomially in both the PDE dimension  $d \in \mathbb{N}$  and the reciprocal of the prescribed approximation accuracy  $\varepsilon > 0$ . There are now also a few rigorous mathematical results in the scientific literature which substantiate this conjecture by proving that DNNs overcome the curse of dimensionality in approximating solutions of PDEs. Each of these results establishes that DNNs overcome the curse of dimensionality in approximating suitable PDE solutions at a fixed time point  $T > 0$  and on a compact cube  $[a, b]^d$  in space but none of these results provides an answer to the question whether the entire PDE solution on  $[0, T] \times [a, b]^d$  can be approximated by DNNs without the curse of dimensionality. It is precisely the subject of this article to overcome this issue. More specifically, the main result of this work in particular proves for every  $a \in \mathbb{R}$ ,  $b \in (a, \infty)$  that solutions of certain

Kolmogorov PDEs can be approximated by DNNs on the space-time region  $[0, T] \times [a, b]^d$  without the curse of dimensionality.

*Keywords:* deep neural network, DNN, artificial neural network, ANN, curse of dimensionality, approximation, partial differential equation, PDE, stochastic differential equation, SDE, Monte Carlo Euler, Feynman–Kac formula

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## 1 Introduction

It is one of the most challenging issues in applied mathematics to approximately solve high-dimensional partial differential equations (PDEs) and most of the numerical approximation methods for PDEs in the scientific literature suffer from the so-called *curse of dimensionality* in the sense that the number of computational operations employed in the corresponding approximation scheme to obtain an approximation precision  $\varepsilon > 0$  grows exponentially in the PDE dimension and/or the reciprocal of  $\varepsilon$  (cf., e.g., [42, Chapter 1] and [43, Chapter 9] for related concepts and cf., e.g., [4, 5, 7, 19, 29, 32, 33] for numerical approximation methods for nonlinear PDEs which do not suffer from the curse of dimensionality). Recently, certain deep learning based approximation methods for PDEs have been proposed and various numerical simulations for such methods suggest (cf., e.g., [1, 2, 3, 8, 9, 10, 12, 13, 14, 15, 17, 18, 21, 26, 27, 28, 30, 34, 39, 40, 41, 44, 45, 46, 48]) that deep neural network (DNN) approximations might have the capacity to indeed overcome the curse of dimensionality in the sense that the number of real parameters used to describe the approximating DNNs grows at most polynomially in both the PDE dimension  $d \in \mathbb{N} = \{1, 2, \dots\}$  and the reciprocal of the prescribed approximation accuracy  $\varepsilon > 0$ . There are now also a few rigorous mathematical results in the scientific literature which substantiate this conjecture by proving that DNNs overcome the curse of dimensionality in approximating solutions of PDEs; cf., e.g., [11, 16, 20, 22, 24, 31, 35, 37, 47]. Each of the references mentioned in the previous sentence establishes that DNNs overcome the curse of dimensionality

in approximating suitable PDE solutions at a fixed time point  $T > 0$  and on a compact cube  $[a, b]^d$  in space but none of the results in these references provides an answer to the question whether the entire PDE solution on  $[0, T] \times [a, b]^d$  can be approximated by DNNs without the curse of dimensionality.

It is precisely the subject of this article to overcome this issue. More specifically, the main result of this work, Theorem 4.13 in Subsection 4.6 below, in particular proves for every  $a \in \mathbb{R}$ ,  $b \in (a, \infty)$  that solutions of certain Kolmogorov PDEs can be approximated by DNNs on the space-time region  $[0, T] \times [a, b]^d$  without the curse of dimensionality. To illustrate the findings of this work in more details we now present in Theorem 1.1 below a special case of Theorem 4.13.

**Theorem 1.1.** *Let  $\mathbf{N} = \cup_{L \in \mathbb{N}} \cup_{(l_0, l_1, \dots, l_L) \in \mathbb{N}^{L+1}} (\times_{k=1}^L (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k}))$ , let  $A_d \in C(\mathbb{R}^d, \mathbb{R}^d)$ ,  $d \in \mathbb{N}$ , satisfy for all  $d \in \mathbb{N}$ ,  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$  that  $A_d(x) = (\max\{x_1, 0\}, \dots, \max\{x_d, 0\})$ , let  $\mathcal{P}: \mathbf{N} \rightarrow \mathbb{N}$  and  $\mathcal{R}: \mathbf{N} \rightarrow (\cup_{k, l \in \mathbb{N}} C(\mathbb{R}^k, \mathbb{R}^l))$  satisfy for all  $L \in \mathbb{N}$ ,  $l_0, l_1, \dots, l_L \in \mathbb{N}$ ,  $\Phi = ((W_1, B_1), \dots, (W_L, B_L)) \in (\times_{k=1}^L (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k}))$ ,  $x_0 \in \mathbb{R}^{l_0}, x_1 \in \mathbb{R}^{l_1}, \dots, x_{L-1} \in \mathbb{R}^{l_{L-1}}$  with  $\forall k \in \mathbb{N} \cap (0, L): x_k = A_{l_k}(W_k x_{k-1} + B_k)$  that  $\mathcal{P}(\Phi) = \sum_{k=1}^L l_k(l_{k-1} + 1)$ ,  $\mathcal{R}(\Phi) \in C(\mathbb{R}^{l_0}, \mathbb{R}^{l_L})$ , and  $(\mathcal{R}(\Phi))(x_0) = W_L x_{L-1} + B_L$ , let  $f_d: \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $d \in \mathbb{N}$ , and  $g_d: \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $d \in \mathbb{N}$ , be functions, let  $T, \kappa, p \in (0, \infty)$ ,  $(f_{d,\varepsilon})_{(d,\varepsilon) \in \mathbb{N} \times (0,1]} \subseteq \mathbf{N}$ ,  $(g_{d,\varepsilon})_{(d,\varepsilon) \in \mathbb{N} \times (0,1]} \subseteq \mathbf{N}$ , assume for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$ ,  $x, y \in \mathbb{R}^d$  that*

$$\mathcal{R}(f_{d,\varepsilon}) \in C(\mathbb{R}^d, \mathbb{R}^d), \quad \mathcal{R}(g_{d,\varepsilon}) \in C(\mathbb{R}^d, \mathbb{R}), \quad \mathcal{P}(f_{d,\varepsilon}) + \mathcal{P}(g_{d,\varepsilon}) \leq \kappa d^\kappa \varepsilon^{-\kappa}, \quad (1)$$

$$\varepsilon |g_d(x)| + \|f_d(x) - (\mathcal{R}(f_{d,\varepsilon}))(x)\| + \|g_d(x) - (\mathcal{R}(g_{d,\varepsilon}))(x)\| \leq \varepsilon \kappa d^\kappa (1 + \|x\|^\kappa), \quad (2)$$

$$\|f_d(x) - f_d(y)\| \leq \kappa \|x - y\|, \quad \|(\mathcal{R}(f_{d,\varepsilon}))(x)\| \leq \kappa (d^\kappa + \|x\|), \quad (3)$$

$$\text{and} \quad |(\mathcal{R}(g_{d,\varepsilon}))(x) - (\mathcal{R}(g_{d,\varepsilon}))(y)| \leq \kappa d^\kappa (1 + \|x\|^\kappa + \|y\|^\kappa) \|x - y\|, \quad (4)$$

and for every  $d \in \mathbb{N}$  let  $u_d \in \{v \in C([0, T] \times \mathbb{R}^d, \mathbb{R}): \inf_{q \in (0, \infty)} \sup_{(t,x) \in [0, T] \times \mathbb{R}^d} \frac{|v(t,x)|}{1 + \|x\|^q} < \infty\}$  be a viscosity solution of

$$\left(\frac{\partial}{\partial t} u_d\right)(t, x) = (\Delta_x u_d)(t, x) + \left(\frac{\partial}{\partial x} u_d\right)(t, x) f_d(x) \quad (5)$$

with  $u_d(0, x) = g_d(x)$  for  $(t, x) \in (0, T) \times \mathbb{R}^d$ . Then there exist  $c \in \mathbb{R}$  and  $(\mathbf{u}_{d,\varepsilon})_{(d,\varepsilon) \in \mathbb{N} \times (0,1]} \subseteq \mathbf{N}$  such that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  it holds that  $\mathcal{R}(\mathbf{u}_{d,\varepsilon}) \in C(\mathbb{R}^{d+1}, \mathbb{R})$ ,  $\mathcal{P}(\mathbf{u}_{d,\varepsilon}) \leq c \varepsilon^{-c} d^c$ , and

$$\left[ \int_{[0, T] \times [0, 1]^d} |u_d(y) - (\mathcal{R}(\mathbf{u}_{d,\varepsilon}))(y)|^p dy \right]^{1/p} \leq \varepsilon. \quad (6)$$

Theorem 1.1 follows from Corollary 4.16 in Subsection 4.6 below. Corollary 4.16, in turn, is a consequence of Theorem 4.13 which is the main result of this article. In the following we add a few comments on some of the mathematical objects appearing in Theorem 1.1 above. Note in Theorem 1.1 that  $\|\cdot\|: (\cup_{d \in \mathbb{N}} \mathbb{R}^d) \rightarrow [0, \infty)$  is the function which satisfies for all  $d \in \mathbb{N}$ ,  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$  that  $\|x\| = [\sum_{j=1}^d |x_j|^2]^{1/2}$  (standard norm, cf. Definition 2.1 below). The set  $\mathbf{N}$  in Theorem 1.1 is a set of tuples of pairs of real matrices and real vectors and we think of  $\mathbf{N}$  as the set of all neural networks (cf. Definition 4.1 below). Observe that Theorem 1.1 is an approximation result for rectified DNNs and the corresponding rectifier functions are described through the functions  $A_d: \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $d \in \mathbb{N}$ , appearing in Theorem 1.1. For every  $\Phi \in \mathbf{N}$  the number  $\mathcal{P}(\Phi) \in \mathbb{N}$  in Theorem 1.1 corresponds to the number of real parameters employed to describe the neural network  $\Phi$  (cf. Definition 4.1 below). For every  $\Phi \in \mathbf{N}$  the function  $\mathcal{R}(\Phi) \in (\cup_{k, l \in \mathbb{N}} C(\mathbb{R}^k, \mathbb{R}^l))$  corresponds to the realization function associated to the neural network  $\Phi$  (cf. Definition 4.3 below). The functions  $f_d: \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $d \in \mathbb{N}$ , describe the drift coefficient functions and the functions  $g_d: \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $d \in \mathbb{N}$ , describe the initial value functions of the PDEs whose solutions we intend to approximate in Theorem 1.1 (see (5))

above). The real number  $T \in (0, \infty)$  denotes the time horizon of the PDEs whose solutions we intend to approximate. The real number  $\kappa \in (0, \infty)$  is a constant which we employ to formulate the assumptions on the drift coefficient functions  $f_d: \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $d \in \mathbb{N}$ , and the initial value functions  $g_d: \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $d \in \mathbb{N}$ , of the PDEs whose solutions we intend to approximate in Theorem 1.1 (see (1)–(4) above). The real number  $p \in (0, \infty)$  is used to describe the way how we measure the error between the exact solutions of the PDEs in (5) and the corresponding DNN approximations in the sense that we measure the error in the strong  $L^p$ -sense (see (6) above). We assume in Theorem 1.1 that the drift coefficient functions  $f_d: \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $d \in \mathbb{N}$ , and the initial value functions  $g_d: \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $d \in \mathbb{N}$ , of the PDEs whose solutions we intend to approximate can be approximated by DNNs without the curse of dimensionality and these approximating DNNs are described by the neural networks  $(\mathbf{f}_{d,\varepsilon})_{(d,\varepsilon) \in \mathbb{N} \times (0,1]} \subseteq \mathbf{N}$  and  $(\mathbf{g}_{d,\varepsilon})_{(d,\varepsilon) \in \mathbb{N} \times (0,1]} \subseteq \mathbf{N}$  (see (1) and (2) above). The functions  $u_d: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $d \in \mathbb{N}$ , in Theorem 1.1 denote the PDE solutions which we intend to approximate by means of DNNs. For every  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  the neural network  $\mathbf{u}_{d,\varepsilon} \in \mathbf{N}$  denotes the DNN whose realization function  $\mathcal{R}(\mathbf{u}_{d,\varepsilon}): \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  approximates the function  $u_d: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  with the precision  $\varepsilon$  (see (6) above). Our proofs of Theorem 1.1 above and Theorem 4.13 below, respectively, are based on an application of Proposition 3.10 in Grohs et al. [23] (see (I)–(VI) in the proof of Proposition 4.8 in Subsection 4.4 below for details).

The remainder of this article is structured in the following way. In Section 2 we establish in Proposition 2.4 and Lemma 2.5 suitable weak and strong error estimates for Euler–Maruyama approximations for a certain class of stochastic differential equations (SDEs). In Section 3 we use these weak and strong error estimates for Euler–Maruyama approximations to establish in Proposition 3.2 below suitable error estimates for Monte Carlo Euler approximations for a class of SDEs with perturbed drift coefficient functions. In Section 4 we use these error estimates for Monte Carlo Euler approximations to establish in Theorem 4.13 below that for every  $T \in (0, \infty)$ ,  $a \in \mathbb{R}$ ,  $b \in (a, \infty)$  it holds that solutions of certain Kolmogorov PDEs can be approximated by DNNs on the space-time region  $[0, T] \times [a, b]^d$  without the curse of dimensionality.

## 2 Numerical approximations for stochastic differential equations (SDEs)

In this section we establish in Proposition 2.4 and Lemma 2.5 below suitable weak and strong error estimates for Euler–Maruyama approximations for a certain class of SDEs. Our proofs of Proposition 2.4 and Lemma 2.5 are based on the elementary a priori moment estimates in Lemmas 2.2–2.3 below. Lemma 2.2 is, e.g., proved as Gonon et al. [20, Lemma 3.1] (see also, e.g., Jentzen et al. [35, Lemma 4.2]) and a slightly modified version of Lemma 2.3 is, e.g., proved as Gonon et al. [20, Lemma 3.4] (see also, e.g., Jentzen et al. [35, Lemma 4.1]).

### 2.1 A priori moment bounds for Gaussian random variables

**Definition 2.1** (Standard norms). *We denote by  $\|\cdot\|: (\cup_{d \in \mathbb{N}} \mathbb{R}^d) \rightarrow [0, \infty)$  the function which satisfies for all  $d \in \mathbb{N}$ ,  $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$  that*

$$\|x\| = \left[ \sum_{j=1}^d |x_j|^2 \right]^{1/2}. \quad (7)$$

**Lemma 2.2.** *Let  $d \in \mathbb{N}$ ,  $p \in (0, \infty)$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, and let  $X: \Omega \rightarrow \mathbb{R}^d$  be a centered Gaussian random variable. Then*

$$\left( \mathbb{E}[\|X\|^p] \right)^{1/p} \leq \sqrt{\max\{1, p-1\} \text{Trace}(\text{Cov}(X))} \quad (8)$$

(cf. Definition 2.1).

## 2.2 A priori moment bounds for solutions of SDEs

**Lemma 2.3.** *Let  $d \in \mathbb{N}$ ,  $\xi \in \mathbb{R}^d$ ,  $p \in [1, \infty)$ ,  $c, C, T \in [0, \infty)$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $\mu: \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\chi: [0, T] \rightarrow [0, T]$  be measurable functions, assume for all  $x \in \mathbb{R}^d$ ,  $t \in [0, T]$  that  $\|\mu(x)\| \leq C + c\|x\|$  and  $\chi(t) \leq t$ , and let  $X, \beta: [0, T] \times \Omega \rightarrow \mathbb{R}^d$  be stochastic processes with continuous sample paths which satisfy for all  $t \in [0, T]$  that*

$$\mathbb{P}\left(X_t = \xi + \int_0^t \mu(X_{\chi(s)}) ds + \beta_t\right) = 1 \quad (9)$$

(cf. Definition 2.1). Then

$$\sup_{t \in [0, T]} (\mathbb{E}[\|X_t\|^p])^{1/p} \leq \left(\|\xi\| + CT + \sup_{t \in [0, T]} (\mathbb{E}[\|\beta_t\|^p])^{1/p}\right) e^{cT}. \quad (10)$$

## 2.3 Weak error estimates for Euler–Maruyama approximations

**Proposition 2.4.** *Let  $d, m \in \mathbb{N}$ ,  $\xi \in \mathbb{R}^d$ ,  $T \in (0, \infty)$ ,  $c, C, \varepsilon_0, \varepsilon_1, \varepsilon_2, \varsigma_0, \varsigma_1, \varsigma_2, L_0, L_1, \ell \in [0, \infty)$ ,  $h \in [0, T]$ ,  $p \in [2, \infty)$ ,  $q \in (1, 2]$  satisfy  $1/p + 1/q = 1$ , let  $B \in \mathbb{R}^{d \times m}$ ,  $(\varpi_r)_{r \in (0, \infty)} \subseteq \mathbb{R}$  satisfy for all  $r \in (0, \infty)$  that*

$$\varpi_r = \max\{1, \sqrt{\max\{1, r-1\} \text{Trace}(B^*B)}\}, \quad (11)$$

let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $W: [0, T] \times \Omega \rightarrow \mathbb{R}^m$  be a standard Brownian motion, let  $F_0: \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $f_1: \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $F_2: \mathbb{R}^d \rightarrow \mathbb{R}^d$ , and  $\chi: [0, T] \rightarrow [0, T]$  be functions, let  $f_0: \mathbb{R}^d \rightarrow \mathbb{R}$  and  $F_1: \mathbb{R}^d \rightarrow \mathbb{R}^d$  be measurable functions, assume for all  $t \in [0, T]$ ,  $x, y \in \mathbb{R}^d$  that

$$|f_0(x) - F_0(x)| \leq \varepsilon_0(1 + \|x\|^{\varsigma_0}), \quad \|f_1(x) - F_1(x)\| \leq \varepsilon_1(1 + \|x\|^{\varsigma_1}), \quad (12)$$

$$|F_0(x) - F_0(y)| \leq L_0 \left[1 + \int_0^1 [r\|x\| + (1-r)\|y\|]^\ell dr\right] \|x - y\|, \quad (13)$$

$$\|f_1(x) - f_1(y)\| \leq L_1\|x - y\|, \quad \|F_1(x)\| \leq C + c\|x\|, \quad \|\xi - F_2(\xi)\| \leq \varepsilon_2(1 + \|\xi\|^{\varsigma_2}), \quad (14)$$

and  $\chi(t) = \max(\{0, h, 2h, \dots\} \cap [0, t])$ , and let  $X, Y: [0, T] \times \Omega \rightarrow \mathbb{R}^d$  be stochastic processes with continuous sample paths which satisfy for all  $t \in [0, T]$  that

$$X_t = \xi + \int_0^t f_1(X_s) ds + BW_t \quad \text{and} \quad Y_t = F_2(\xi) + \int_0^t F_1(Y_{\chi(s)}) ds + BW_t \quad (15)$$

(cf. Definition 2.1). Then it holds for all  $t \in [0, T]$  that

$$\begin{aligned} & \left| \mathbb{E}[f_0(X_t)] - \mathbb{E}[F_0(Y_t)] \right| \leq (\varepsilon_2(1 + \|\xi\|^{\varsigma_2}) + \varepsilon_0 + \varepsilon_1 + h + h^{1/2}) \\ & \cdot e^{[\max\{\varsigma_0, 1\}L_1 + 1 - 1/p + \ell \max\{L_1, c\} + \max\{\varsigma_1, 1\}c]T} (\varpi_{\max\{\varsigma_0, \ell q, p\varsigma_1, p\}})^{\max\{\varsigma_0, \ell + \max\{1, \varsigma_1\}\}} \\ & \cdot (\max\{T, 1\})^{\max\{\varsigma_0, \ell + \max\{\varsigma_1, 1\} + 1/p\}} \max\{L_0, 1\} \max\{L_1, 1\} 2^{\max\{\ell - 1, 0\}} \\ & \cdot \left[ \max\{C, 1\} + 5 \max\{C, c, 1\} (\|\xi\| + \varepsilon_2(1 + \|\xi\|^{\varsigma_2}) + 2 \max\{\|f_1(0)\|, C, 1\})^{\max\{\varsigma_0, \ell + \max\{\varsigma_1, 1\}\}} \right]. \end{aligned} \quad (16)$$

*Proof of Proposition 2.4.* Observe that (11), Lemma 2.2, the fact that for all  $t \in [0, T]$  it holds that  $BW_t$  is a centered Gaussian random variable, and the fact that for all  $t \in [0, T]$  it holds that  $\text{Cov}(BW_t) = BB^*t$  assure that for all  $r \in (0, \infty)$ ,  $t \in [0, T]$  it holds that

$$\begin{aligned} (\mathbb{E}[\|BW_t\|^r])^{1/r} & \leq \sqrt{\max\{1, r-1\} \text{Trace}(\text{Cov}(BW_t))} \\ & = \sqrt{\max\{1, r-1\} \text{Trace}(BB^*)t} \\ & \leq \max\{t^{1/2}, \sqrt{\max\{1, r-1\} \text{Trace}(B^*B)t}\} = \varpi_r t^{1/2}. \end{aligned} \quad (17)$$

In addition, note that (14) shows that for all  $x \in \mathbb{R}^d$  it holds that

$$\|f_1(x)\| \leq \|f_1(x) - f_1(0)\| + \|f_1(0)\| \leq \|f_1(0)\| + L_1\|x\|. \quad (18)$$

Hölder's inequality, (14), Lemma 2.3, and (17) hence demonstrate that for all  $r \in (0, \infty)$ ,  $t \in [0, T]$  it holds that

$$\begin{aligned} \sup_{s \in [0, t]} (\mathbb{E}[\|Y_s\|^r])^{1/r} &\leq \sup_{s \in [0, t]} (\mathbb{E}[\|Y_s\|^{\max\{r, 1\}}])^{1/\max\{r, 1\}} \\ &\leq \left[ \|F_2(\xi)\| + Ct + \sup_{s \in [0, t]} (\mathbb{E}[\|BW_s\|^{\max\{r, 1\}}])^{1/\max\{r, 1\}} \right] e^{ct} \\ &\leq [\|F_2(\xi)\| + Ct + \varpi_{\max\{r, 1\}} t^{1/2}] e^{ct} \end{aligned} \quad (19)$$

and

$$\begin{aligned} \sup_{s \in [0, t]} (\mathbb{E}[\|X_s\|^r])^{1/r} &\leq \sup_{s \in [0, t]} (\mathbb{E}[\|X_s\|^{\max\{r, 1\}}])^{1/\max\{r, 1\}} \\ &\leq \left[ \|\xi\| + \|f_1(0)\|t + \sup_{s \in [0, t]} (\mathbb{E}[\|BW_s\|^{\max\{r, 1\}}])^{1/\max\{r, 1\}} \right] e^{L_1 t} \\ &\leq [\|\xi\| + \|f_1(0)\|t + \varpi_{\max\{r, 1\}} t^{1/2}] e^{L_1 t}. \end{aligned} \quad (20)$$

Hölder's inequality, (14), and the triangle inequality therefore imply that for all  $r \in (0, \infty)$ ,  $t \in [0, T]$  it holds that

$$\begin{aligned} \sup_{s \in [0, t]} (\mathbb{E}[\|F_1(Y_s)\|^r])^{1/r} &\leq \sup_{s \in [0, t]} (\mathbb{E}[\|F_1(Y_s)\|^{\max\{r, 1\}}])^{1/\max\{r, 1\}} \\ &\leq C + c \sup_{s \in [0, t]} (\mathbb{E}[\|Y_s\|^{\max\{r, 1\}}])^{1/\max\{r, 1\}} \\ &\leq C + c (\|F_2(\xi)\| + Ct + \varpi_{\max\{r, 1\}} t^{1/2}) e^{ct}. \end{aligned} \quad (21)$$

Moreover, note that [35, Lemma 4.5] assures for all  $t \in [0, T]$  that

$$\begin{aligned} &|\mathbb{E}[f_0(X_t)] - \mathbb{E}[F_0(Y_t)]| \\ &\leq \varepsilon_0 (1 + \mathbb{E}[\|X_t\|^{s_0}]) + L_0 2^{\max\{\ell-1, 0\}} e^{[L_1+1-1/p]t} \left[ 1 + (\mathbb{E}[\|X_t\|^{\ell q}])^{1/q} + (\mathbb{E}[\|Y_t\|^{\ell q}])^{1/q} \right] \\ &\cdot \left[ \|\xi - F_2(\xi)\| + \varepsilon_1 t^{1/p} \left[ 1 + \sup_{s \in [0, t]} (\mathbb{E}[\|Y_s\|^{ps_1}])^{1/p} \right] + ht^{1/p} L_1 \left[ \sup_{s \in [0, t]} (\mathbb{E}[\|F_1(Y_s)\|^p])^{1/p} \right] \right] \\ &+ t^{1/p} L_1 (\mathbb{E}[\|BW_h\|^p])^{1/p}. \end{aligned} \quad (22)$$

This, (17), (19), (20), and (21) prove that for all  $t \in [0, T]$  it holds that

$$\begin{aligned} &|\mathbb{E}[f_0(X_t)] - \mathbb{E}[F_0(Y_t)]| \\ &\leq \varepsilon_0 \left[ 1 + (\|\xi\| + \|f_1(0)\|t + \varpi_{\max\{s_0, 1\}} t^{1/2})^{s_0} e^{s_0 L_1 t} \right] + L_0 2^{\max\{\ell-1, 0\}} e^{[L_1+1-1/p]t} \\ &\cdot \left[ 1 + (\|\xi\| + \|f_1(0)\|t + \varpi_{\max\{\ell q, 1\}} t^{1/2})^\ell e^{\ell L_1 t} + (\|F_2(\xi)\| + Ct + \varpi_{\max\{\ell q, 1\}} t^{1/2})^\ell e^{\ell ct} \right] \\ &\cdot \left[ \|\xi - F_2(\xi)\| + \varepsilon_1 t^{1/p} \left[ 1 + (\|F_2(\xi)\| + Ct + \varpi_{\max\{ps_1, 1\}} t^{1/2})^{s_1} e^{s_1 ct} \right] \right] \\ &+ ht^{1/p} L_1 [C + c (\|F_2(\xi)\| + Ct + \varpi_p t^{1/2}) e^{ct}] + h^{1/2} t^{1/p} L_1 \varpi_p. \end{aligned} \quad (23)$$

In addition, observe that the fact that  $(0, \infty) \ni r \mapsto \varpi_r \in (0, \infty)$  is non-decreasing and the hypothesis that  $p \in [2, \infty)$  imply that

$$\varpi_{\max\{\varsigma_0, \ell q, p\varsigma_1, p\}} \geq \max\{\varpi_{\max\{\varsigma_0, 1\}}, \varpi_{\max\{\ell q, 1\}}, \varpi_{\max\{p\varsigma_1, 1\}}, \varpi_p\}. \quad (24)$$

This and (23) ensure that for all  $t \in [0, T]$  it holds that

$$\begin{aligned} & |\mathbb{E}[f_0(X_t)] - \mathbb{E}[F_0(Y_t)]| \\ & \leq \varepsilon_0 e^{\varsigma_0 L_1 T} \left[ 1 + (\|\xi\| + \|f_1(0)\|T + \varpi_{\max\{\varsigma_0, \ell q, p\varsigma_1, p\}} T^{1/2})^{\varsigma_0} \right] \\ & + L_0 2^{\max\{\ell-1, 0\}} e^{[L_1+1-1/p+\ell \max\{L_1, c\}+\max\{\varsigma_1, 1\}]cT} \\ & \cdot \left[ 1 + (\|\xi\| + \|f_1(0)\|T + \varpi_{\max\{\varsigma_0, \ell q, p\varsigma_1, p\}} T^{1/2})^\ell + (\|F_2(\xi)\| + CT + \varpi_{\max\{\varsigma_0, \ell q, p\varsigma_1, p\}} T^{1/2})^\ell \right] \quad (25) \\ & \cdot \left[ \|\xi - F_2(\xi)\| + \varepsilon_1 T^{1/p} \left[ 1 + (\|F_2(\xi)\| + CT + \varpi_{\max\{\varsigma_0, \ell q, p\varsigma_1, p\}} T^{1/2})^{\varsigma_1} \right] \right] \\ & + hT^{1/p} L_1 \left[ C + c(\|F_2(\xi)\| + CT + \varpi_{\max\{\varsigma_0, \ell q, p\varsigma_1, p\}} T^{1/2}) \right] + h^{1/2} T^{1/p} L_1 \varpi_{\max\{\varsigma_0, \ell q, p\varsigma_1, p\}}. \end{aligned}$$

Combining this with the fact that  $\varpi_{\max\{\varsigma_0, \ell q, p\varsigma_1, p\}} \geq 1$  and the fact that  $T^{1/2} \leq (\max\{T, 1\})^{1/2} \leq \max\{T, 1\}$  assures that for all  $t \in [0, T]$  it holds that

$$\begin{aligned} & |\mathbb{E}[f_0(X_t)] - \mathbb{E}[F_0(Y_t)]| \\ & \leq \varepsilon_0 e^{\varsigma_0 L_1 T} (\varpi_{\max\{\varsigma_0, \ell q, p\varsigma_1, p\}})^{\varsigma_0} \left[ 1 + (\|\xi\| + \|f_1(0)\|T + T^{1/2})^{\varsigma_0} \right] \\ & + L_0 2^{\max\{\ell-1, 0\}} e^{[L_1+1-1/p+\ell \max\{L_1, c\}+\max\{\varsigma_1, 1\}]cT} (\varpi_{\max\{\varsigma_0, \ell q, p\varsigma_1, p\}})^{\ell+\max\{1, \varsigma_1\}} \\ & \cdot \left[ 1 + (\|\xi\| + \|f_1(0)\|T + T^{1/2})^\ell + (\|F_2(\xi)\| + CT + T^{1/2})^\ell \right] \quad (26) \\ & \cdot \left[ \|\xi - F_2(\xi)\| + \varepsilon_1 T^{1/p} \left[ 1 + (\|F_2(\xi)\| + CT + T^{1/2})^{\varsigma_1} \right] \right] \\ & + hT^{1/p} L_1 \left[ C + c(\|F_2(\xi)\| + CT + T^{1/2}) \right] + h^{1/2} T^{1/p} L_1. \end{aligned}$$

Therefore, we obtain that for all  $t \in [0, T]$  it holds that

$$\begin{aligned} & |\mathbb{E}[f_0(X_t)] - \mathbb{E}[F_0(Y_t)]| \\ & \leq \varepsilon_0 e^{\varsigma_0 L_1 T} (\varpi_{\max\{\varsigma_0, \ell q, p\varsigma_1, p\}})^{\varsigma_0} (1 + (\max\{T, 1\})^{\varsigma_0} (\|\xi\| + \|f_1(0)\| + 1)^{\varsigma_0}) \\ & + L_0 2^{\max\{\ell-1, 0\}} e^{[L_1+1-1/p+\ell \max\{L_1, c\}+\max\{\varsigma_1, 1\}]cT} (\varpi_{\max\{\varsigma_0, \ell q, p\varsigma_1, p\}})^{\ell+\max\{1, \varsigma_1\}} \\ & \cdot \left[ 1 + (\max\{T, 1\})^\ell \left[ (\|\xi\| + \|f_1(0)\| + 1)^\ell + (\|F_2(\xi)\| + C + 1)^\ell \right] \right] \quad (27) \\ & \cdot \left[ \|\xi - F_2(\xi)\| + \varepsilon_1 T^{1/p} [1 + (\max\{T, 1\})^{\varsigma_1} (\|F_2(\xi)\| + C + 1)^{\varsigma_1}] \right] \\ & + hT^{1/p} \max\{T, 1\} L_1 [C + c(\|F_2(\xi)\| + C + 1)] + h^{1/2} T^{1/p} L_1. \end{aligned}$$

Hence, we obtain that for all  $t \in [0, T]$  it holds that

$$\begin{aligned} & |\mathbb{E}[f_0(X_t)] - \mathbb{E}[F_0(Y_t)]| \\ & \leq \varepsilon_0 e^{\varsigma_0 L_1 T} (\varpi_{\max\{\varsigma_0, \ell q, p\varsigma_1, p\}})^{\varsigma_0} (\max\{T, 1\})^{\varsigma_0} (1 + (\|\xi\| + \|f_1(0)\| + 1)^{\varsigma_0}) \\ & + L_0 2^{\max\{\ell-1, 0\}} e^{[L_1+1-1/p+\ell \max\{L_1, c\}+\max\{\varsigma_1, 1\}]cT} (\varpi_{\max\{\varsigma_0, \ell q, p\varsigma_1, p\}})^{\ell+\max\{1, \varsigma_1\}} \\ & \cdot (\max\{T, 1\})^{\ell+\max\{\varsigma_1, 1\}+1/p} \left[ 1 + (\|\xi\| + \|f_1(0)\| + 1)^\ell + (\|F_2(\xi)\| + C + 1)^\ell \right] \\ & \cdot \left[ \|\xi - F_2(\xi)\| + \varepsilon_1 [1 + (\|F_2(\xi)\| + C + 1)^{\varsigma_1}] + hL_1 [C + c(\|F_2(\xi)\| + C + 1)] + h^{1/2} L_1 \right]. \quad (28) \end{aligned}$$



This implies that for all  $t \in [0, T]$  it holds that

$$\begin{aligned}
& |\mathbb{E}[f_0(X_t)] - \mathbb{E}[F_0(Y_t)]| \\
& \leq \varepsilon_0 e^{\varepsilon_0 L_1 T} (\varpi_{\max\{s_0, \ell q, p s_1, p\}})^{s_0} (\max\{T, 1\})^{s_0} \\
& \cdot (1 + (\max\{\|F_2(\xi)\|, \|\xi\|\} + 2 \max\{\|f_1(0)\|, C, 1\})^{s_0}) \\
& + L_0 2^{\max\{\ell-1, 0\}} e^{[L_1+1-1/p+\ell \max\{L_1, c\}+\max\{s_1, 1\}]cT} (\varpi_{\max\{s_0, \ell q, p s_1, p\}})^{\ell+\max\{1, s_1\}} \\
& \cdot (\max\{T, 1\})^{\ell+\max\{s_1, 1\}+1/p} \left[ 1 + 2(\max\{\|F_2(\xi)\|, \|\xi\|\} + 2 \max\{\|f_1(0)\|, C, 1\})^\ell \right] \\
& \cdot \left[ \|\xi - F_2(\xi)\| + \varepsilon_1 [1 + (\max\{\|F_2(\xi)\|, \|\xi\|\} + 2 \max\{\|f_1(0)\|, C, 1\})^{s_1}] \right. \\
& \left. + h L_1 [C + c(\max\{\|F_2(\xi)\|, \|\xi\|\} + 2 \max\{\|f_1(0)\|, C, 1\})] + h^{1/2} L_1 \right].
\end{aligned} \tag{29}$$

Therefore, we obtain that for all  $t \in [0, T]$  it holds that

$$\begin{aligned}
& |\mathbb{E}[f_0(X_t)] - \mathbb{E}[F_0(Y_t)]| \\
& \leq \varepsilon_0 e^{\varepsilon_0 L_1 T} (\varpi_{\max\{s_0, \ell q, p s_1, p\}})^{s_0} (\max\{T, 1\})^{s_0} \\
& \cdot (1 + (\max\{\|F_2(\xi)\|, \|\xi\|\} + 2 \max\{\|f_1(0)\|, C, 1\})^{s_0}) \\
& + L_0 2^{\max\{\ell-1, 0\}} e^{[L_1+1-1/p+\ell \max\{L_1, c\}+\max\{s_1, 1\}]cT} (\varpi_{\max\{s_0, \ell q, p s_1, p\}})^{\ell+\max\{1, s_1\}} \\
& \cdot (\max\{T, 1\})^{\ell+\max\{s_1, 1\}+1/p} \left[ 1 + 2(\max\{\|F_2(\xi)\|, \|\xi\|\} + 2 \max\{\|f_1(0)\|, C, 1\})^\ell \right] \\
& \cdot (\|\xi - F_2(\xi)\| + \varepsilon_1 + h + h^{1/2}) \max\{L_1, 1\} \\
& \cdot \left[ \max\{C, 1\} + \max\{c, 1\} (\max\{\|F_2(\xi)\|, \|\xi\|\} + 2 \max\{\|f_1(0)\|, C, 1\})^{\max\{s_1, 1\}} \right].
\end{aligned} \tag{30}$$

Combining this with the fact that for all  $a, b \in [0, \infty)$ ,  $z \in [1, \infty)$  it holds that

$$\begin{aligned}
(1 + 2z^\ell)(a + bz^{\max\{s_1, 1\}}) &= a + bz^{\max\{s_1, 1\}} + 2az^\ell + 2bz^{\ell+\max\{s_1, 1\}} \\
&\leq a + (3b + 2a)z^{\ell+\max\{s_1, 1\}} \\
&\leq a + 5 \max\{a, b\} z^{\ell+\max\{s_1, 1\}}
\end{aligned} \tag{31}$$

demonstrates that for all  $t \in [0, T]$  it holds that

$$\begin{aligned}
& |\mathbb{E}[f_0(X_t)] - \mathbb{E}[F_0(Y_t)]| \\
& \leq \varepsilon_0 e^{\varepsilon_0 L_1 T} (\varpi_{\max\{s_0, \ell q, p s_1, p\}})^{s_0} (\max\{T, 1\})^{s_0} \\
& \cdot (1 + (\max\{\|F_2(\xi)\|, \|\xi\|\} + 2 \max\{\|f_1(0)\|, C, 1\})^{s_0}) \\
& + L_0 2^{\max\{\ell-1, 0\}} e^{[L_1+1-1/p+\ell \max\{L_1, c\}+\max\{s_1, 1\}]cT} (\varpi_{\max\{s_0, \ell q, p s_1, p\}})^{\ell+\max\{1, s_1\}} \\
& \cdot (\max\{T, 1\})^{\ell+\max\{s_1, 1\}+1/p} (\|\xi - F_2(\xi)\| + \varepsilon_1 + h + h^{1/2}) \max\{L_1, 1\} \\
& \cdot \left[ \max\{C, 1\} + 5 \max\{C, c, 1\} (\max\{\|F_2(\xi)\|, \|\xi\|\} + 2 \max\{\|f_1(0)\|, C, 1\})^{\ell+\max\{s_1, 1\}} \right].
\end{aligned} \tag{32}$$

Therefore, we obtain that for all  $t \in [0, T]$  it holds that

$$\begin{aligned}
& |\mathbb{E}[f_0(X_t)] - \mathbb{E}[F_0(Y_t)]| \leq (\|\xi - F_2(\xi)\| + \varepsilon_0 + \varepsilon_1 + h + h^{1/2}) \\
& \cdot e^{[\max\{s_0, 1\}L_1+1-1/p+\ell \max\{L_1, c\}+\max\{s_1, 1\}]cT} (\varpi_{\max\{s_0, \ell q, p s_1, p\}})^{\max\{s_0, \ell+\max\{1, s_1\}\}} \\
& \cdot (\max\{T, 1\})^{\max\{s_0, \ell+\max\{s_1, 1\}+1/p\}} \max\{L_0, 1\} \max\{L_1, 1\} 2^{\max\{\ell-1, 0\}} \\
& \cdot \left[ \max\{C, 1\} + 5 \max\{C, c, 1\} (\max\{\|F_2(\xi)\|, \|\xi\|\} + 2 \max\{\|f_1(0)\|, C, 1\})^{\max\{s_0, \ell+\max\{s_1, 1\}\}} \right].
\end{aligned} \tag{33}$$

The hypothesis that  $\|F_2(\xi) - \xi\| \leq \varepsilon_2(1 + \|\xi\|^2)$  and the fact that

$$\max\{\|\xi\|, \|F_2(\xi)\|\} \leq \|\xi\| + \|F_2(\xi) - \xi\| \leq \|\xi\| + \varepsilon_2(1 + \|\xi\|^2) \quad (34)$$

hence imply that for all  $t \in [0, T]$  it holds that

$$\begin{aligned} & \left| \mathbb{E}[f_0(X_t)] - \mathbb{E}[F_0(Y_t)] \right| \leq (\varepsilon_2(1 + \|\xi\|^2) + \varepsilon_0 + \varepsilon_1 + h + h^{1/2}) \\ & \cdot e^{[\max\{s_0, 1\}L_1 + 1 - 1/p + \ell \max\{L_1, c\} + \max\{s_1, 1\}c]T} (\varpi_{\max\{s_0, \ell q, p s_1, p\}})^{\max\{s_0, \ell + \max\{1, s_1\}\}} \\ & \cdot (\max\{T, 1\})^{\max\{s_0, \ell + \max\{s_1, 1\} + 1/p\}} \max\{L_0, 1\} \max\{L_1, 1\} 2^{\max\{\ell - 1, 0\}} \\ & \cdot \left[ \max\{C, 1\} + 5 \max\{C, c, 1\} (\|\xi\| + \varepsilon_2(1 + \|\xi\|^2) + 2 \max\{\|f_1(0)\|, C, 1\})^{\max\{s_0, \ell + \max\{s_1, 1\}\}} \right]. \end{aligned} \quad (35)$$

This completes the proof of Proposition 2.4.  $\square$

## 2.4 Strong error estimates for linearly interpolated Euler–Maruyama approximations

**Lemma 2.5.** *Let  $d, m, N \in \mathbb{N}$ ,  $T, p \in (0, \infty)$ ,  $C, c \in [0, \infty)$ ,  $q \in [1, \infty)$ ,  $x \in \mathbb{R}^d$ ,  $B \in \mathbb{R}^{d \times m}$ ,  $\tau_0, \tau_1, \dots, \tau_N \in [0, T]$  satisfy that  $0 = \tau_0 < \tau_1 < \dots < \tau_{N-1} < \tau_N = T$ , let  $\mu: \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a measurable function, assume for all  $x \in \mathbb{R}^d$  that  $\|\mu(x)\| \leq C + c\|x\|$ , let  $[\cdot]: [0, T] \rightarrow [0, T]$  satisfy for all  $t \in [0, T]$  that  $[t] = \max(\{\tau_0, \tau_1, \dots, \tau_N\} \cap [0, t])$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $W: [0, T] \times \Omega \rightarrow \mathbb{R}^m$  be a standard Brownian motion, let  $\mathcal{Y}: [0, T] \times \Omega \rightarrow \mathbb{R}^d$  satisfy for all  $t \in [0, T]$  that*

$$\mathcal{Y}_t = x + \int_0^t \mu(\mathcal{Y}_{[s]}) ds + BW_t, \quad (36)$$

and let  $Y: [0, T] \times \Omega \rightarrow \mathbb{R}^d$  satisfy for all  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in [\tau_n, \tau_{n+1}]$  that  $Y_0 = x$  and

$$Y_t = Y_{\tau_n} + \frac{t - \tau_n}{\tau_{n+1} - \tau_n} [\mu(Y_{\tau_n})(\tau_{n+1} - \tau_n) + B(W_{\tau_{n+1}} - W_{\tau_n})] \quad (37)$$

(cf. Definition 2.1). Then

- (i) it holds that  $\mathcal{Y}$  and  $Y$  are stochastic processes,
- (ii) it holds for all  $n \in \{0, 1, \dots, N\}$  that  $Y_{\tau_n} = \mathcal{Y}_{\tau_n}$ ,
- (iii) it holds for all  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in (\tau_n, \tau_{n+1})$  that

$$(\mathbb{E}[\|Y_t - \mathcal{Y}_t\|^p])^{1/p} \leq \frac{1}{2} \sqrt{\max\{1, p-1\}(\tau_{n+1} - \tau_n) \text{Trace}(BB^*)}, \quad (38)$$

and

- (iv) it holds for all  $t \in [0, T]$  that

$$\begin{aligned} & \max\left\{ (\mathbb{E}[\|\mathcal{Y}_t\|^q])^{1/q}, (\mathbb{E}[\|Y_t\|^q])^{1/q} \right\} \\ & \leq \left[ \|x\| + CT + \sqrt{\max\{1, q-1\}T \text{Trace}(BB^*)} \right] e^{cT}. \end{aligned} \quad (39)$$

*Proof of Lemma 2.5.* Throughout this proof for every  $\mathfrak{d} \in \mathbb{N}$  let  $\mathbf{I}_{\mathfrak{d}} \in \mathbb{R}^{\mathfrak{d} \times \mathfrak{d}}$  be the identity matrix in  $\mathbb{R}^{\mathfrak{d} \times \mathfrak{d}}$ , let  $[\cdot]: [0, T] \rightarrow [0, T]$  satisfy for all  $t \in [0, T]$  that  $[t] = \min(\{\tau_0, \tau_1, \dots, \tau_N\} \cap [t, T])$ , and let  $\rho: [0, T] \rightarrow [0, 1]$  satisfy for all  $t \in [0, T]$  that

$$\rho(t) = \begin{cases} \frac{t - [t]}{[t] - [t]} & : t \notin \{\tau_0, \tau_1, \dots, \tau_N\} \\ 0 & : t \in \{\tau_0, \tau_1, \dots, \tau_N\} \end{cases}. \quad (40)$$

Observe that (36), the fact that for all  $t \in [0, T]$  it holds that  $\Omega \ni \omega \mapsto W_t(\omega) \in \mathbb{R}^m$  is measurable, and induction imply that for all  $t \in [0, T]$  it holds that  $\Omega \ni \omega \mapsto \mathcal{Y}_t(\omega) \in \mathbb{R}^d$  is measurable. Moreover, note that (37), the fact that for all  $t \in [0, T]$  it holds that  $\Omega \ni \omega \mapsto W_t(\omega) \in \mathbb{R}^m$  is measurable, and induction prove that for all  $t \in [0, T]$  it holds that  $\Omega \ni \omega \mapsto Y_t(\omega) \in \mathbb{R}^d$  is measurable. Combining this with the fact that for all  $t \in [0, T]$  it holds that  $\Omega \ni \omega \mapsto \mathcal{Y}_t(\omega) \in \mathbb{R}^d$  is measurable establishes item (i). Next we claim that for all  $n \in \{0, 1, \dots, N\}$ ,  $t \in [\tau_{\max\{n-1, 0\}}, \tau_n]$  it holds that  $Y_{\tau_n} = \mathcal{Y}_{\tau_n}$  and

$$Y_t = x + \int_0^t \mu(Y_{[s]}) ds + BW_{[t]} + \rho(t)B(W_{[t]} - W_{[t]}). \quad (41)$$

We prove (41) by induction on  $n \in \{0, 1, \dots, N\}$ . Note that the fact that  $Y_{\tau_0} = x$ , the fact that  $\rho(\tau_0) = 0$ , and the fact that  $W_{\tau_0} = 0$  demonstrate that

$$Y_{\tau_0} = x = x + \int_0^{\tau_0} \mu(Y_{[s]}) ds + BW_{[\tau_0]} + \rho(\tau_0)B(W_{[\tau_0]} - W_{[\tau_0]}). \quad (42)$$

This and the fact that  $\mathcal{Y}_{\tau_0} = x$  prove (41) in the base case  $n = 0$ . For the induction step  $\{0, 1, \dots, N-1\} \ni n \rightarrow n+1 \in \{1, 2, \dots, N\}$  assume that there exists  $n \in \{0, 1, \dots, N-1\}$  which satisfies that for all  $m \in \{0, 1, \dots, n\}$ ,  $t \in [\tau_{\max\{n-1, 0\}}, \tau_n]$  it holds that  $Y_{\tau_m} = \mathcal{Y}_{\tau_m}$  and

$$Y_t = x + \int_0^t \mu(Y_{[s]}) ds + BW_{[t]} + \rho(t)B(W_{[t]} - W_{[t]}). \quad (43)$$

Note that (36) and (43) imply that

$$Y_{\tau_n} = \mathcal{Y}_{\tau_n} = x + \int_0^{\tau_n} \mu(\mathcal{Y}_{[s]}) ds + BW_{\tau_n} = x + \int_0^{\tau_n} \mu(Y_{[s]}) ds + BW_{\tau_n}. \quad (44)$$

Combining this with (37) ensures that for all  $t \in [\tau_n, \tau_{n+1}] = [\tau_{\max\{n, 0\}}, \tau_{n+1}]$  it holds that

$$\begin{aligned} Y_t &= Y_{\tau_n} + (t - \tau_n)\mu(Y_{\tau_n}) + \frac{t - \tau_n}{\tau_{n+1} - \tau_n} B(W_{\tau_{n+1}} - W_{\tau_n}) \\ &= x + \int_0^{\tau_n} \mu(Y_{[s]}) ds + BW_{\tau_n} + (t - \tau_n)\mu(Y_{\tau_n}) + \frac{t - \tau_n}{\tau_{n+1} - \tau_n} B(W_{\tau_{n+1}} - W_{\tau_n}) \\ &= x + \int_0^t \mu(Y_{[s]}) ds + BW_{\tau_n} + \frac{t - \tau_n}{\tau_{n+1} - \tau_n} B(W_{\tau_{n+1}} - W_{\tau_n}). \end{aligned} \quad (45)$$

Therefore, we obtain that for all  $t \in [\tau_{\max\{n, 0\}}, \tau_{n+1}]$  it holds that

$$Y_t = x + \int_0^t \mu(Y_{[s]}) ds + BW_{[t]} + \rho(t)B(W_{[t]} - W_{[t]}). \quad (46)$$

This, (43), and (36) assure that

$$Y_{\tau_{n+1}} = x + \int_0^{\tau_{n+1}} \mu(Y_{[s]}) ds + BW_{\tau_{n+1}} = x + \int_0^{\tau_{n+1}} \mu(\mathcal{Y}_{[s]}) ds + BW_{\tau_{n+1}} = \mathcal{Y}_{\tau_{n+1}}. \quad (47)$$

Combining this with (46) implies that for all  $t \in [\tau_{\max\{n, 0\}}, \tau_{n+1}]$  it holds that  $Y_{\tau_{n+1}} = \mathcal{Y}_{\tau_{n+1}}$  and

$$Y_t = x + \int_0^t \mu(Y_{[s]}) ds + BW_{[t]} + \rho(t)B(W_{[t]} - W_{[t]}). \quad (48)$$

Induction thus proves (41). Next observe that (41) establishes item (ii). Moreover, note that (36), (37), (40), and (41) demonstrate that for all  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in (\tau_n, \tau_{n+1})$  it holds that

$$\begin{aligned}
& Y_t - \mathcal{Y}_t \\
&= Y_{\tau_n} + \rho(t) [(\tau_{n+1} - \tau_n)\mu(Y_{\tau_n}) + B(W_{\tau_{n+1}} - W_{\tau_n})] - \left[ x + \int_0^t \mu(\mathcal{Y}_{[s]}) ds + BW_t \right] \\
&= Y_{\tau_n} + (t - \tau_n)\mu(Y_{\tau_n}) + \rho(t)B(W_{\tau_{n+1}} - W_{\tau_n}) - \left[ \mathcal{Y}_{\tau_n} + \int_{\tau_n}^t \mu(\mathcal{Y}_{[s]}) ds + B(W_t - W_{\tau_n}) \right] \\
&= (t - \tau_n)\mu(Y_{\tau_n}) - (t - \tau_n)\mu(\mathcal{Y}_{\tau_n}) + \rho(t)B(W_{\tau_{n+1}} - W_{\tau_n}) - B(W_t - W_{\tau_n}).
\end{aligned} \tag{49}$$

This and (41) prove that for all  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in (\tau_n, \tau_{n+1})$  it holds that

$$\begin{aligned}
Y_t - \mathcal{Y}_t &= \rho(t)B(W_{\tau_{n+1}} - W_{\tau_n}) + BW_{\tau_n} - BW_t \\
&= -[\rho(t) - 1]BW_{\tau_n} + [(\rho(t) - 1) - \rho(t)]BW_t + \rho(t)BW_{\tau_{n+1}} \\
&= [\rho(t) - 1]B(W_t - W_{\tau_n}) + \rho(t)B(W_{\tau_{n+1}} - W_t).
\end{aligned} \tag{50}$$

In addition, note that the hypothesis that  $W : [0, T] \times \Omega \rightarrow \mathbb{R}^m$  is a standard Brownian motion ensures that

(A) it holds for all  $a, \mathbf{a} \in \mathbb{R}$ ,  $r, s, t \in [0, T]$  with  $r \leq s \leq t$  that  $aB(W_t - W_s) + \mathbf{a}B(W_s - W_r)$  is a centered Gaussian random variable and

(B) it holds for all  $a, \mathbf{a} \in \mathbb{R}$ ,  $r, s, t \in [0, T]$  with  $r \leq s \leq t$  that

$$\text{Cov}(aB(W_t - W_s) + \mathbf{a}B(W_s - W_r)) = [a^2(t - s) + \mathbf{a}^2(s - r)] BB^*. \tag{51}$$

Combining this with (50) ensures that for all  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in (\tau_n, \tau_{n+1})$  it holds that  $Y_t - \mathcal{Y}_t$  is a centered Gaussian random variable. Moreover, note that (50) and (51) demonstrate that for all  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in (\tau_n, \tau_{n+1})$  it holds that

$$\text{Cov}(Y_t - \mathcal{Y}_t) = ([\rho(t) - 1]^2(t - \tau_n) + [\rho(t)]^2(\tau_{n+1} - t)) BB^*. \tag{52}$$

In addition, observe that (40) implies that for all  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in (\tau_n, \tau_{n+1})$  it holds that

$$\begin{aligned}
& [\rho(t) - 1]^2(t - \tau_n) + [\rho(t)]^2(\tau_{n+1} - t) \\
&= [\rho(t)]^2(\tau_{n+1} - \tau_n) + [1 - 2\rho(t)](t - \tau_n) \\
&= \frac{(t - \tau_n)^2}{(\tau_{n+1} - \tau_n)} + (t - \tau_n) - \frac{2(t - \tau_n)^2}{(\tau_{n+1} - \tau_n)} = (t - \tau_n) \left[ 1 - \frac{(t - \tau_n)}{(\tau_{n+1} - \tau_n)} \right] \\
&= \frac{(t - \tau_n)(\tau_{n+1} - t)}{(\tau_{n+1} - \tau_n)}.
\end{aligned} \tag{53}$$

This and the fact that for all  $a \in \mathbb{R}$ ,  $b \in (a, \infty)$ ,  $r \in [a, b]$  it holds that

$$(r - a)(b - r) \leq \left(\frac{1}{2}(b + a) - a\right) \left(b - \frac{1}{2}(b + a)\right) = \frac{1}{4}(b - a)^2 \tag{54}$$

show that for all  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in (\tau_n, \tau_{n+1})$  it holds that

$$[\rho(t) - 1]^2(t - \tau_n) + [\rho(t)]^2(\tau_{n+1} - t) \leq \frac{1}{4}(\tau_{n+1} - \tau_n). \tag{55}$$

The fact that  $BB^*$  is a symmetric positive semidefinite matrix and (52) therefore imply that for all  $t \in [0, T]$  it holds that

$$\text{Trace}(\text{Cov}(Y_t - \mathcal{Y}_t)) \leq \frac{1}{4}(\tau_{n+1} - \tau_n) \text{Trace}(BB^*). \quad (56)$$

Lemma 2.2 hence demonstrates that for all  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in (\tau_n, \tau_{n+1})$  it holds that

$$\begin{aligned} (\mathbb{E}[\|Y_t - \mathcal{Y}_t\|^p])^{1/p} &\leq \sqrt{\max\{1, p-1\} \text{Trace}(\text{Cov}(Y_t - \mathcal{Y}_t))} \\ &\leq \frac{1}{2} \sqrt{\max\{1, p-1\} (\tau_{n+1} - \tau_n) \text{Trace}(BB^*)}. \end{aligned} \quad (57)$$

This establishes item (iii). Next note that Lemma 2.2, Lemma 2.3, the fact that for all  $t \in [0, T]$  it holds that  $BW_t$  is a centered Gaussian random variable, and the fact that for all  $t \in [0, T]$  it holds that  $\text{Cov}(BW_t) = BB^*t$  ensure that for all  $t \in [0, T]$  it holds that

$$\begin{aligned} (\mathbb{E}[\|\mathcal{Y}_t\|^q])^{1/q} &\leq \left[ \|x\| + CT + \sup_{t \in [0, T]} (\mathbb{E}[\|BW_t\|^q])^{1/q} \right] e^{cT} \\ &\leq \left[ \|x\| + CT + \sup_{t \in [0, T]} \sqrt{\max\{1, q-1\} \text{Trace}(BB^*)t} \right] e^{cT} \\ &= \left[ \|x\| + CT + \sqrt{\max\{1, q-1\} \text{Trace}(BB^*)T} \right] e^{cT}. \end{aligned} \quad (58)$$

Next note that (51), the fact that  $W_0 = 0$ , the fact that  $BB^*$  is a symmetric positive semidefinite matrix, and the fact that  $\forall t \in [0, T]: 0 \leq \rho(t) \leq 1$  imply that

- a) it holds for all  $t \in [0, T]$  that  $BW_{[t]} + \rho(t)B(W_{[t]} - W_{[t]})$  is a centered Gaussian random variable and
- b) it holds for all  $t \in [0, T]$  that

$$\begin{aligned} &\text{Trace}(\text{Cov}(BW_{[t]} + \rho(t)B(W_{[t]} - W_{[t]}))) \\ &= \text{Trace}(BB^*) [ [t] + [\rho(t)]^2 ([t] - [t]) ] \leq \text{Trace}(BB^*) [t]. \end{aligned} \quad (59)$$

Combining this with (41), Lemma 2.2, and Lemma 2.3 demonstrates that for all  $t \in [0, T]$  it holds that

$$\begin{aligned} (\mathbb{E}[\|Y_t\|^q])^{1/q} &\leq \left[ \|x\| + CT + \sup_{t \in [0, T]} (\mathbb{E}[\|BW_{[t]} + \rho(t)B(W_{[t]} - W_{[t]})\|^q])^{1/q} \right] e^{cT} \\ &\leq \left[ \|x\| + CT + \sup_{t \in [0, T]} \sqrt{\max\{1, q-1\} \text{Trace}(BB^*) [t]} \right] e^{cT} \\ &= \left[ \|x\| + CT + \sqrt{\max\{1, q-1\} \text{Trace}(BB^*)T} \right] e^{cT}. \end{aligned} \quad (60)$$

This and (58) establish item (iv). This completes the proof of Lemma 2.5.  $\square$

### 3 Numerical approximations for partial differential equations (PDEs)

In this section we use the weak and strong error estimates which we have presented in Proposition 2.4 and Lemma 2.5 in Section 2 above to establish in Proposition 3.2 below suitable error estimates for Monte Carlo Euler approximations for a class of SDEs with perturbed drift

coefficient functions. Besides Proposition 2.4 and Lemma 2.5, our proof of Proposition 3.2 also employs a special case of the famous Feynman–Kac formula, which provides a connection between solutions of SDEs and solutions of deterministic Kolmogorov PDEs. For completeness we briefly recall in Proposition 3.1 below this special case of the Feynman–Kac formula. Proposition 3.1 is, e.g., proved as Beck et al. [6, Theorem 1.1] (cf. also, e.g., Hairer et al. [25, Subsection 4.4] and Jentzen et al. [35, Theorem 3.1]).

### 3.1 On the Feynman–Kac formula for additive noise driven SDEs

**Proposition 3.1.** *Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $T \in (0, \infty)$ ,  $d, m \in \mathbb{N}$ ,  $B \in \mathbb{R}^{d \times m}$ ,  $\varphi \in C(\mathbb{R}^d, \mathbb{R})$ , let  $W: [0, T] \times \Omega \rightarrow \mathbb{R}^m$  be a standard Brownian motion, let  $\langle \cdot, \cdot \rangle: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  be the standard scalar product on  $\mathbb{R}^d$ , let  $\mu: \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a locally Lipschitz continuous function, and assume that*

$$\inf_{p \in (0, \infty)} \sup_{x \in \mathbb{R}^d} \left[ \frac{|\varphi(x)|}{(1 + \|x\|^p)} + \frac{\|\mu(x)\|}{(1 + \|x\|)} \right] < \infty \quad (61)$$

(cf. Definition 2.1). Then

- (i) *there exist unique stochastic processes  $X^x: [0, T] \times \Omega \rightarrow \mathbb{R}^d$ ,  $x \in \mathbb{R}^d$ , with continuous sample paths which satisfy for all  $x \in \mathbb{R}^d$ ,  $t \in [0, T]$  that*

$$X_t^x = x + \int_0^t \mu(X_s^x) ds + BW_t, \quad (62)$$

- (ii) *there exists a unique viscosity solution  $u \in \{v \in C([0, T] \times \mathbb{R}^d, \mathbb{R}) : \inf_{p \in (0, \infty)} \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \frac{|v(t, x)|}{1 + \|x\|^p} < \infty\}$  of*

$$\left(\frac{\partial}{\partial t} u\right)(t, x) = \langle (\nabla_x u)(t, x), \mu(x) \rangle + \frac{1}{2} \text{Trace}(BB^*(\text{Hess}_x u)(t, x)) \quad (63)$$

with  $u(0, x) = \varphi(x)$  for  $(t, x) \in (0, T) \times \mathbb{R}^d$ , and

- (iii) *it holds for all  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that  $\mathbb{E}[|\varphi(X_t^x)|] < \infty$  and  $u(t, x) = \mathbb{E}[\varphi(X_t^x)]$ .*

### 3.2 Approximation error estimates for Monte Carlo Euler approximations

**Proposition 3.2.** *Let  $T, \kappa \in (0, \infty)$ ,  $\eta \in [1, \infty)$ ,  $p \in [2, \infty)$ , let  $A_d = (a_{d, i, j})_{(i, j) \in \{1, 2, \dots, d\}^2} \in \mathbb{R}^{d \times d}$ ,  $d \in \mathbb{N}$ , be symmetric positive semidefinite matrices, let  $\nu_d: \mathcal{B}([0, T] \times \mathbb{R}^d) \rightarrow [0, \infty)$ ,  $d \in \mathbb{N}$ , be finite measures which satisfy for all  $d \in \mathbb{N}$  that*

$$\left[ \int_{[0, T] \times \mathbb{R}^d} \|x\|^{2p \max\{2\kappa, 3\}} \nu_d(dt, dx) \right]^{1/p} \leq \eta d^n, \quad (64)$$

let  $f_d^m \in C(\mathbb{R}^d, \mathbb{R}^{md-m+1})$ ,  $m \in \{0, 1\}$ ,  $d \in \mathbb{N}$ , and  $F_{d, \varepsilon}^m \in C(\mathbb{R}^d, \mathbb{R}^{md-m+1})$ ,  $m \in \{0, 1\}$ ,  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$ , satisfy for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$ ,  $m \in \{0, 1\}$ ,  $x, y \in \mathbb{R}^d$  that

$$|f_d^0(x)| + \text{Trace}(A_d) \leq \kappa d^\kappa (1 + \|x\|^\kappa), \quad \|f_d^1(x) - f_d^1(y)\| \leq \kappa \|x - y\|, \quad (65)$$

$$\|f_d^m(x) - F_{d, \varepsilon}^m(x)\| \leq \varepsilon \kappa d^\kappa (1 + \|x\|^\kappa), \quad \|F_{d, \varepsilon}^1(x)\| \leq \kappa (d^\kappa + \|x\|), \quad (66)$$

$$\text{and} \quad |F_{d, \varepsilon}^0(x) - F_{d, \varepsilon}^0(y)| \leq \kappa d^\kappa (1 + \|x\|^\kappa + \|y\|^\kappa) \|x - y\|, \quad (67)$$

let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $W^{d, m}: [0, T] \times \Omega \rightarrow \mathbb{R}^d$ ,  $d, m \in \mathbb{N}$ , be independent standard Brownian motions, and let  $Y^{N, d, m, x}: [0, T] \times \Omega \rightarrow \mathbb{R}^d$ ,  $x \in \mathbb{R}^d$ ,  $N, d, m \in \mathbb{N}$ , be stochastic

processes which satisfy for all  $N, d, m \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$ ,  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in [\frac{nT}{N}, \frac{(n+1)T}{N}]$  that  $Y_0^{N,d,m,x} = x$  and

$$Y_t^{N,d,m,x} = Y_{\frac{nT}{N}}^{N,d,m,x} + \left(\frac{tN}{T} - n\right) \left[ \frac{T}{N} F_{d, \min\{(T/N)^{1/2}, 1\}}^1 \left( Y_{\frac{nT}{N}}^{N,d,m,x} \right) + \sqrt{2A_d} \left( W_{\frac{(n+1)T}{N}}^{d,m} - W_{\frac{nT}{N}}^{d,m} \right) \right] \quad (68)$$

(cf. Definition 2.1). Then

(i) for every  $d \in \mathbb{N}$  there exists a unique viscosity solution  $u_d \in \{v \in C([0, T] \times \mathbb{R}^d, \mathbb{R}) : \inf_{q \in (0, \infty)} \sup_{(t,x) \in [0, T] \times \mathbb{R}^d} \frac{|v(t,x)|}{1+|x|^q} < \infty\}$  of

$$\left(\frac{\partial}{\partial t} u_d\right)(t, x) = \left(\frac{\partial}{\partial x} u_d\right)(t, x) f_d^1(x) + \sum_{i,j=1}^d a_{d,i,j} \left(\frac{\partial^2}{\partial x_i \partial x_j} u_d\right)(t, x) \quad (69)$$

with  $u_d(0, x) = f_d^0(x)$  for  $(t, x) \in (0, T) \times \mathbb{R}^d$  and

(ii) there exists  $\mathcal{C} \in \mathbb{R}$  such that for all  $d, N, M \in \mathbb{N}$  it holds that

$$\begin{aligned} & \left( \mathbb{E} \left[ \int_{[0, T] \times \mathbb{R}^d} |u_d(t, x) - \frac{1}{M} \left[ \sum_{m=1}^M F_{d, \min\{(T/N)^{1/2}, 1\}}^0 \left( Y_t^{N,d,m,x} \right) \right]|^p \nu_d(dt, dx) \right] \right)^{1/p} \\ & \leq \mathcal{C} \left[ \frac{d^{\kappa(\kappa+4) + \max\{\eta, \kappa(2\kappa+1)\}}}{N^{1/2}} + \frac{d^{\kappa + \max\{\eta, \kappa^2\}}}{M^{1/2}} \right] [\max\{1, \nu_d([0, T] \times \mathbb{R}^d)\}]^{1/p}. \end{aligned} \quad (70)$$

*Proof of Proposition 3.2.* Throughout this proof let  $\iota \in \mathbb{R}$  satisfy that  $\iota = \max\{\kappa, 1\}$ , let  $C, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C} \in (0, \infty)$  satisfy that

$$C = e^{\kappa^2 T} 2^{\max\{0, \kappa-1\}} \left( \eta + \left[ \kappa T + \max\left\{1, \sqrt{2(p\iota - 1)\kappa}\right\} T^{1/2} \right]^\kappa \right), \quad (71)$$

$$\begin{aligned} \mathcal{C}_1 &= \iota^2 2^\iota (\kappa + 1) \left[ \max\left\{1, \sqrt{2 \max\{1, 2\kappa - 1\}\kappa}\right\} \right]^{2\iota} e^{[3\iota^2 + 1/2]T} (\max\{T, 1\})^{\kappa + \iota + 3/2} \\ & \cdot \left[ \max\{2\kappa(\kappa + 1), 1\} \right] \left[ 1 + 5\eta 2^{\kappa + \iota - 1} + 5(4\iota)^{\kappa + \iota} 2^{\kappa + \iota - 1} \right], \end{aligned} \quad (72)$$

$$\mathcal{C}_2 = \frac{1}{\sqrt{2}} \kappa^{3/2} e^{\kappa^2 T} 2^\iota (\max\{T, 1\})^{\kappa + 1/2} \left( \eta + 1 + \left[ \kappa + \max\left\{1, \sqrt{2 \max\{1, 2\kappa - 1\}\kappa}\right\} \right]^\kappa \right), \quad (73)$$

and

$$\mathcal{C} = \max\{\mathcal{C}_1 + \mathcal{C}_2, 8\kappa(1 + C)\sqrt{p-1}\}, \quad (74)$$

let  $N, M \in \mathbb{N}$ ,  $\delta \in (0, \infty)$  satisfy that  $\delta = \sqrt{T/N}$ , let  $\mathcal{A}_d \in \mathbb{R}^{d \times d}$ ,  $d \in \mathbb{N}$ , satisfy for all  $d \in \mathbb{N}$  that  $\mathcal{A}_d = \sqrt{2A_d}$ , let  $\varpi_{d,q} \in \mathbb{R}$ ,  $d \in \mathbb{N}$ ,  $q \in (0, \infty)$ , satisfy for all  $q \in (0, \infty)$ ,  $d \in \mathbb{N}$  that

$$\varpi_{d,q} = \max\left\{1, \sqrt{\max\{1, q-1\} \text{Trace}((\mathcal{A}_d)^* \mathcal{A}_d)}\right\}, \quad (75)$$

let  $X^{d,x} : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ ,  $x \in \mathbb{R}^d$ ,  $d \in \mathbb{N}$ , be stochastic processes with continuous sample paths which satisfy for all  $d \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$ ,  $t \in [0, T]$  that

$$X_t^{d,x} = x + \int_0^t f_d^1(X_s^{d,x}) ds + \mathcal{A}_d W_t^{d,1} \quad (76)$$

(cf. item (i) in Proposition 3.1), let  $[\cdot] : [0, T] \rightarrow [0, T]$  satisfy for all  $t \in [0, T]$  that

$$[t] = \max(\{0, \delta^2, 2\delta^2, \dots\} \cap [0, t]), \quad (77)$$

let  $[\cdot]: [0, T] \rightarrow [0, T]$  satisfy for all  $t \in [0, T]$  that

$$[t] = \min(\{0, \delta^2, 2\delta^2, \dots\} \cap [t, T]), \quad (78)$$

and let  $\mathcal{Y}^{d,x}: [0, T] \times \Omega \rightarrow \mathbb{R}^d$ ,  $x \in \mathbb{R}^d$ ,  $d \in \mathbb{N}$ , be stochastic processes with continuous sample paths which satisfy for all  $d \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$ ,  $t \in [0, T]$  that

$$\mathcal{Y}_t^{d,x} = x + \int_0^t F_{d, \min\{\delta, 1\}}^1(\mathcal{Y}_{[s]}^{d,x}) ds + \mathcal{A}_d W_t^{d,1}. \quad (79)$$

Note that Hölder's inequality and (64) imply that for all  $d \in \mathbb{N}$ ,  $r \in (0, 2 \max\{2\kappa, 3\})$  it holds that

$$\begin{aligned} & \left[ \int_{[0, T] \times \mathbb{R}^d} \|x\|^{pr} \nu_d(dt, dx) \right]^{1/p} \\ & \leq \left[ \int_{[0, T] \times \mathbb{R}^d} \|x\|^{2p \max\{2\kappa, 3\}} \nu_d(dt, dx) \right]^{r/(2p \max\{2\kappa, 3\})} [\nu_d([0, T] \times \mathbb{R}^d)]^{(1-r/(2 \max\{2\kappa, 3\}))/p} \\ & \leq (\eta d^n)^{r/(2 \max\{2\kappa, 3\})} \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\} \\ & \leq \eta d^n \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\}. \end{aligned} \quad (80)$$

Furthermore, observe that (65) and Proposition 3.1 establish item (i). It thus remains to prove item (ii). For this note that the triangle inequality and Proposition 3.1 ensure that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned} & \left[ \int_{[0, T] \times \mathbb{R}^d} \mathbb{E} \left[ \left| u_d(t, x) - \frac{1}{M} \left[ \sum_{m=1}^M F_{d, \min\{\delta, 1\}}^0(Y_t^{N, d, m, x}) \right] \right|^p \right] \nu_d(dt, dx) \right]^{1/p} \\ & \leq \left[ \int_{[0, T] \times \mathbb{R}^d} \mathbb{E} \left[ \left| u_d(t, x) - \mathbb{E} \left[ F_{d, \min\{\delta, 1\}}^0(Y_t^{N, d, 1, x}) \right] \right|^p \right] \nu_d(dt, dx) \right]^{1/p} \\ & + \left[ \int_{[0, T] \times \mathbb{R}^d} \mathbb{E} \left[ \left| \mathbb{E} \left[ F_{d, \min\{\delta, 1\}}^0(Y_t^{N, d, 1, x}) \right] - \frac{1}{M} \left[ \sum_{m=1}^M F_{d, \min\{\delta, 1\}}^0(Y_t^{N, d, m, x}) \right] \right|^p \right] \nu_d(dt, dx) \right]^{1/p} \\ & = \left[ \int_{[0, T] \times \mathbb{R}^d} \left| \mathbb{E} \left[ f_d^0(X_t^{d, x}) \right] - \mathbb{E} \left[ F_{d, \min\{\delta, 1\}}^0(Y_t^{N, d, 1, x}) \right] \right|^p \nu_d(dt, dx) \right]^{1/p} \\ & + \left[ \int_{[0, T] \times \mathbb{R}^d} \mathbb{E} \left[ \left| \mathbb{E} \left[ F_{d, \min\{\delta, 1\}}^0(Y_t^{N, d, 1, x}) \right] - \frac{1}{M} \left[ \sum_{m=1}^M F_{d, \min\{\delta, 1\}}^0(Y_t^{N, d, m, x}) \right] \right|^p \right] \nu_d(dt, dx) \right]^{1/p}. \end{aligned} \quad (81)$$

This and, e.g., [22, Corollary 2.5] prove that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned} & \left[ \int_{[0, T] \times \mathbb{R}^d} \mathbb{E} \left[ \left| u_d(t, x) - \frac{1}{M} \left[ \sum_{m=1}^M F_{d, \min\{\delta, 1\}}^0(Y_t^{N, d, m, x}) \right] \right|^p \right] \nu_d(dt, dx) \right]^{1/p} \\ & \leq \left[ \int_{[0, T] \times \mathbb{R}^d} \left| \mathbb{E} \left[ f_d^0(X_t^{d, x}) \right] - \mathbb{E} \left[ F_{d, \min\{\delta, 1\}}^0(Y_t^{N, d, 1, x}) \right] \right|^p \nu_d(dt, dx) \right]^{1/p} \\ & + \frac{2\sqrt{p-1}}{M^{1/2}} \left[ \int_{[0, T] \times \mathbb{R}^d} \mathbb{E} \left[ \left| F_{d, \min\{\delta, 1\}}^0(Y_t^{N, d, 1, x}) - \mathbb{E} \left[ F_{d, \min\{\delta, 1\}}^0(Y_t^{N, d, 1, x}) \right] \right|^p \right] \nu_d(dt, dx) \right]^{1/p}. \end{aligned} \quad (82)$$



Next note that the triangle inequality and Hölder's inequality imply for all  $d \in \mathbb{N}$  that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} \left[ \left| F_{d,\min\{\delta,1\}}^0(Y_t^{N,d,1,x}) - \mathbb{E} \left[ F_{d,\min\{\delta,1\}}^0(Y_t^{N,d,1,x}) \right] \right|^p \nu_d(dt, dx) \right]^{1/p} \\
& \leq \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} \left[ \left| F_{d,\min\{\delta,1\}}^0(Y_t^{N,d,1,x}) \right|^p \nu_d(dt, dx) \right]^{1/p} \\
& + \left[ \int_{[0,T] \times \mathbb{R}^d} \left| \mathbb{E} \left[ F_{d,\min\{\delta,1\}}^0(Y_t^{N,d,1,x}) \right] \right|^p \nu_d(dt, dx) \right]^{1/p} \\
& \leq 2 \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} \left[ \left| F_{d,\min\{\delta,1\}}^0(Y_t^{N,d,1,x}) \right|^p \nu_d(dt, dx) \right]^{1/p}.
\end{aligned} \tag{83}$$

Combining this and (82) demonstrates for all  $d \in \mathbb{N}$  that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} \left[ \left| u_d(t, x) - \frac{1}{M} \left[ \sum_{m=1}^M F_{d,\min\{\delta,1\}}^0(Y_t^{N,d,m,x}) \right] \right|^p \nu_d(dt, dx) \right]^{1/p} \\
& \leq \left[ \int_{[0,T] \times \mathbb{R}^d} \left| \mathbb{E} \left[ f_d^0(X_t^{d,x}) \right] - \mathbb{E} \left[ F_{d,\min\{\delta,1\}}^0(Y_t^{N,d,1,x}) \right] \right|^p \nu_d(dt, dx) \right]^{1/p} \\
& + \frac{4\sqrt{p-1}}{M^{1/2}} \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} \left[ \left| F_{d,\min\{\delta,1\}}^0(Y_t^{N,d,1,x}) \right|^p \nu_d(dt, dx) \right]^{1/p}.
\end{aligned} \tag{84}$$

Next observe that the fact that  $\forall a, b, q \in [0, \infty): a^q + b^q \leq (a+b)^q + (a+b)^q = 2(a+b)^q$  proves that for all  $d \in \mathbb{N}, x, y \in \mathbb{R}^d$  it holds that

$$\begin{aligned}
& 2 \int_0^1 [r\|x\| + (1-r)\|y\|]^\kappa dr \geq \int_0^1 [r^\kappa\|x\|^\kappa + (1-r)^\kappa\|y\|^\kappa] dr \\
& = [\|x\|^\kappa + \|y\|^\kappa] \int_0^1 r^\kappa dr = \frac{[\|x\|^\kappa + \|y\|^\kappa]}{\kappa + 1}.
\end{aligned} \tag{85}$$

This and (67) show that for all  $d \in \mathbb{N}, x, y \in \mathbb{R}^d$  it holds that

$$\begin{aligned}
& |F_{d,\min\{\delta,1\}}^0(x) - F_{d,\min\{\delta,1\}}^0(y)| \leq \kappa d^\kappa (1 + \|x\|^\kappa + \|y\|^\kappa) \|x - y\| \\
& \leq \kappa d^\kappa \left[ 1 + 2(\kappa + 1) \int_0^1 [r\|x\| + (1-r)\|y\|]^\kappa dr \right] \|x - y\| \\
& \leq 2\kappa(\kappa + 1)d^\kappa \left[ 1 + \int_0^1 [r\|x\| + (1-r)\|y\|]^\kappa dr \right] \|x - y\|.
\end{aligned} \tag{86}$$

Proposition 2.4 (applied with  $d \leftarrow d, m \leftarrow d, \xi \leftarrow x, T \leftarrow T, c \leftarrow \kappa, C \leftarrow \kappa d^\kappa, \varepsilon_0 \leftarrow \min\{\delta, 1\}\kappa d^\kappa, \varepsilon_1 \leftarrow \min\{\delta, 1\}\kappa d^\kappa, \varepsilon_2 \leftarrow 0, \varsigma_0 \leftarrow \kappa, \varsigma_1 \leftarrow \kappa, \varsigma_2 \leftarrow 0, L_0 \leftarrow 2\kappa(\kappa + 1)d^\kappa, L_1 \leftarrow \kappa, \ell \leftarrow \kappa, h \leftarrow \delta^2, p \leftarrow 2, q \leftarrow 2, B \leftarrow \mathcal{A}_d, (\varpi_r)_{r \in (0, \infty)} \leftarrow (\varpi_{d,r})_{r \in (0, \infty)}, \|\cdot\| \leftarrow \|\cdot\|, (\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P}), W \leftarrow W^{d,1}, F_0 \leftarrow F_{d,\min\{\delta,1\}}^0, f_1 \leftarrow f_d^1, F_2 \leftarrow \text{id}_{\mathbb{R}^d}, \chi(s) \leftarrow \lfloor s \rfloor, f_0 \leftarrow f_d^0, F_1 \leftarrow F_{d,\min\{\delta,1\}}^1, X \leftarrow X^{d,x}, Y \leftarrow \mathcal{Y}^{d,x}$  for  $d \in \mathbb{N}, x \in \mathbb{R}^d, s \in [0, T]$  in the notation of Proposition 2.4) hence ensures that for all  $d \in \mathbb{N}, x \in \mathbb{R}^d, t \in [0, T]$  it holds that

$$\begin{aligned}
& \left| \mathbb{E} \left[ f_d^0(X_t^{d,x}) \right] - \mathbb{E} \left[ F_{d,\min\{\delta,1\}}^0(\mathcal{Y}_t^{d,x}) \right] \right| \leq (\min\{\delta, 1\}\kappa d^\kappa + \min\{\delta, 1\}\kappa d^\kappa + \delta^2 + \delta) \\
& \cdot e^{[\max\{\kappa, 1\}\kappa + 1 - 1/2 + \kappa \max\{\kappa, \kappa\} + \max\{\kappa, 1\}\kappa]T} (\varpi_{d,\max\{\kappa, 2\kappa, 2\kappa, 2\}})^{\max\{\kappa, \kappa + \max\{1, \kappa\}\}} \\
& \cdot (\max\{T, 1\})^{\max\{\kappa, \kappa + \max\{\kappa, 1\} + 1/2\}} \max\{2\kappa(\kappa + 1)d^\kappa, 1\} \max\{\kappa, 1\} 2^{\max\{\kappa - 1, 0\}} \\
& \cdot \left[ \max\{\kappa d^\kappa, 1\} + 5 \max\{\kappa d^\kappa, \kappa, 1\} (\|x\| + 2 \max\{\|f_d^1(0)\|, \kappa d^\kappa, 1\})^{\max\{\kappa, \kappa + \max\{\kappa, 1\}\}} \right].
\end{aligned} \tag{87}$$

Next note that (65) demonstrates that for all  $d \in \mathbb{N}$ ,  $q \in (0, \infty)$  it holds that

$$\begin{aligned}
\varpi_{d,q} &= \max \left\{ 1, \sqrt{\max\{1, q-1\} \text{Trace}((\mathcal{A}_d)^* \mathcal{A}_d)} \right\} \\
&= \max \left\{ 1, \sqrt{2 \max\{1, q-1\} \text{Trace}(A_d)} \right\} \\
&\leq \max \left\{ 1, \sqrt{2 \max\{1, q-1\} \kappa d^\kappa} \right\} \\
&\leq d^{\kappa/2} \max \left\{ 1, \sqrt{2 \max\{1, q-1\} \kappa} \right\}.
\end{aligned} \tag{88}$$

Moreover, observe that (66) implies that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned}
\|f_d^1(0)\| &\leq \|f_d^1(0) - F_{d,\min\{\delta,1\}}^1(0)\| + \|F_{d,\min\{\delta,1\}}^1(0)\| \\
&\leq \min\{\delta, 1\} \kappa d^\kappa + \kappa d^\kappa \leq 2\kappa d^\kappa.
\end{aligned} \tag{89}$$

This, (87), (88) and the fact that  $\iota = \max\{\kappa, 1\}$  ensure that for all  $d \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$ ,  $t \in [0, T]$  it holds that

$$\begin{aligned}
&|\mathbb{E}[f_d^0(X_t^{d,x})] - \mathbb{E}[F_{d,\min\{\delta,1\}}^0(\mathcal{Y}_t^{d,x})]| \\
&\leq (2\delta\kappa d^\kappa + \delta^2 + \delta) e^{[3\iota^2+1/2]T} \left[ d^{\kappa/2} \max \left\{ 1, \sqrt{2 \max\{1, 2\kappa-1\} \kappa} \right\} \right]^{2\iota} \\
&\quad \cdot (\max\{T, 1\})^{\kappa+\iota+1/2} \max\{2\kappa(\kappa+1)d^\kappa, 1\} \iota 2^{\iota-1} \\
&\quad \cdot \left[ \max\{\kappa d^\kappa, 1\} + 5 \max\{\kappa d^\kappa, 1\} (\|x\| + 2 \max\{2\kappa d^\kappa, 1\})^{\kappa+\iota} \right].
\end{aligned} \tag{90}$$

The triangle inequality hence ensures that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned}
&\left[ \int_{[0,T] \times \mathbb{R}^d} |\mathbb{E}[f_d^0(X_t^{d,x})] - \mathbb{E}[F_{d,\min\{\delta,1\}}^0(\mathcal{Y}_t^{d,x})]|^p \nu_d(dt, dx) \right]^{1/p} \\
&\leq (2\delta\kappa d^\kappa + \delta^2 + \delta) e^{[3\iota^2+1/2]T} \left[ d^{\kappa/2} \max \left\{ 1, \sqrt{2 \max\{1, 2\kappa-1\} \kappa} \right\} \right]^{2\iota} \\
&\quad \cdot (\max\{T, 1\})^{\kappa+\iota+1/2} \max\{2\kappa(\kappa+1)d^\kappa, 1\} \iota 2^{\iota-1} \left[ \max\{\kappa d^\kappa, 1\} [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p} \right. \\
&\quad \left. + 5 \max\{\kappa d^\kappa, 1\} \left[ \int_{[0,T] \times \mathbb{R}^d} (\|x\| + 2 \max\{2\kappa d^\kappa, 1\})^{p(\kappa+\iota)} \nu_d(dt, dx) \right]^{1/p} \right].
\end{aligned} \tag{91}$$

Moreover, observe that the fact that  $\forall y, z \in \mathbb{R}, \alpha \in [1, \infty): |y+z|^\alpha \leq 2^{\alpha-1}(|y|^\alpha + |z|^\alpha)$  and the triangle inequality demonstrate that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned}
&\left[ \int_{[0,T] \times \mathbb{R}^d} (\|x\| + 2 \max\{2\kappa d^\kappa, 1\})^{p(\kappa+\iota)} \nu_d(dt, dx) \right]^{1/p} \\
&\leq 2^{\kappa+\iota-1} \left[ \int_{[0,T] \times \mathbb{R}^d} \left[ \|x\|^{\kappa+\iota} + (2 \max\{2\kappa d^\kappa, 1\})^{\kappa+\iota} \right]^p \nu_d(dt, dx) \right]^{1/p} \\
&\leq 2^{\kappa+\iota-1} \left[ \int_{[0,T] \times \mathbb{R}^d} \|x\|^{p(\kappa+\iota)} \nu_d(dt, dx) \right]^{1/p} + 2^{\kappa+\iota-1} [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p} (2 \max\{2\kappa d^\kappa, 1\})^{\kappa+\iota}.
\end{aligned} \tag{92}$$

This, the fact that  $\kappa + \iota = \max\{2\kappa, \kappa + 1\} < 2 \max\{2\kappa, 3\}$ , and (80) ensure that for all  $d \in \mathbb{N}$

it holds that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} (\|x\| + 2 \max\{2\kappa d^\kappa, 1\})^{p(\kappa+\iota)} \nu_d(dt, dx) \right]^{1/p} \\
& \leq 2^{\kappa+\iota-1} \eta d^\eta \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\} \\
& + 2^{\kappa+\iota-1} [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p} (2 \max\{2\kappa d^\kappa, 1\})^{\kappa+\iota} \\
& \leq 2^{\kappa+\iota-1} \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\} [\eta d^\eta + (2 \max\{2\kappa d^\kappa, 1\})^{\kappa+\iota}].
\end{aligned} \tag{93}$$

In addition, note that the fact that  $\delta = \sqrt{T/N}$  proves that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned}
2\delta\kappa d^\kappa + \delta^2 + \delta & \leq 2\delta\kappa d^\kappa + \delta\sqrt{T} + \delta \\
& \leq 2\delta\kappa d^\kappa + 2 \max\{\sqrt{T}, 1\} \delta \leq 2 \max\{\sqrt{T}, 1\} (\delta\kappa d^\kappa + \delta).
\end{aligned} \tag{94}$$

Combining this, (91), and (93) shows that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} |\mathbb{E}[f_d^0(X_t^{d,x})] - \mathbb{E}[F_{d,\min\{\delta,1\}}^0(\mathcal{Y}_t^{d,x})]|^p \nu_d(dt, dx) \right]^{1/p} \\
& \leq \iota 2^\iota \max\{\sqrt{T}, 1\} (\delta\kappa d^\kappa + \delta) \left[ d^{\kappa/2} \max\left\{1, \sqrt{2 \max\{1, 2\kappa - 1\} \kappa}\right\} \right]^{2\iota} \\
& \cdot e^{[3\iota^2+1/2]T} (\max\{T, 1\})^{\kappa+\iota+1/2} \max\{2\kappa(\kappa+1)d^\kappa, 1\} \max\{\kappa d^\kappa, 1\} \\
& \cdot \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\} \left(1 + 5[\eta d^\eta + (2 \max\{2\kappa d^\kappa, 1\})^{\kappa+\iota}] 2^{\kappa+\iota-1}\right).
\end{aligned} \tag{95}$$

Hence, we obtain that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} |\mathbb{E}[f_d^0(X_t^{d,x})] - \mathbb{E}[F_{d,\min\{\delta,1\}}^0(\mathcal{Y}_t^{d,x})]|^p \nu_d(dt, dx) \right]^{1/p} \\
& \leq \iota 2^\iota d^{3\kappa+\kappa\iota} \delta (\kappa+1) \left[ \max\left\{1, \sqrt{2 \max\{1, 2\kappa - 1\} \kappa}\right\} \right]^{2\iota} \\
& \cdot e^{[3\iota^2+1/2]T} (\max\{T, 1\})^{\kappa+\iota+1} \max\{2\kappa(\kappa+1), 1\} \max\{\kappa, 1\} \\
& \cdot \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\} \left(1 + 5[\eta d^\eta + (2 \max\{2\kappa d^\kappa, 1\})^{\kappa+\iota}] 2^{\kappa+\iota-1}\right) \\
& \leq \iota^2 2^\iota d^{3\kappa+\kappa\iota+\max\{\eta, \kappa(\kappa+\iota)\}} \delta (\kappa+1) \left[ \max\left\{1, \sqrt{2 \max\{1, 2\kappa - 1\} \kappa}\right\} \right]^{2\iota} \\
& \cdot e^{[3\iota^2+1/2]T} (\max\{T, 1\})^{\kappa+\iota+1} \max\{2\kappa(\kappa+1), 1\} \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\} \\
& \cdot [1 + 5\eta 2^{\kappa+\iota-1} + 5(4\iota)^{\kappa+\iota} 2^{\kappa+\iota-1}].
\end{aligned} \tag{96}$$

The fact that  $\delta = \sqrt{T/N}$  and (72) hence prove that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} |\mathbb{E}[f_d^0(X_t^{d,x})] - \mathbb{E}[F_{d,\min\{\delta,1\}}^0(\mathcal{Y}_t^{d,x})]|^p \nu_d(dt, dx) \right]^{1/p} \\
& \leq N^{-1/2} \iota^2 2^\iota d^{3\kappa+\kappa\iota+\max\{\eta, \kappa(\kappa+\iota)\}} (\kappa+1) \left[ \max\left\{1, \sqrt{2 \max\{1, 2\kappa - 1\} \kappa}\right\} \right]^{2\iota} \\
& \cdot e^{[3\iota^2+1/2]T} (\max\{T, 1\})^{\kappa+\iota+3/2} \max\{2\kappa(\kappa+1), 1\} \\
& \cdot [1 + 5\eta 2^{\kappa+\iota-1} + 5(4\iota)^{\kappa+\iota} 2^{\kappa+\iota-1}] \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\} \\
& = N^{-1/2} \mathcal{C}_1 d^{3\kappa+\kappa\iota+\max\{\eta, \kappa(\kappa+\iota)\}} \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\}.
\end{aligned} \tag{97}$$

Furthermore, observe that (67), the Cauchy-Schwarz inequality, and the triangle inequality ensure that for all  $d \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$ ,  $t \in [0, T]$  it holds that

$$\begin{aligned}
& \left| \mathbb{E}[F_{d,\min\{\delta,1\}}^0(\mathcal{Y}_t^{d,x})] - \mathbb{E}[F_{d,\min\{\delta,1\}}^0(Y_t^{N,d,1,x})] \right| \\
& \leq \mathbb{E} \left[ \left| F_{d,\min\{\delta,1\}}^0(\mathcal{Y}_t^{d,x}) - F_{d,\min\{\delta,1\}}^0(Y_t^{N,d,1,x}) \right| \right] \\
& \leq \kappa d^\kappa \mathbb{E} \left[ (1 + \|\mathcal{Y}_t^{d,x}\|^\kappa + \|Y_t^{N,d,1,x}\|^\kappa) \|\mathcal{Y}_t^{d,x} - Y_t^{N,d,1,x}\| \right] \\
& \leq \kappa d^\kappa (\mathbb{E}[(1 + \|\mathcal{Y}_t^{d,x}\|^\kappa + \|Y_t^{N,d,1,x}\|^\kappa)^2])^{1/2} (\mathbb{E}[\|\mathcal{Y}_t^{d,x} - Y_t^{N,d,1,x}\|^2])^{1/2} \\
& \leq \kappa d^\kappa \left[ 1 + (\mathbb{E}[\|\mathcal{Y}_t^{d,x}\|^{2\kappa}])^{1/2} + (\mathbb{E}[\|Y_t^{N,d,1,x}\|^{2\kappa}])^{1/2} \right] (\mathbb{E}[\|\mathcal{Y}_t^{d,x} - Y_t^{N,d,1,x}\|^2])^{1/2}.
\end{aligned} \tag{98}$$

Next note that (66) and items (ii)–(iii) in Lemma 2.5 prove that for all  $d \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$ ,  $t \in [0, T]$  it holds that

$$\begin{aligned}
(\mathbb{E}[\|\mathcal{Y}_t^{d,x} - Y_t^{N,d,1,x}\|^2])^{1/2} & \leq \frac{1}{2} \sqrt{(\lceil t \rceil - \lfloor t \rfloor) \text{Trace}(\mathcal{A}_d \mathcal{A}_d^*)} \\
& = \frac{1}{2} \sqrt{2(\lceil t \rceil - \lfloor t \rfloor) \text{Trace}(A_d)} \leq \frac{1}{\sqrt{2}} \delta \sqrt{\kappa d^\kappa}.
\end{aligned} \tag{99}$$

Moreover, observe that Hölder's inequality, (66), and item (iv) in Lemma 2.5 show that for all  $q \in (0, \infty)$ ,  $d, m \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$ ,  $t \in [0, T]$  it holds that

$$\begin{aligned}
& \max \left\{ (\mathbb{E}[\|\mathcal{Y}_t^{d,x}\|^q])^{1/q}, (\mathbb{E}[\|Y_t^{N,d,m,x}\|^q])^{1/q} \right\} \\
& \leq \max \left\{ (\mathbb{E}[\|\mathcal{Y}_t^{d,x}\|^{\max\{q,1\}}])^{1/\max\{q,1\}}, (\mathbb{E}[\|Y_t^{N,d,m,x}\|^{\max\{q,1\}}])^{1/\max\{q,1\}} \right\} \\
& \leq \left[ \|x\| + \kappa d^\kappa T + \sqrt{\max\{1, \max\{1, q\} - 1\} T \text{Trace}(\mathcal{A}_d \mathcal{A}_d^*)} \right] e^{\kappa T}.
\end{aligned} \tag{100}$$

Combining this with (75) ensures that for all  $q \in (0, \infty)$ ,  $d, m \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$ ,  $t \in [0, T]$  it holds that

$$\max \left\{ (\mathbb{E}[\|\mathcal{Y}_t^{d,x}\|^q])^{1/q}, (\mathbb{E}[\|Y_t^{N,d,m,x}\|^q])^{1/q} \right\} \leq \left[ \|x\| + \kappa d^\kappa T + \varpi_{d,\max\{q,1\}} T^{1/2} \right] e^{\kappa T}. \tag{101}$$

The fact that  $\forall y, z \in \mathbb{R}$ ,  $\alpha \in (0, \infty)$ :  $|y + z|^\alpha \leq 2^{\max\{0, \alpha - 1\}} (|y|^\alpha + |z|^\alpha)$ , (98), and (99) hence demonstrate that for all  $d \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$ ,  $t \in [0, T]$  it holds that

$$\begin{aligned}
& \left| \mathbb{E}[F_{d,\min\{\delta,1\}}^0(\mathcal{Y}_t^{d,x})] - \mathbb{E}[F_{d,\min\{\delta,1\}}^0(Y_t^{N,d,1,x})] \right| \\
& \leq \kappa d^\kappa \left[ 1 + 2 \left( \|x\| + \kappa d^\kappa T + \varpi_{d,\max\{2\kappa,1\}} T^{1/2} \right)^\kappa e^{\kappa^2 T} \right] \frac{1}{\sqrt{2}} \delta \sqrt{\kappa d^\kappa} \\
& \leq \frac{1}{\sqrt{2}} \delta (\kappa d^\kappa)^{3/2} e^{\kappa^2 T} \left[ 1 + 2^{1+\max\{0, \kappa - 1\}} \left( \|x\|^\kappa + [\kappa d^\kappa T + \varpi_{d,\max\{2\kappa,1\}} T^{1/2}]^\kappa \right) \right] \\
& \leq \frac{1}{\sqrt{2}} \delta (\kappa d^\kappa)^{3/2} e^{\kappa^2 T} 2^{\max\{1, \kappa\}} \left[ \|x\|^\kappa + \left( 1 + [\kappa d^\kappa T + \varpi_{d,\max\{2\kappa,1\}} T^{1/2}]^\kappa \right) \right].
\end{aligned} \tag{102}$$

Combining this with the triangle inequality assures that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} \left| \mathbb{E}[F_{d,\min\{\delta,1\}}^0(\mathcal{Y}_t^{d,x})] - \mathbb{E}[F_{d,\min\{\delta,1\}}^0(Y_t^{N,d,1,x})] \right|^p \nu_d(dt, dx) \right]^{1/p} \\
& \leq \frac{1}{\sqrt{2}} \delta (\kappa d^\kappa)^{3/2} e^{\kappa^2 T} 2^\kappa \left[ \int_{[0,T] \times \mathbb{R}^d} \left[ \|x\|^\kappa + \left( 1 + [\kappa d^\kappa T + \varpi_{d,\max\{2\kappa,1\}} T^{1/2}]^\kappa \right) \right]^p \nu_d(dt, dx) \right]^{1/p} \\
& \leq \frac{1}{\sqrt{2}} \delta (\kappa d^\kappa)^{3/2} e^{\kappa^2 T} 2^\kappa \left[ \int_{[0,T] \times \mathbb{R}^d} \|x\|^{p\kappa} \nu_d(dt, dx) \right]^{1/p} \\
& + \frac{1}{\sqrt{2}} \delta (\kappa d^\kappa)^{3/2} e^{\kappa^2 T} 2^\kappa \left( 1 + [\kappa d^\kappa T + \varpi_{d,\max\{2\kappa,1\}} T^{1/2}]^\kappa \right) [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}.
\end{aligned} \tag{103}$$

Next note that (88) ensures for all  $d \in \mathbb{N}$  that

$$\begin{aligned}
& 1 + [\kappa d^\kappa T + \varpi_{d, \max\{2\kappa, 1\}} T^{1/2}]^\kappa \\
& \leq 1 + \left[ \kappa d^\kappa T + d^{\kappa/2} \max\left\{1, \sqrt{2 \max\{1, 2\kappa - 1\} \kappa}\right\} T^{1/2} \right]^\kappa \\
& \leq 1 + d^{(\kappa^2)} (\max\{T, 1\})^\kappa \left[ \kappa + \max\left\{1, \sqrt{2 \max\{1, 2\kappa - 1\} \kappa}\right\} \right]^\kappa.
\end{aligned} \tag{104}$$

Hence, we obtain for all  $d \in \mathbb{N}$  that

$$\begin{aligned}
& 1 + [\kappa d^\kappa T + \varpi_{d, \max\{2\kappa, 1\}} T^{1/2}]^\kappa \\
& \leq d^{(\kappa^2)} (\max\{T, 1\})^\kappa \left( 1 + \left[ \kappa + \max\left\{1, \sqrt{2 \max\{1, 2\kappa - 1\} \kappa}\right\} \right]^\kappa \right).
\end{aligned} \tag{105}$$

This, (103), (80), and the fact that  $\kappa < 2 \max\{2\kappa, 3\}$  establish that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned}
& \left[ \int_{[0, T] \times \mathbb{R}^d} \left| \mathbb{E}[F_{d, \min\{\delta, 1\}}^0(\mathcal{Y}_t^{d, x})] - \mathbb{E}[F_{d, \min\{\delta, 1\}}^0(Y_t^{N, d, 1, x})] \right|^p \nu_d(dt, dx) \right]^{1/p} \\
& \leq \frac{1}{\sqrt{2}} \delta (\kappa d^\kappa)^{3/2} e^{\kappa^2 T} 2^\ell \eta d^\eta \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\} \\
& + \frac{1}{\sqrt{2}} \delta (\kappa d^\kappa)^{3/2} e^{\kappa^2 T} 2^\ell d^{(\kappa^2)} (\max\{T, 1\})^\kappa [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p} \\
& \cdot \left( 1 + \left[ \kappa + \max\left\{1, \sqrt{2 \max\{1, 2\kappa - 1\} \kappa}\right\} \right]^\kappa \right).
\end{aligned} \tag{106}$$

Combining this with (73) and the fact that  $\delta = \sqrt{T/N}$  implies that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned}
& \left[ \int_{[0, T] \times \mathbb{R}^d} \left| \mathbb{E}[F_{d, \min\{\delta, 1\}}^0(\mathcal{Y}_t^{d, x})] - \mathbb{E}[F_{d, \min\{\delta, 1\}}^0(Y_t^{N, d, 1, x})] \right|^p \nu_d(dt, dx) \right]^{1/p} \\
& \leq \delta d^{3\kappa/2 + \eta} \frac{1}{\sqrt{2}} \kappa^{3/2} e^{\kappa^2 T} 2^\ell \eta \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\} \\
& + \delta d^{3\kappa/2 + \kappa^2} \frac{1}{\sqrt{2}} \kappa^{3/2} e^{\kappa^2 T} 2^\ell [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p} \\
& \cdot (\max\{T, 1\})^\kappa \left( 1 + \left[ \kappa + \max\left\{1, \sqrt{2 \max\{1, 2\kappa - 1\} \kappa}\right\} \right]^\kappa \right) \\
& \leq \delta d^{3\kappa/2 + \max\{\eta, \kappa^2\}} \frac{1}{\sqrt{2}} \kappa^{3/2} e^{\kappa^2 T} 2^\ell \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\} \\
& \cdot (\max\{T, 1\})^\kappa \left( \eta + 1 + \left[ \kappa + \max\left\{1, \sqrt{2 \max\{1, 2\kappa - 1\} \kappa}\right\} \right]^\kappa \right) \\
& \leq N^{-1/2} d^{3\kappa/2 + \max\{\eta, \kappa^2\}} \frac{1}{\sqrt{2}} \kappa^{3/2} e^{\kappa^2 T} 2^\ell \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\} \\
& \cdot (\max\{T, 1\})^{\kappa+1/2} \left( \eta + 1 + \left[ \kappa + \max\left\{1, \sqrt{2 \max\{1, 2\kappa - 1\} \kappa}\right\} \right]^\kappa \right) \\
& = N^{-1/2} d^{3\kappa/2 + \max\{\eta, \kappa^2\}} \mathcal{C}_2 \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\}.
\end{aligned} \tag{107}$$

Next note that the triangle inequality proves that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned}
& \left[ \int_{[0, T] \times \mathbb{R}^d} \left| \mathbb{E}[f_d^0(X_t^{d, x})] - \mathbb{E}[F_{d, \min\{\delta, 1\}}^0(Y_t^{N, d, 1, x})] \right|^p \nu_d(dt, dx) \right]^{1/p} \\
& \leq \left[ \int_{[0, T] \times \mathbb{R}^d} \left| \mathbb{E}[f_d^0(X_t^{d, x})] - \mathbb{E}[F_{d, \min\{\delta, 1\}}^0(\mathcal{Y}_t^{d, x})] \right|^p \nu_d(dt, dx) \right]^{1/p} \\
& + \left[ \int_{[0, T] \times \mathbb{R}^d} \left| \mathbb{E}[F_{d, \min\{\delta, 1\}}^0(\mathcal{Y}_t^{d, x})] - \mathbb{E}[F_{d, \min\{\delta, 1\}}^0(Y_t^{N, d, 1, x})] \right|^p \nu_d(dt, dx) \right]^{1/p}.
\end{aligned} \tag{108}$$

This, (97), and (107) ensure that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned} & \left[ \int_{[0,T] \times \mathbb{R}^d} |\mathbb{E}[f_d^0(X_t^{d,x})] - \mathbb{E}[F_{d,\min\{\delta,1\}}^0(Y_t^{N,d,1,x})]|^p \nu_d(dt, dx) \right]^{1/p} \\ & \leq N^{-1/2} \mathcal{C}_1 d^{3\kappa+\kappa\iota+\max\{\eta,\kappa(\kappa+\iota)\}} \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\} \\ & \quad + N^{-1/2} d^{3\kappa/2+\max\{\eta,\kappa^2\}} \mathcal{C}_2 \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\}. \end{aligned} \quad (109)$$

The fact that  $\iota \leq \kappa + 1$ , the fact that  $3\kappa/2 + \max\{\eta, \kappa^2\} \leq \kappa(\kappa + 4) + \max\{\eta, \kappa(2\kappa + 1)\}$ , and (74) hence demonstrate that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned} & \left[ \int_{[0,T] \times \mathbb{R}^d} |\mathbb{E}[f_d^0(X_t^{d,x})] - \mathbb{E}[F_{d,\min\{\delta,1\}}^0(Y_t^{N,d,1,x})]|^p \nu_d(dt, dx) \right]^{1/p} \\ & \leq N^{-1/2} d^{\kappa(\kappa+4)+\max\{\eta,\kappa(2\kappa+1)\}} [\mathcal{C}_1 + \mathcal{C}_2] \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\} \\ & \leq N^{-1/2} d^{\kappa(\kappa+4)+\max\{\eta,\kappa(2\kappa+1)\}} \mathcal{C} \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\}. \end{aligned} \quad (110)$$

Next observe that (65) and (66) prove for all  $d \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} |F_{d,\min\{\delta,1\}}^0(x)| & \leq |F_{d,\min\{\delta,1\}}^0(x) - f_d^0(x)| + |f_d^0(x)| \\ & \leq \min\{\delta, 1\} \kappa d^\kappa (1 + \|x\|^\kappa) + \kappa d^\kappa (1 + \|x\|^\kappa) \leq 2\kappa d^\kappa (1 + \|x\|^\kappa). \end{aligned} \quad (111)$$

Moreover, note that (101) and the fact that  $\forall y, z \in \mathbb{R}, \alpha \in (0, \infty): |y+z|^\alpha \leq 2^{\max\{0,\alpha-1\}}(|y|^\alpha + |z|^\alpha)$  imply that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned} & \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E}[\|Y_t^{N,d,1,x}\|^{p\kappa}] \nu_d(dt, dx) \right]^{1/p} \\ & \leq e^{\kappa^2 T} \left[ \int_{[0,T] \times \mathbb{R}^d} [\|x\| + \kappa d^\kappa T + \varpi_{d,\max\{p\kappa,1\}} T^{1/2}]^{p\kappa} \nu_d(dt, dx) \right]^{1/p} \\ & \leq e^{\kappa^2 T} 2^{\max\{0,\kappa-1\}} \left[ \int_{[0,T] \times \mathbb{R}^d} [\|x\|^\kappa + (\kappa d^\kappa T + \varpi_{d,\max\{p\kappa,1\}} T^{1/2})^\kappa]^p \nu_d(dt, dx) \right]^{1/p}. \end{aligned} \quad (112)$$

Combining this with the triangle inequality ensures that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned} & \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E}[\|Y_t^{N,d,1,x}\|^{p\kappa}] \nu_d(dt, dx) \right]^{1/p} \\ & \leq e^{\kappa^2 T} 2^{\max\{0,\kappa-1\}} \left[ \int_{[0,T] \times \mathbb{R}^d} \|x\|^{p\kappa} \nu_d(dt, dx) \right]^{1/p} \\ & \quad + e^{\kappa^2 T} 2^{\max\{0,\kappa-1\}} (\kappa d^\kappa T + \varpi_{d,\max\{p\kappa,1\}} T^{1/2})^\kappa [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}. \end{aligned} \quad (113)$$

This, (80), and the fact that  $\kappa < 2 \max\{2\kappa, 3\}$  demonstrate that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned} & \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E}[\|Y_t^{N,d,1,x}\|^{p\kappa}] \nu_d(dt, dx) \right]^{1/p} \\ & \leq e^{\kappa^2 T} 2^{\max\{0,\kappa-1\}} (\eta d^\eta + (\kappa d^\kappa T + \varpi_{d,\max\{p\kappa,1\}} T^{1/2})^\kappa) \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\}. \end{aligned} \quad (114)$$

In addition, observe that (88) and the fact that  $\max\{p\kappa, 1\} \leq p\iota$  prove that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned} \varpi_{d,\max\{p\kappa,1\}} & \leq d^{\kappa/2} \max\left\{1, \sqrt{2 \max\{1, \max\{p\kappa, 1\} - 1\} \kappa}\right\} \\ & \leq d^{\kappa/2} \max\left\{1, \sqrt{2 \max\{1, p\iota - 1\} \kappa}\right\} \\ & = d^{\kappa/2} \max\left\{1, \sqrt{2(p\iota - 1)\kappa}\right\}. \end{aligned} \quad (115)$$

This and (114) ensure that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} [\|Y_t^{N,d,1,x}\|^{p\kappa}] \nu_d(dt, dx) \right]^{1/p} \\
& \leq e^{\kappa^2 T} 2^{\max\{0, \kappa-1\}} \left( \eta d^\eta + \left[ \kappa d^\kappa T + d^{\kappa/2} \max\{1, \sqrt{2(p\iota-1)\kappa}\} T^{1/2} \right]^\kappa \right) \\
& \cdot \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\} \\
& \leq e^{\kappa^2 T} 2^{\max\{0, \kappa-1\}} d^{\max\{\eta, \kappa^2\}} \left( \eta + \left[ \kappa T + \max\{1, \sqrt{2(p\iota-1)\kappa}\} T^{1/2} \right]^\kappa \right) \\
& \cdot \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\} \\
& = C d^{\max\{\eta, \kappa^2\}} \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\}.
\end{aligned} \tag{116}$$

This, the triangle inequality, and (111) assure that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} [ |F_{d, \min\{\delta, 1\}}^0(Y_t^{N,d,1,x})|^p ] \nu_d(dt, dx) \right]^{1/p} \\
& \leq 2\kappa d^\kappa \left( [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p} + C d^{\max\{\eta, \kappa^2\}} \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\} \right) \\
& \leq 2\kappa d^{\kappa + \max\{\eta, \kappa^2\}} (1 + C) \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\}.
\end{aligned} \tag{117}$$

Combining this, Fubini's theorem, (110), (74), and (84) proves that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned}
& \left( \mathbb{E} \left[ \int_{[0,T] \times \mathbb{R}^d} |u_d(t, x) - \frac{1}{M} [\sum_{m=1}^M F_{d, \min\{\delta, 1\}}^0(Y_t^{N,d,m,x})]|^p \nu_d(dt, dx) \right] \right)^{1/p} \\
& = N^{-1/2} \mathcal{C} d^{\kappa(\kappa+4) + \max\{\eta, \kappa(2\kappa+1)\}} \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\} \\
& + \frac{4\sqrt{p-1}}{M^{1/2}} 2\kappa d^{\kappa + \max\{\eta, \kappa^2\}} (1 + C) \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\} \\
& \leq \mathcal{C} \left[ \frac{d^{\kappa(\kappa+4) + \max\{\eta, \kappa(2\kappa+1)\}}}{N^{1/2}} + \frac{d^{\kappa + \max\{\eta, \kappa^2\}}}{M^{1/2}} \right] [\max\{1, \nu_d([0, T] \times \mathbb{R}^d)\}]^{1/p}.
\end{aligned} \tag{118}$$

This establishes item (ii). This completes the proof of Proposition 3.2.  $\square$

## 4 Deep neural network (DNN) approximations for PDEs

In this section we establish in Theorem 4.13 in Subsection 4.6 below the main result of this article. Theorem 4.13, in particular, proves that for every  $T \in (0, \infty)$ ,  $a \in \mathbb{R}$ ,  $b \in (a, \infty)$  it holds that solutions of certain Kolmogorov PDEs can be approximated by DNNs on the space-time region  $[0, T] \times [a, b]^d$  without the curse of dimensionality. In our proof of Theorem 4.13 we employ the auxiliary intermediate result in Proposition 4.12 in Subsection 4.5 below. Our proof of Proposition 4.12, in turn, uses the error estimates for Monte Carlo Euler approximations which we have presented in Proposition 3.2 in Section 3 above as well as the DNN approximation result for Monte Carlo Euler approximations in Corollary 4.11 in Subsection 4.4 below. Our proof of Corollary 4.11 employs the auxiliary results in Proposition 4.9 and Lemma 4.10 in Subsection 4.4 below. Our proof of Proposition 4.9, in turn, uses the DNN approximation result for Monte Carlo Euler approximations in Proposition 4.8 in Subsection 4.4 below. Our proof of Proposition 4.8 is based on an application of [23, Proposition 3.10] and is very similar to the proof of [23, Theorem 3.12]. Our proof of Theorem 4.13 in Subsection 4.6 below also employs several well-known concepts and results from an appropriate calculus for DNNs from

the scientific literature which we briefly recall in Subsections 4.1–4.3 below. In particular, Definition 4.1 is, e.g., [23, Definition 2.1], Definition 4.2 is, e.g., [23, Definition 2.2], Definition 4.3 is, e.g., [23, Definition 2.3], Definition 4.5 is, e.g., [23, Definition 2.5], Lemma 4.6 is, e.g., [23, Lemma 2.8]), and Definition 4.7 is, e.g., [23, Definition 2.15].

## 4.1 DNNs

**Definition 4.1** (DNNs). *We denote by  $\mathbf{N}$  the set given by*

$$\mathbf{N} = \cup_{L \in \mathbb{N}} \cup_{(l_0, l_1, \dots, l_L) \in \mathbb{N}^{L+1}} \left( \times_{k=1}^L (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k}) \right) \quad (119)$$

and we denote by  $\mathcal{P}, \mathcal{L}, \mathcal{I}, \mathcal{O}: \mathbf{N} \rightarrow \mathbb{N}$  and  $\mathcal{D}: \mathbf{N} \rightarrow \cup_{L=2}^{\infty} \mathbb{N}^L$  the functions which satisfy for all  $L \in \mathbb{N}$ ,  $l_0, l_1, \dots, l_L \in \mathbb{N}$ ,  $\Phi \in (\times_{k=1}^L (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k}))$  that  $\mathcal{P}(\Phi) = \sum_{k=1}^L l_k(l_{k-1} + 1)$ ,  $\mathcal{L}(\Phi) = L$ ,  $\mathcal{I}(\Phi) = l_0$ ,  $\mathcal{O}(\Phi) = l_L$ , and  $\mathcal{D}(\Phi) = (l_0, l_1, \dots, l_L)$ .

## 4.2 Realizations of DNNs

**Definition 4.2** (Multidimensional versions). *Let  $d \in \mathbb{N}$  and let  $\psi: \mathbb{R} \rightarrow \mathbb{R}$  be a function. Then we denote by  $\mathfrak{M}_{\psi, d}: \mathbb{R}^d \rightarrow \mathbb{R}^d$  the function which satisfies for all  $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$  that*

$$\mathfrak{M}_{\psi, d}(x) = (\psi(x_1), \psi(x_2), \dots, \psi(x_d)). \quad (120)$$

**Definition 4.3** (Realizations associated to DNNs). *Let  $a \in C(\mathbb{R}, \mathbb{R})$ . Then we denote by  $\mathcal{R}_a: \mathbf{N} \rightarrow (\cup_{k, l \in \mathbb{N}} C(\mathbb{R}^k, \mathbb{R}^l))$  the function which satisfies for all  $L \in \mathbb{N}$ ,  $l_0, l_1, \dots, l_L \in \mathbb{N}$ ,  $\Phi = ((W_1, B_1), (W_2, B_2), \dots, (W_L, B_L)) \in (\times_{k=1}^L (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k}))$ ,  $x_0 \in \mathbb{R}^{l_0}$ ,  $x_1 \in \mathbb{R}^{l_1}, \dots, x_{L-1} \in \mathbb{R}^{l_{L-1}}$  with  $\forall k \in \mathbb{N} \cap (0, L): x_k = \mathfrak{M}_{a, l_k}(W_k x_{k-1} + B_k)$  that*

$$\mathcal{R}_a(\Phi) \in C(\mathbb{R}^{l_0}, \mathbb{R}^{l_L}) \quad \text{and} \quad (\mathcal{R}_a(\Phi))(x_0) = W_L x_{L-1} + B_L \quad (121)$$

(cf. Definitions 4.1 and 4.2).

**Definition 4.4** (Rectifier function). *We denote by  $\mathfrak{r}: \mathbb{R} \rightarrow \mathbb{R}$  the function which satisfies for all  $x \in \mathbb{R}$  that  $\mathfrak{r}(x) = \max\{x, 0\}$ .*

## 4.3 Compositions of DNNs

**Definition 4.5** (Standard compositions of DNNs). *We denote by  $(\cdot) \bullet (\cdot): \{(\Phi_1, \Phi_2) \in \mathbf{N} \times \mathbf{N}: \mathcal{I}(\Phi_1) = \mathcal{O}(\Phi_2)\} \rightarrow \mathbf{N}$  the function which satisfies for all  $L, \mathfrak{L} \in \mathbb{N}$ ,  $l_0, l_1, \dots, l_L, l_0, l_1, \dots, l_{\mathfrak{L}} \in \mathbb{N}$ ,  $\Phi_1 = ((W_1, B_1), (W_2, B_2), \dots, (W_L, B_L)) \in (\times_{k=1}^L (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k}))$ ,  $\Phi_2 = ((\mathcal{W}_1, \mathfrak{B}_1), (\mathcal{W}_2, \mathfrak{B}_2), \dots, (\mathcal{W}_{\mathfrak{L}}, \mathfrak{B}_{\mathfrak{L}})) \in (\times_{k=1}^{\mathfrak{L}} (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k}))$  with  $l_0 = \mathcal{I}(\Phi_1) = \mathcal{O}(\Phi_2) = l_{\mathfrak{L}}$  that*

$$\Phi_1 \bullet \Phi_2 = \begin{cases} ((\mathcal{W}_1, \mathfrak{B}_1), (\mathcal{W}_2, \mathfrak{B}_2), \dots, (\mathcal{W}_{\mathfrak{L}-1}, \mathfrak{B}_{\mathfrak{L}-1}), (W_1 \mathcal{W}_{\mathfrak{L}}, W_1 \mathfrak{B}_{\mathfrak{L}} + B_1), \\ \quad \quad \quad (W_2, B_2), (W_3, B_3), \dots, (W_L, B_L)) & : L > 1 < \mathfrak{L} \\ ((W_1 \mathcal{W}_1, W_1 \mathfrak{B}_1 + B_1), (W_2, B_2), (W_3, B_3), \dots, (W_L, B_L)) & : L > 1 = \mathfrak{L} \\ ((\mathcal{W}_1, \mathfrak{B}_1), (\mathcal{W}_2, \mathfrak{B}_2), \dots, (\mathcal{W}_{\mathfrak{L}-1}, \mathfrak{B}_{\mathfrak{L}-1}), (W_1 \mathcal{W}_{\mathfrak{L}}, W_1 \mathfrak{B}_{\mathfrak{L}} + B_1)) & : L = 1 < \mathfrak{L} \\ (W_1 \mathcal{W}_1, W_1 \mathfrak{B}_1 + B_1) & : L = 1 = \mathfrak{L} \end{cases} \quad (122)$$

(cf. Definition 4.1).



**Lemma 4.6.** Let  $\Phi_1, \Phi_2, \Phi_3 \in \mathbf{N}$  satisfy that  $\mathcal{I}(\Phi_1) = \mathcal{O}(\Phi_2)$  and  $\mathcal{I}(\Phi_2) = \mathcal{O}(\Phi_3)$  (cf. Definition 4.1). Then

$$(\Phi_1 \bullet \Phi_2) \bullet \Phi_3 = \Phi_1 \bullet (\Phi_2 \bullet \Phi_3) \quad (123)$$

(cf. Definition 4.5).

**Definition 4.7** (Compositions of DNNs involving artificial identities). Let  $\Psi \in \mathbf{N}$ . Then we denote by

$$(\cdot) \odot_{\Psi} (\cdot) : \{(\Phi_1, \Phi_2) \in \mathbf{N} \times \mathbf{N} : \mathcal{I}(\Phi_1) = \mathcal{O}(\Psi) \text{ and } \mathcal{O}(\Phi_2) = \mathcal{I}(\Psi)\} \rightarrow \mathbf{N} \quad (124)$$

the function which satisfies for all  $\Phi_1, \Phi_2 \in \mathbf{N}$  with  $\mathcal{I}(\Phi_1) = \mathcal{O}(\Psi)$  and  $\mathcal{O}(\Phi_2) = \mathcal{I}(\Psi)$  that

$$\Phi_1 \odot_{\Psi} \Phi_2 = \Phi_1 \bullet (\Psi \bullet \Phi_2) = (\Phi_1 \bullet \Psi) \bullet \Phi_2 \quad (125)$$

(cf. Definitions 4.1 and 4.5 and Lemma 4.6).

#### 4.4 DNN approximations for Monte Carlo Euler approximations

**Proposition 4.8.** Let  $N, d \in \mathbb{N}$ ,  $c, C \in [0, \infty)$ ,  $T, \mathfrak{D} \in (0, \infty)$ ,  $q \in (2, \infty)$ ,  $\varepsilon \in (0, 1]$ ,  $(\tau_n)_{n \in \{0, 1, \dots, N\}} \subseteq \mathbb{R}$  satisfy for all  $n \in \{0, 1, \dots, N\}$  that  $\tau_n = \frac{nT}{N}$  and

$$\mathfrak{D} = \left\lceil \frac{720q}{(q-2)} \right\rceil [\log_2(\varepsilon^{-1}) + q + 1] - 504, \quad (126)$$

let  $\Phi \in \mathbf{N}$  satisfy for all  $x \in \mathbb{R}^d$  that  $\mathcal{I}(\Phi) = \mathcal{O}(\Phi) = d$  and  $\|(\mathcal{R}_{\tau}(\Phi))(x)\| \leq C + c\|x\|$ , let  $Y = (Y_t^{x,y})_{(t,x,y) \in [0,T] \times \mathbb{R}^d \times (\mathbb{R}^d)^N} : [0, T] \times \mathbb{R}^d \times (\mathbb{R}^d)^N \rightarrow \mathbb{R}^d$  satisfy for all  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in [\tau_n, \tau_{n+1}]$ ,  $x \in \mathbb{R}^d$ ,  $y = (y_1, y_2, \dots, y_N) \in (\mathbb{R}^d)^N$  that  $Y_0^{x,y} = x$  and

$$Y_t^{x,y} = Y_{\tau_n}^{x,y} + \left(\frac{tN}{T} - n\right) \left[\frac{T}{N} (\mathcal{R}_{\tau}(\Phi))(Y_{\tau_n}^{x,y}) + y_{n+1}\right], \quad (127)$$

and let  $g_n : \mathbb{R}^d \times (\mathbb{R}^d)^N \rightarrow [0, \infty)$ ,  $n \in \{0, 1, \dots, N\}$ , satisfy for all  $n \in \{0, 1, \dots, N\}$ ,  $x \in \mathbb{R}^d$ ,  $y = (y_1, y_2, \dots, y_N) \in (\mathbb{R}^d)^N$  that

$$g_n(x, y) = \left[ \|x\| + C\tau_n + \max_{m \in \{0, 1, \dots, n\}} \left\| \sum_{k=1}^m y_k \right\| \right] \exp(c\tau_n) \quad (128)$$

(cf. Definitions 2.1, 4.1, 4.3, and 4.4). Then there exist  $\Psi_y \in \mathbf{N}$ ,  $y \in (\mathbb{R}^d)^N$ , such that

(i) it holds for all  $y \in (\mathbb{R}^d)^N$  that  $\mathcal{R}_{\tau}(\Psi_y) \in C(\mathbb{R}^{d+1}, \mathbb{R}^d)$ ,

(ii) it holds for all  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in [\tau_n, \tau_{n+1}]$ ,  $x \in \mathbb{R}^d$ ,  $y \in (\mathbb{R}^d)^N$  that

$$\|Y_t^{x,y} - (\mathcal{R}_{\tau}(\Psi_y))(t, x)\| \leq \varepsilon [2\sqrt{d} + (g_n(x, y))^q + (g_{n+1}(x, y))^q], \quad (129)$$

(iii) it holds for all  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in [\tau_n, \tau_{n+1}]$ ,  $x \in \mathbb{R}^d$ ,  $y \in (\mathbb{R}^d)^N$  that

$$\|(\mathcal{R}_{\tau}(\Psi_y))(t, x)\| \leq 6\sqrt{d} + 2[(g_n(x, y))^2 + (g_{n+1}(x, y))^2], \quad (130)$$

(iv) it holds for all  $y \in (\mathbb{R}^d)^N$  that

$$\mathcal{P}(\Psi_y) \leq \frac{9}{2} N^6 d^{16} [2(\mathcal{L}(\Phi) - 1) + \mathfrak{D} + (24 + 6\mathcal{L}(\Phi) + [4 + \mathcal{P}(\Phi)]^2)^2], \quad (131)$$

(v) it holds for all  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that  $[(\mathbb{R}^d)^N \ni y \mapsto (\mathcal{R}_{\tau}(\Psi_y))(t, x) \in \mathbb{R}^d] \in C((\mathbb{R}^d)^N, \mathbb{R}^d)$ , and

(vi) it holds for all  $n \in \{0, 1, \dots, N\}$ ,  $t \in [0, \tau_n]$ ,  $x \in \mathbb{R}^d$ ,  $y = (y_1, y_2, \dots, y_N)$ ,  $z = (z_1, z_2, \dots, z_N) \in (\mathbb{R}^d)^N$  with  $\forall k \in \mathbb{N} \cap [0, n]: y_k = z_k$  that

$$(\mathcal{R}_\tau(\Psi_y))(t, x) = (\mathcal{R}_\tau(\Psi_z))(t, x). \quad (132)$$

*Proof of Proposition 4.8.* Throughout this proof let  $\Psi_y \in \mathbf{N}$ ,  $y \in (\mathbb{R}^d)^N$ , satisfy that

(I) it holds for all  $y \in (\mathbb{R}^d)^N$  that  $\mathcal{R}_\tau(\Psi_y) \in C(\mathbb{R}^{d+1}, \mathbb{R}^d)$ ,

(II) it holds for all  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in [\tau_n, \tau_{n+1}]$ ,  $x \in \mathbb{R}^d$ ,  $y \in (\mathbb{R}^d)^N$  that

$$\|Y_t^{x,y} - (\mathcal{R}_\tau(\Psi_y))(t, x)\| \leq \varepsilon(2\sqrt{d} + \|Y_{\tau_n}^{x,y}\|^q + \|Y_{\tau_{n+1}}^{x,y}\|^q), \quad (133)$$

(III) it holds for all  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in [\tau_n, \tau_{n+1}]$ ,  $x \in \mathbb{R}^d$ ,  $y \in (\mathbb{R}^d)^N$  that

$$\|(\mathcal{R}_\tau(\Psi_y))(t, x)\| \leq 6\sqrt{d} + 2(\|Y_{\tau_n}^{x,y}\|^2 + \|Y_{\tau_{n+1}}^{x,y}\|^2), \quad (134)$$

(IV) it holds for all  $y \in (\mathbb{R}^d)^N$  that

$$\begin{aligned} \mathcal{P}(\Psi_y) & \quad (135) \\ & \leq \frac{1}{2} \left[ 6d^2 N^2 (\mathcal{L}(\Phi) - 1) + 3N [d^2 \mathfrak{D} + (23 + 6N(\mathcal{L}(\Phi) - 1) + 7d^2 + N[4d^2 + \mathcal{P}(\Phi)]^2)^2] \right]^2, \end{aligned}$$

(V) it holds for all  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that  $[(\mathbb{R}^d)^N \ni y \mapsto (\mathcal{R}_\tau(\Psi_y))(t, x) \in \mathbb{R}^d] \in C((\mathbb{R}^d)^N, \mathbb{R}^d)$ , and

(VI) it holds for all  $n \in \{0, 1, \dots, N\}$ ,  $t \in [0, \tau_n]$ ,  $x \in \mathbb{R}^d$ ,  $y = (y_1, y_2, \dots, y_N)$ ,  $z = (z_1, z_2, \dots, z_N) \in (\mathbb{R}^d)^N$  with  $\forall k \in \mathbb{N} \cap [0, n]: y_k = z_k$  that

$$(\mathcal{R}_\tau(\Psi_y))(t, x) = (\mathcal{R}_\tau(\Psi_z))(t, x) \quad (136)$$

(cf. Grohs et al. [23, Proposition 3.10] (applied with  $N \leftarrow N$ ,  $d \leftarrow d$ ,  $a \leftarrow \mathfrak{r}$ ,  $T \leftarrow T$ ,  $t_0 \leftarrow \tau_0$ ,  $t_1 \leftarrow \tau_1$ ,  $\dots$ ,  $t_N \leftarrow \tau_N$ ,  $\mathfrak{D} \leftarrow \mathfrak{D}$ ,  $\varepsilon \leftarrow \varepsilon$ ,  $q \leftarrow q$ ,  $Y \leftarrow Y$  in the notation of Grohs et al. [23, Proposition 3.10])). Note that (IV) ensures that for all  $y \in (\mathbb{R}^d)^N$  it holds that

$$\begin{aligned} \mathcal{P}(\Psi_y) & \\ & \leq \frac{1}{2} \left[ 6d^2 N^2 (\mathcal{L}(\Phi) - 1) + 3N [d^2 \mathfrak{D} + (23 + 6N(\mathcal{L}(\Phi) - 1) + 7d^2 + Nd^4 [4 + \mathcal{P}(\Phi)]^2)^2] \right]^2 \\ & \leq \frac{1}{2} \left[ 6d^2 N^2 (\mathcal{L}(\Phi) - 1) + 3N [d^2 \mathfrak{D} + N^2 d^8 (23 + 6(\mathcal{L}(\Phi) - 1) + 7 + [4 + \mathcal{P}(\Phi)]^2)^2] \right]^2 \\ & = \frac{1}{2} \left[ 6d^2 N^2 (\mathcal{L}(\Phi) - 1) + 3N [d^2 \mathfrak{D} + N^2 d^8 (24 + 6\mathcal{L}(\Phi) + [4 + \mathcal{P}(\Phi)]^2)^2] \right]^2. \end{aligned} \quad (137)$$

Hence, we obtain that for all  $y \in (\mathbb{R}^d)^N$  it holds that

$$\begin{aligned} \mathcal{P}(\Psi_y) & \leq \frac{1}{2} \left[ 6d^2 N^2 (\mathcal{L}(\Phi) - 1) + 3N^3 d^8 [\mathfrak{D} + (24 + 6\mathcal{L}(\Phi) + [4 + \mathcal{P}(\Phi)]^2)^2] \right]^2 \\ & \leq \frac{9}{2} N^6 d^{16} [2(\mathcal{L}(\Phi) - 1) + \mathfrak{D} + (24 + 6\mathcal{L}(\Phi) + [4 + \mathcal{P}(\Phi)]^2)^2]^2. \end{aligned} \quad (138)$$

In addition, observe that, e.g., Grohs et al. [23, Lemma 3.11] (applied with  $N \leftarrow N$ ,  $d \leftarrow d$ ,  $c \leftarrow c$ ,  $C \leftarrow C$ ,  $A_1 \leftarrow \frac{T}{N} \text{id}_{\mathbb{R}^d}$ ,  $A_2 \leftarrow \frac{T}{N} \text{id}_{\mathbb{R}^d}$ ,  $\dots$ ,  $A_N \leftarrow \frac{T}{N} \text{id}_{\mathbb{R}^d}$ ,  $\mu \leftarrow \mathcal{R}_\tau(\Phi)$ ,  $Y_0 \leftarrow (Y_{\tau_0}^{x,y})_{(x,y) \in \mathbb{R}^d \times (\mathbb{R}^d)^N}$ ,  $Y_1 \leftarrow (Y_{\tau_1}^{x,y})_{(x,y) \in \mathbb{R}^d \times (\mathbb{R}^d)^N}$ ,  $\dots$ ,  $Y_N \leftarrow (Y_{\tau_N}^{x,y})_{(x,y) \in \mathbb{R}^d \times (\mathbb{R}^d)^N}$  in the notation

of Grohs et al. [23, Lemma 3.11]) and the hypothesis that  $\forall n \in \{0, 1, \dots, N\}: \tau_n = \frac{nT}{N}$  demonstrate that for all  $n \in \{0, 1, \dots, N\}$ ,  $x \in \mathbb{R}^d$ ,  $y = (y_1, y_2, \dots, y_N) \in (\mathbb{R}^d)^N$  it holds that

$$\begin{aligned} \|Y_{\tau_n}^{x,y}\| &\leq \left[ \|x\| + \frac{cnT}{N} + \max_{m \in \{0,1,\dots,n\}} \left\| \sum_{k=1}^m y_k \right\| \right] \exp\left(\frac{cnT}{N}\right) \\ &= \left[ \|x\| + C\tau_n + \max_{m \in \{0,1,\dots,n\}} \left\| \sum_{k=1}^m y_k \right\| \right] \exp(c\tau_n) = g_n(x, y). \end{aligned} \quad (139)$$

Combining this with (II) and (III) ensures that for all  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in [\tau_n, \tau_{n+1}]$ ,  $x \in \mathbb{R}^d$ ,  $y \in (\mathbb{R}^d)^N$  it holds that

$$\begin{aligned} \|Y_t^{x,y} - (\mathcal{R}_\tau(\Psi_y))(t, x)\| &\leq \varepsilon [2\sqrt{d} + \|Y_{\tau_n}^{x,y}\|^q + \|Y_{\tau_{n+1}}^{x,y}\|^q] \\ &\leq \varepsilon [2\sqrt{d} + (g_n(x, y))^q + (g_{n+1}(x, y))^q] \end{aligned} \quad (140)$$

and

$$\begin{aligned} \|(\mathcal{R}_\tau(\Psi_y))(t, x)\| &\leq 6\sqrt{d} + 2(\|Y_{\tau_n}^{x,y}\|^2 + \|Y_{\tau_{n+1}}^{x,y}\|^2) \\ &\leq 6\sqrt{d} + 2[(g_n(x, y))^2 + (g_{n+1}(x, y))^2]. \end{aligned} \quad (141)$$

This, (I), (V), (VI), and (138) establish items (i)–(vi). The proof of Proposition 4.8 is thus completed.  $\square$

**Proposition 4.9.** *Let  $M, N, d, \mathfrak{d} \in \mathbb{N}$ ,  $\alpha, c, C, \mathfrak{C} \in [0, \infty)$ ,  $T, \mathfrak{D} \in (0, \infty)$ ,  $q \in (2, \infty)$ ,  $\varepsilon \in (0, 1]$ ,  $\mathfrak{f}^1, \mathfrak{f}^0 \in \mathbb{N}$  satisfy that  $\mathcal{I}(\mathfrak{f}^1) = \mathcal{O}(\mathfrak{f}^1) = \mathcal{I}(\mathfrak{f}^0) = d$ ,  $\mathcal{O}(\mathfrak{f}^0) = \mathfrak{d}$ , and*

$$\mathfrak{D} = \left\lceil \frac{720q}{q-2} \right\rceil [\log_2(\varepsilon^{-1}) + q + 1] - 504, \quad (142)$$

assume for all  $x, y \in \mathbb{R}^d$  that  $\|(\mathcal{R}_\tau(\mathfrak{f}^1))(x)\| \leq C + c\|x\|$  and

$$\|(\mathcal{R}_\tau(\mathfrak{f}^0))(x) - (\mathcal{R}_\tau(\mathfrak{f}^0))(y)\| \leq \mathfrak{C}(1 + \|x\|^\alpha + \|y\|^\alpha)\|x - y\|, \quad (143)$$

let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $W^m = (W_n^m)_{n \in \{0,1,\dots,N\}}: \{0, 1, \dots, N\} \times \Omega \rightarrow \mathbb{R}^d$ ,  $m \in \{1, 2, \dots, M\}$ , be stochastic processes which satisfy for all  $m \in \{1, 2, \dots, M\}$  that  $W_0^m = 0$ , let  $Y^m = (Y_t^{m,x}(\omega))_{(t,x,\omega) \in [0,T] \times \mathbb{R}^d \times \Omega}: [0, T] \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ ,  $m \in \{1, 2, \dots, M\}$ , satisfy for all  $m \in \{1, 2, \dots, M\}$ ,  $x \in \mathbb{R}^d$ ,  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in [\frac{nT}{N}, \frac{(n+1)T}{N}]$  that  $Y_0^{m,x} = x$  and

$$Y_t^{m,x} = Y_{\frac{nT}{N}}^{m,x} + \left(\frac{tN}{T} - n\right) \left[ \frac{T}{N} (\mathcal{R}_\tau(\mathfrak{f}^1))(Y_{\frac{nT}{N}}^{m,x}) + W_{n+1}^m - W_n^m \right], \quad (144)$$

and let  $h_{m,r}: \mathbb{R}^d \times \Omega \rightarrow [0, \infty)$ ,  $m \in \{1, 2, \dots, M\}$ ,  $r \in (0, \infty)$ , satisfy for all  $m \in \{1, 2, \dots, M\}$ ,  $r \in (0, \infty)$ ,  $x \in \mathbb{R}^d$  that

$$h_{m,r}(x) = 1 + \left[ \|x\| + CT + \max_{n \in \{0,1,\dots,N\}} \|W_n^m\| \right]^r \exp(rcT) \quad (145)$$

(cf. Definitions 2.1, 4.1, 4.3, and 4.4). Then there exists  $(\Psi_\omega)_{\omega \in \Omega} \subseteq \mathbb{N}$  such that

(i) it holds for all  $\omega \in \Omega$  that  $\mathcal{R}_\tau(\Psi_\omega) \in C(\mathbb{R}^{d+1}, \mathbb{R}^{\mathfrak{d}})$ ,

(ii) it holds for all  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $\omega \in \Omega$  that

$$\begin{aligned} &\left\| (\mathcal{R}_\tau(\Psi_\omega))(t, x) - \frac{1}{M} \left[ \sum_{m=1}^M (\mathcal{R}_\tau(\mathfrak{f}^0))(Y_t^{m,x}(\omega)) \right] \right\| \\ &\leq \frac{2\varepsilon \mathfrak{C} \sqrt{d}}{M} \left[ \sum_{m=1}^M [1 + 2d^{\alpha/2} 6^\alpha |h_{m,2}(x, \omega)|^\alpha] h_{m,q}(x, \omega) \right], \end{aligned} \quad (146)$$

(iii) it holds for all  $\omega \in \Omega$  that

$$\mathcal{P}(\Psi_\omega) \leq 2M^2\mathcal{P}(\mathfrak{f}^0) + 9M^2N^6d^{16} [2\mathcal{L}(\mathfrak{f}^1) + \mathfrak{D} + (24 + 6\mathcal{L}(\mathfrak{f}^1) + [4 + \mathcal{P}(\mathfrak{f}^1)]^2)^2]^2, \quad (147)$$

and

(iv) it holds for all  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that  $\Omega \ni \omega \mapsto (\mathcal{R}_\tau(\Psi_\omega))(t, x) \in \mathbb{R}^{\mathfrak{D}}$  is measurable.

*Proof of Proposition 4.9.* Throughout this proof let  $\tau_0, \tau_1, \dots, \tau_N \in \mathbb{R}$  satisfy for all  $n \in \{0, 1, \dots, N\}$  that  $\tau_n = \frac{nT}{N}$ , let  $g_m: \mathbb{R}^d \times \Omega \rightarrow [0, \infty)$ ,  $m \in \{1, 2, \dots, M\}$ , satisfy for all  $m \in \{1, 2, \dots, M\}$ ,  $x \in \mathbb{R}^d$  that

$$g_m(x) = \left[ \|x\| + CT + \max_{n \in \{0, 1, \dots, N\}} \left\| \sum_{l=1}^n (W_l^m - W_{l-1}^m) \right\| \right] \exp(cT), \quad (148)$$

let  $(\Psi_{\omega, m})_{\omega \in \Omega, m \in \{1, 2, \dots, M\}} \subseteq \mathbf{N}$  satisfy that

(I) it holds for all  $m \in \{1, 2, \dots, M\}$ ,  $\omega \in \Omega$  that  $\mathcal{R}_\tau(\Psi_{\omega, m}) \in C(\mathbb{R}^{d+1}, \mathbb{R}^d)$ ,

(II) it holds for all  $m \in \{1, 2, \dots, M\}$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $\omega \in \Omega$  that

$$\|Y_t^{m, x}(\omega) - (\mathcal{R}_\tau(\Psi_{\omega, m}))(t, x)\| \leq 2\varepsilon\sqrt{d} [1 + (g_m(x, \omega))^q], \quad (149)$$

(III) it holds for all  $m \in \{1, 2, \dots, M\}$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $\omega \in \Omega$  that

$$\|(\mathcal{R}_\tau(\Psi_{\omega, m}))(t, x)\| \leq 6\sqrt{d} [1 + (g_m(x, \omega))^2], \quad (150)$$

(IV) it holds for all  $m \in \{1, 2, \dots, M\}$ ,  $\omega \in \Omega$  that

$$\mathcal{P}(\Psi_{\omega, m}) \leq \frac{9}{2} N^6 d^{16} [2\mathcal{L}(\mathfrak{f}^1) + \mathfrak{D} + (24 + 6\mathcal{L}(\mathfrak{f}^1) + [4 + \mathcal{P}(\mathfrak{f}^1)]^2)^2]^2, \quad (151)$$

and

(V) it holds for all  $m \in \{1, 2, \dots, M\}$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that  $\Omega \ni \omega \mapsto (\mathcal{R}_\tau(\Psi_{\omega, m}))(t, x) \in \mathbb{R}^d$  is measurable

(cf. Proposition 4.8 (applied with  $N \leftarrow N$ ,  $d \leftarrow d$ ,  $c \leftarrow c$ ,  $C \leftarrow C$ ,  $T \leftarrow T$ ,  $\mathfrak{D} \leftarrow \mathfrak{D}$ ,  $q \leftarrow q$ ,  $\varepsilon \leftarrow \varepsilon$ ,  $\tau_0 \leftarrow \tau_0$ ,  $\tau_1 \leftarrow \tau_1, \dots, \tau_N \leftarrow \tau_N$ ,  $\Phi \leftarrow \mathfrak{f}^1$  in the notation of Proposition 4.8)), let  $\mathbb{I} \in \mathbf{N}$  satisfy for all  $x \in \mathbb{R}^d$  that  $\mathcal{R}_\tau(\mathbb{I}) \in C(\mathbb{R}^d, \mathbb{R}^d)$ ,  $\mathcal{D}(\mathbb{I}) = (d, 2d, d)$ , and  $(\mathcal{R}_\tau(\mathbb{I}))(x) = x$  (cf. [35, Lemma 5.4]), let  $(\psi_{\omega, m})_{\omega \in \Omega, m \in \{1, 2, \dots, M\}} \subseteq \mathbf{N}$  satisfy for all  $m \in \{1, 2, \dots, M\}$ ,  $\omega \in \Omega$  that  $\psi_{\omega, m} = \mathfrak{f}^0 \odot_{\mathbb{I}} \Psi_{\omega, m}$  (cf. Definition 4.7), and let  $(\Phi_\omega)_{\omega \in \Omega} \subseteq \mathbf{N}$  satisfy that

(A) it holds for all  $\omega \in \Omega$  that  $\mathcal{R}_\tau(\Phi_\omega) \in C(\mathbb{R}^{\mathcal{I}(\psi_{\omega, 1})}, \mathbb{R}^{\mathcal{O}(\psi_{\omega, 1})})$ ,

(B) it holds for all  $\omega \in \Omega$  that  $\mathcal{P}(\Phi_\omega) \leq M^2\mathcal{P}(\psi_{\omega, 1})$ , and

(C) it holds for all  $t \in \mathbb{R}$ ,  $x \in \mathbb{R}^d$ ,  $\omega \in \Omega$  that

$$(\mathcal{R}_\tau(\Phi_\omega))(t, x) = \frac{1}{M} \sum_{m=1}^M (\mathcal{R}_\tau(\psi_{\omega, m}))(t, x) \quad (152)$$

(cf. Grohs et al. [23, Proposition 2.25]). Note that (B), (IV), Grohs et al. [23, item (iii) in Proposition 2.16], and the fact that  $\mathcal{D}(\mathbb{I}) = (d, 2d, d)$  demonstrate that for all  $\omega \in \Omega$  it holds that

$$\begin{aligned} \mathcal{P}(\Phi_\omega) &\leq M^2 \mathcal{P}(\psi_{\omega,1}) \leq 2M^2 [\mathcal{P}(f^0) + \mathcal{P}(\Psi_{\omega,1})] \\ &\leq 2M^2 \mathcal{P}(f^0) + 9M^2 N^6 d^{16} [2\mathcal{L}(f^1) + \mathfrak{D} + (24 + 6\mathcal{L}(f^1) + [4 + \mathcal{P}(f^1)]^2)^2]. \end{aligned} \quad (153)$$

Moreover, observe that (A), (C), and Grohs et al. [23, item (iv) in Proposition 2.16] imply that for all  $t \in \mathbb{R}$ ,  $x \in \mathbb{R}^d$ ,  $\omega \in \Omega$  it holds that  $\mathcal{R}_\tau(\Phi_\omega) \in C(\mathbb{R}^{d+1}, \mathbb{R}^0)$  and

$$(\mathcal{R}_\tau(\Phi_\omega))(t, x) = \frac{1}{M} \sum_{m=1}^M (\mathcal{R}_\tau(f^0))((\mathcal{R}_\tau(\Psi_{\omega,m}))(t, x)). \quad (154)$$

Next note that the fact that  $\forall m \in \{1, 2, \dots, M\}: W_0^m = 0$  ensures that for all  $n \in \{1, 2, \dots, N\}$ ,  $m \in \{1, 2, \dots, M\}$  it holds that

$$\sum_{l=1}^n (W_l^m - W_{l-1}^m) = W_n^m - W_0^m = W_n^m. \quad (155)$$

Combining this with (148) proves that for all  $m \in \{1, 2, \dots, M\}$ ,  $x \in \mathbb{R}^d$  it holds that

$$g_m(x) = \left[ \|x\| + CT + \max_{n \in \{0, 1, \dots, N\}} \|W_n^m\| \right] \exp(cT). \quad (156)$$

In addition, observe that (144) and the fact that  $\forall n \in \{0, 1, \dots, N\}: \tau_n = \frac{nT}{N}$  assure that for all  $m \in \{1, 2, \dots, M\}$ ,  $n \in \{0, 1, \dots, N-1\}$ ,  $x \in \mathbb{R}^d$  it holds that

$$Y_{\tau_{n+1}}^{m,x} = Y_{\tau_n}^{m,x} + \frac{T}{N} (\mathcal{R}_\tau(f^1))(Y_{\tau_n}^{m,x}) + W_{n+1}^m - W_n^m. \quad (157)$$

Induction and (155) hence show that for all  $m \in \{1, 2, \dots, M\}$ ,  $n \in \{0, 1, \dots, N-1\}$ ,  $x \in \mathbb{R}^d$  it holds that

$$\begin{aligned} Y_{\tau_{n+1}}^{m,x} &= Y_{\tau_0}^{m,x} + \frac{T}{N} \left[ \sum_{l=0}^n (\mathcal{R}_\tau(f^1))(Y_{\tau_l}^{m,x}) \right] + \left[ \sum_{l=0}^n (W_{l+1}^m - W_l^m) \right] \\ &= x + \frac{T}{N} \left[ \sum_{l=0}^n (\mathcal{R}_\tau(f^1))(Y_{\tau_l}^{m,x}) \right] + W_{n+1}^m. \end{aligned} \quad (158)$$

This and the assumption that  $\forall x \in \mathbb{R}^d: \|(\mathcal{R}_\tau(f^1))(x)\| \leq C + c\|x\|$  establish that for all  $m \in \{1, 2, \dots, M\}$ ,  $n \in \{0, 1, \dots, N\}$ ,  $x \in \mathbb{R}^d$  it holds that

$$\begin{aligned} \|Y_{\tau_n}^{m,x}\| &\leq \|x\| + \frac{T}{N} \left[ \sum_{l=0}^{n-1} \|(\mathcal{R}_\tau(f^1))(Y_{\tau_l}^{m,x})\| \right] + \|W_n^m\| \\ &\leq \|x\| + \frac{T}{N} \left[ \sum_{l=0}^{n-1} (C + c\|Y_{\tau_l}^{m,x}\|) \right] + \|W_n^m\| \\ &\leq \|x\| + CT + \left[ \max_{k \in \{0, 1, \dots, N\}} \|W_k^m\| \right] + \frac{cT}{N} \left[ \sum_{l=0}^{n-1} \|Y_{\tau_l}^{m,x}\| \right]. \end{aligned} \quad (159)$$

The time-discrete Gronwall inequality, e.g., in Hutzenthaler et al. [33, Lemma 2.1] (applied with  $N \leftarrow N$ ,  $\alpha \leftarrow (\|x\| + CT + \max_{k \in \{0, 1, \dots, N\}} \|W_k^m(\omega)\|)$ ,  $\beta_0 \leftarrow \frac{cT}{N}$ ,  $\beta_1 \leftarrow \frac{cT}{N}$ ,  $\dots$ ,  $\beta_{N-1} \leftarrow \frac{cT}{N}$ ,  $\epsilon_0 \leftarrow \|x\|$ ,  $\epsilon_1 \leftarrow \|Y_{\tau_1}^{m,x}(\omega)\|$ ,  $\dots$ ,  $\epsilon_N \leftarrow \|Y_{\tau_N}^{m,x}(\omega)\|$  for  $m \in \{1, 2, \dots, M\}$ ,  $x \in \mathbb{R}^d$ ,  $\omega \in \Omega$  in

the notation of Hutzenthaler et al. [33, Lemma 2.1]) and (156) hence demonstrate that for all  $m \in \{1, 2, \dots, M\}$ ,  $n \in \{0, 1, \dots, N\}$ ,  $x \in \mathbb{R}^d$  it holds that

$$\|Y_{\tau_n}^{m,x}\| \leq \left[ \|x\| + CT + \max_{k \in \{0,1,\dots,N\}} \|W_k^m\| \right] \exp(cT) = g_m(x). \quad (160)$$

In addition, note that (144) and the fact that  $\forall n \in \{0, 1, \dots, N-1\}, t \in [\tau_n, \tau_{n+1}]$ :  $\frac{t-\tau_n}{\tau_{n+1}-\tau_n} = \frac{tN}{T} - n$  ensure that for all  $m \in \{1, 2, \dots, M\}$ ,  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in [\tau_n, \tau_{n+1}]$ ,  $x \in \mathbb{R}^d$  it holds that

$$\begin{aligned} & Y_{\tau_{n+1}}^{m,x} \left[ \frac{t-\tau_n}{\tau_{n+1}-\tau_n} \right] + Y_{\tau_n}^{m,x} \left[ 1 - \frac{t-\tau_n}{\tau_{n+1}-\tau_n} \right] \\ &= \left( Y_{\tau_n}^{m,x} + \left[ \frac{\tau_{n+1}N}{T} - n \right] \left[ \frac{T}{N} (\mathcal{R}_\tau(\mathbf{f}^1))(Y_{\tau_n}^{m,x}) + W_{n+1}^m - W_n^m \right] \right) \frac{t-\tau_n}{\tau_{n+1}-\tau_n} + Y_{\tau_n}^{m,x} \left[ 1 - \frac{t-\tau_n}{\tau_{n+1}-\tau_n} \right] \\ &= \left( Y_{\tau_n}^{m,x} + \left[ \frac{T}{N} (\mathcal{R}_\tau(\mathbf{f}^1))(Y_{\tau_n}^{m,x}) + W_{n+1}^m - W_n^m \right] \right) \frac{t-\tau_n}{\tau_{n+1}-\tau_n} + Y_{\tau_n}^{m,x} \left[ 1 - \frac{t-\tau_n}{\tau_{n+1}-\tau_n} \right] \\ &= Y_{\tau_n}^{m,x} + \left[ \frac{T}{N} (\mathcal{R}_\tau(\mathbf{f}^1))(Y_{\tau_n}^{m,x}) + W_{n+1}^m - W_n^m \right] \frac{t-\tau_n}{\tau_{n+1}-\tau_n} \\ &= Y_t^{m,x}. \end{aligned} \quad (161)$$

Combining this with (160) implies that for all  $m \in \{1, 2, \dots, M\}$ ,  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in [\tau_n, \tau_{n+1}]$ ,  $x \in \mathbb{R}^d$  it holds that

$$\begin{aligned} \|Y_t^{m,x}\| &\leq \|Y_{\tau_n}^{m,x}\| \left[ 1 - \frac{t-\tau_n}{\tau_{n+1}-\tau_n} \right] + \|Y_{\tau_{n+1}}^{m,x}\| \left[ \frac{t-\tau_n}{\tau_{n+1}-\tau_n} \right] \\ &\leq \max\{\|Y_{\tau_n}^{m,x}\|, \|Y_{\tau_{n+1}}^{m,x}\|\} \leq g_m(x) \leq 1 + |g_m(x)|^2. \end{aligned} \quad (162)$$

This, (143), (II), and (III) ensure that for all  $m \in \{1, 2, \dots, M\}$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $\omega \in \Omega$  it holds that

$$\begin{aligned} & \|(\mathcal{R}_\tau(\mathbf{f}^0))((\mathcal{R}_\tau(\Psi_{\omega,m}))(t, x)) - (\mathcal{R}_\tau(\mathbf{f}^0))(Y_t^{m,x}(\omega))\| \\ &\leq \mathfrak{C} \left( 1 + \|(\mathcal{R}_\tau(\Psi_{\omega,m}))(t, x)\|^\alpha + \|Y_t^{m,x}(\omega)\|^\alpha \right) \|(\mathcal{R}_\tau(\Psi_{\omega,m}))(t, x) - Y_t^{m,x}(\omega)\| \\ &\leq \mathfrak{C} \left[ 1 + 6^\alpha d^{\alpha/2} (1 + |g_m(x, \omega)|^2)^\alpha + (1 + |g_m(x, \omega)|^2)^\alpha \right] 2\varepsilon \sqrt{d} (1 + |g_m(x, \omega)|^q). \end{aligned} \quad (163)$$

Combining this, (145), and (156) demonstrates that for all  $m \in \{1, 2, \dots, M\}$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $\omega \in \Omega$  it holds that

$$\begin{aligned} & \|(\mathcal{R}_\tau(\mathbf{f}^0))((\mathcal{R}_\tau(\Psi_{\omega,m}))(t, x)) - (\mathcal{R}_\tau(\mathbf{f}^0))(Y_t^{m,x}(\omega))\| \\ &\leq 2\varepsilon \mathfrak{C} \sqrt{d} \left[ 1 + 2d^{\alpha/2} 6^\alpha (1 + |g_m(x, \omega)|^2)^\alpha \right] (1 + |g_m(x, \omega)|^q) \\ &= 2\varepsilon \mathfrak{C} \sqrt{d} \left[ 1 + 2d^{\alpha/2} 6^\alpha |h_{m,2}(x, \omega)|^\alpha \right] h_{m,q}(x, \omega). \end{aligned} \quad (164)$$

This and (154) show that for all  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $\omega \in \Omega$  it holds that

$$\begin{aligned} & \left\| (\mathcal{R}_\tau(\Phi_\omega))(t, x) - \frac{1}{M} \left[ \sum_{m=1}^M (\mathcal{R}_\tau(\mathbf{f}^0))(Y_t^{m,x}(\omega)) \right] \right\| \\ &= \left\| \frac{1}{M} \left[ \sum_{m=1}^M (\mathcal{R}_\tau(\mathbf{f}^0))((\mathcal{R}_\tau(\Psi_{\omega,m}))(t, x)) \right] - \frac{1}{M} \left[ \sum_{m=1}^M (\mathcal{R}_\tau(\mathbf{f}^0))(Y_t^{m,x}(\omega)) \right] \right\| \\ &\leq \frac{1}{M} \left[ \sum_{m=1}^M \|(\mathcal{R}_\tau(\mathbf{f}^0))((\mathcal{R}_\tau(\Psi_{\omega,m}))(t, x)) - (\mathcal{R}_\tau(\mathbf{f}^0))(Y_t^{m,x}(\omega))\| \right] \\ &\leq \frac{2\varepsilon \mathfrak{C} \sqrt{d}}{M} \left[ \sum_{m=1}^M \left[ 1 + 2d^{\alpha/2} 6^\alpha |h_{m,2}(x, \omega)|^\alpha \right] h_{m,q}(x, \omega) \right]. \end{aligned} \quad (165)$$

Moreover, observe that (V), the fact that  $\mathcal{R}_\tau(\mathbf{f}^0)$  is continuous, and (154) ensure that for all  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  it holds that  $\Omega \ni \omega \mapsto (\mathcal{R}_\tau(\Phi_\omega))(t, x) \in \mathbb{R}^d$  is measurable. Combining this with (153), (165), and the fact that  $\forall \omega \in \Omega$ :  $\mathcal{R}_\tau(\Phi_\omega) \in C(\mathbb{R}^{d+1}, \mathbb{R}^d)$  establishes items (i)–(iv). This completes the proof of Proposition 4.9.  $\square$

**Lemma 4.10.** *Let  $N, d \in \mathbb{N}$ ,  $T \in (0, \infty)$ ,  $\alpha, c, C, \mathcal{C} \in [0, \infty)$ ,  $p \in (1, \infty)$ , let  $\nu: \mathcal{B}([0, T] \times \mathbb{R}^d) \rightarrow [0, \infty)$  be a finite measure which satisfies that*

$$\mathcal{C} = \max \left\{ \left[ \int_{[0, T] \times \mathbb{R}^d} \|x\|^{\max\{4p\alpha, 6p\}} \nu(dt, dx) \right]^{1/\max\{4p\alpha, 6p\}}, 1 \right\}, \quad (166)$$

let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_n)_{n \in \{0, 1, \dots, N\}})$  be a filtered probability space, let  $\mathcal{M} = (\mathcal{M}_n)_{n \in \{0, 1, \dots, N\}}: \{0, 1, \dots, N\} \times \Omega \rightarrow [0, \infty)$  be an  $(\mathbb{F}_n)_{n \in \{0, 1, \dots, N\}}$ -submartingale which satisfies for all  $n \in \{0, 1, \dots, N\}$  that  $\mathbb{E}[|\mathcal{M}_n|^{\max\{4p\alpha, 6p\}}] < \infty$ , and let  $h_r: \mathbb{R}^d \times \Omega \rightarrow [0, \infty)$ ,  $r \in (0, \infty)$ , satisfy for all  $r \in (0, \infty)$ ,  $x \in \mathbb{R}^d$  that

$$h_r(x) = 1 + e^{rc} \left[ \|x\| + C + \max_{n \in \{0, 1, \dots, N\}} \mathcal{M}_n \right]^r \quad (167)$$

(cf. Definition 2.1). Then

(i) it holds for all  $q, r \in (0, \infty)$  with  $1 < qr \leq \max\{4\alpha p, 6p\}$  that

$$\begin{aligned} & \left[ \int_{[0, T] \times \mathbb{R}^d} \mathbb{E}[|h_r(x)|^q] \nu(dt, dx) \right]^{1/q} \\ & \leq 2e^{rc} \max\{2^{(1/q)-1}, 1\} \left[ \mathcal{C} + C + \frac{qr}{qr-1} |\mathbb{E}[|\mathcal{M}_N|^{qr}]|^{1/qr} \right]^r \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/q} \} \end{aligned} \quad (168)$$

and

(ii) it holds that

$$\begin{aligned} & \left[ \int_{[0, T] \times \mathbb{R}^d} \mathbb{E}[|[h_2(x)]^\alpha h_3(x)|^p] \nu(dt, dx) \right]^{1/p} \\ & \leq \left[ \mathcal{C} + C + \frac{\max\{4p\alpha, 6p\}}{\max\{4p\alpha, 6p\}-1} |\mathbb{E}[|\mathcal{M}_N|^{\max\{4p\alpha, 6p\}}]|^{1/\max\{4p\alpha, 6p\}} \right]^{2\alpha+3} \\ & \cdot 2^{\alpha+1} e^{(2\alpha+3)c} \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/p}\}. \end{aligned} \quad (169)$$

*Proof of Lemma 4.10.* Throughout this proof let  $\mathcal{C}_q \in \mathbb{R}$ ,  $q \in (0, \max\{4p\alpha, 6p\}]$ , satisfy for all  $q \in (0, \max\{4p\alpha, 6p\}]$  that

$$\mathcal{C}_q = \max \left\{ \left[ \int_{[0, T] \times \mathbb{R}^d} \|x\|^q \nu(dt, dx) \right]^{1/q}, 1 \right\} \quad (170)$$

and let  $g: \mathbb{R}^d \times \Omega \rightarrow [0, \infty)$  satisfy for all  $x \in \mathbb{R}^d$  that

$$g(x) = \left[ \|x\| + C + \max_{n \in \{0, 1, \dots, N\}} \mathcal{M}_n \right]. \quad (171)$$

Observe that Doob's inequality (cf., e.g., Klenke [36, Theorem 11.2]), Hölder's inequality, the hypothesis that  $\mathcal{M}$  is a submartingale, the hypothesis that  $\forall n \in \{0, 1, \dots, N\}: \mathcal{M}_n \geq 0$ , and the hypothesis that  $\forall n \in \{0, 1, \dots, N\}: \mathbb{E}[|\mathcal{M}_n|^{\max\{4p\alpha, 6p\}}] < \infty$  demonstrate that for all  $q \in (1, \max\{4p\alpha, 6p\}]$  it holds that

$$\begin{aligned} \left| \mathbb{E} \left[ \max_{n \in \{0, 1, \dots, N\}} |\mathcal{M}_n|^q \right] \right|^{1/q} & \leq \frac{q}{q-1} |\mathbb{E}[|\mathcal{M}_N|^q]|^{1/q} \\ & \leq \frac{q}{q-1} |\mathbb{E}[|\mathcal{M}_N|^{\max\{4p\alpha, 6p\}}]|^{1/\max\{4p\alpha, 6p\}} < \infty. \end{aligned} \quad (172)$$

Moreover, note that the triangle inequality and (171) prove that for all  $q, r \in (0, \infty)$  with  $1 < qr \leq \max\{4\alpha p, 6p\}$  it holds that

$$\begin{aligned} & \left[ \int_{[0, T] \times \mathbb{R}^d} \mathbb{E}[|g(x)|^{qr}] \nu(dt, dx) \right]^{1/qr} \\ & \leq \left[ \int_{[0, T] \times \mathbb{R}^d} \mathbb{E}[||x||^{qr}] \nu(dt, dx) \right]^{1/qr} + \left[ \int_{[0, T] \times \mathbb{R}^d} \mathbb{E} \left[ \left| C + \max_{n \in \{0, 1, \dots, N\}} \mathcal{M}_n \right|^{qr} \right] \nu(dt, dx) \right]^{1/qr}. \end{aligned} \quad (173)$$

The triangle inequality, (170), and (172) hence show that for all  $q, r \in (0, \infty)$  with  $1 < qr \leq \max\{4\alpha p, 6p\}$  it holds that

$$\begin{aligned} & \left[ \int_{[0, T] \times \mathbb{R}^d} \mathbb{E}[|g(x)|^{qr}] \nu(dt, dx) \right]^{1/qr} \\ & \leq \left[ \int_{[0, T] \times \mathbb{R}^d} ||x||^{qr} \nu(dt, dx) \right]^{1/qr} + \left[ C + \left| \mathbb{E} \left[ \max_{n \in \{0, 1, \dots, N\}} |\mathcal{M}_n|^{qr} \right] \right|^{1/qr} \right] [\nu([0, T] \times \mathbb{R}^d)]^{1/qr} \\ & \leq \mathcal{C}_{qr} + \left[ C + \frac{qr}{qr-1} \left| \mathbb{E} [|\mathcal{M}_N|^{qr}] \right|^{1/qr} \right] [\nu([0, T] \times \mathbb{R}^d)]^{1/qr}. \end{aligned} \quad (174)$$

Combining this with (167), (171), and the fact that  $\forall a, b \in [0, \infty)$ ,  $q \in (0, \infty)$ :  $(a + b)^q \leq 2^{\max\{q-1, 0\}}(a^q + b^q)$  ensures that for all  $q, r \in (0, \infty)$  with  $1 < qr \leq \max\{4\alpha p, 6p\}$  it holds that

$$\begin{aligned} & \left[ \int_{[0, T] \times \mathbb{R}^d} \mathbb{E}[|h_r(x)|^q] \nu(dt, dx) \right]^{1/q} \\ & \leq \max\{2^{(1/q)-1}, 1\} \left[ \int_{[0, T] \times \mathbb{R}^d} \mathbb{1}_{[0, T] \times \mathbb{R}^d}(t, x) \nu(dt, dx) \right]^{1/q} \\ & \quad + e^{rc} \max\{2^{(1/q)-1}, 1\} \left[ \int_{[0, T] \times \mathbb{R}^d} \mathbb{E}[|g(x)|^{rq}] \nu(dt, dx) \right]^{1/q} \\ & \leq \max\{2^{(1/q)-1}, 1\} [\nu([0, T] \times \mathbb{R}^d)]^{1/q} \\ & \quad + e^{rc} \max\{2^{(1/q)-1}, 1\} \left[ \mathcal{C}_{qr} + \left[ C + \frac{qr}{qr-1} \left| \mathbb{E} [|\mathcal{M}_N|^{qr}] \right|^{1/qr} \right] [\nu([0, T] \times \mathbb{R}^d)]^{1/qr} \right]^r. \end{aligned} \quad (175)$$

In addition, observe that (166) and Hölder's inequality show that for all  $q, r \in (0, \infty)$  with  $1 < qr \leq \max\{4\alpha p, 6p\}$  it holds that

$$\begin{aligned} & \left[ \int_{[0, T] \times \mathbb{R}^d} ||x||^{qr} \nu(dt, dx) \right]^{1/qr} \\ & \leq \left[ \int_{[0, T] \times \mathbb{R}^d} ||x||^{\max\{4\alpha p, 6p\}} \nu(dt, dx) \right]^{1/\max\{4\alpha p, 6p\}} [\nu([0, T] \times \mathbb{R}^d)]^{\left[ \frac{1}{qr} - \frac{1}{\max\{4\alpha p, 6p\}} \right]} \\ & \leq \mathcal{C} \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/qr}\}. \end{aligned} \quad (176)$$

This, the fact that  $\mathcal{C} \geq 1$ , and (170) prove that for all  $q, r \in (0, \infty)$  with  $1 < qr \leq \max\{4\alpha p, 6p\}$  it holds that

$$\mathcal{C}_{qr} \leq \mathcal{C} \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/qr}\}. \quad (177)$$

Combining this with (175) implies that for all  $q, r \in (0, \infty)$  with  $1 < qr \leq \max\{4\alpha p, 6p\}$  it



holds that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} [|h_r(x)|^q] \nu(dt, dx) \right]^{1/q} \\
& \leq \max\{2^{(1/q)-1}, 1\} [\nu([0, T] \times \mathbb{R}^d)]^{1/q} \\
& + e^{rc} \max\{2^{(1/q)-1}, 1\} \left[ \mathcal{C} + C + \frac{qr}{qr-1} |\mathbb{E}[|\mathcal{M}_N|^{qr}]|^{1/qr} \right]^r \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/q}\}.
\end{aligned} \tag{178}$$

Therefore, we obtain that for all  $q, r \in (0, \infty)$  with  $1 < qr \leq \max\{4\alpha p, 6p\}$  it holds that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} [|h_r(x)|^q] \nu(dt, dx) \right]^{1/q} \\
& \leq 2e^{rc} \max\{2^{(1/q)-1}, 1\} \left[ \mathcal{C} + C + \frac{qr}{qr-1} |\mathbb{E}[|\mathcal{M}_N|^{qr}]|^{1/qr} \right]^r \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/q}\}.
\end{aligned} \tag{179}$$

This establishes item (i). Next observe that Hölder's inequality assures that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} [|h_2(x)|^\alpha h_3(x)^p] \nu(dt, dx) \right]^{1/p} \\
& \leq \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} [|h_2(x)|^{2p\alpha}] \nu(dt, dx) \right]^{1/2p} \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} [|h_3(x)|^{2p}] \nu(dt, dx) \right]^{1/2p}.
\end{aligned} \tag{180}$$

Moreover, note that Hölder's inequality demonstrates that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} [|h_2(x)|^{2p\alpha}] \nu(dt, dx) \right]^{1/2p\alpha} \\
& \leq \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} [|h_2(x)|^{\max\{2p\alpha, 3p\}}] \nu(dt, dx) \right]^{1/\max\{2p\alpha, 3p\}} [\nu([0, T] \times \mathbb{R}^d)]^{\left[\frac{1}{2p\alpha} - \frac{1}{\max\{2p\alpha, 3p\}}\right]}
\end{aligned} \tag{181}$$

and

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} [|h_3(x)|^{2p}] \nu(dt, dx) \right]^{1/2p} \\
& \leq \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} [|h_3(x)|^{\max\{(4/3)p\alpha, 2p\}}] \nu(dt, dx) \right]^{1/\max\{(4/3)p\alpha, 2p\}} [\nu([0, T] \times \mathbb{R}^d)]^{\left[\frac{1}{2p} - \frac{1}{\max\{(4/3)p\alpha, 2p\}}\right]}.
\end{aligned} \tag{182}$$

Combining this with (179) implies that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} [|h_2(x)|^{2p\alpha}] \nu(dt, dx) \right]^{1/2p\alpha} \\
& \leq 2e^{2c} \left[ \mathcal{C} + C + \frac{\max\{4p\alpha, 6p\}}{\max\{4p\alpha, 6p\}-1} |\mathbb{E}[|\mathcal{M}_N|^{\max\{4p\alpha, 6p\}}]|^{1/\max\{4p\alpha, 6p\}} \right]^2 \\
& \cdot \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/\max\{2p\alpha, 3p\}}\} [\nu([0, T] \times \mathbb{R}^d)]^{\left[\frac{1}{2p\alpha} - \frac{1}{\max\{2p\alpha, 3p\}}\right]} \\
& \leq 2e^{2c} \left[ \mathcal{C} + C + \frac{\max\{4p\alpha, 6p\}}{\max\{4p\alpha, 6p\}-1} |\mathbb{E}[|\mathcal{M}_N|^{\max\{4p\alpha, 6p\}}]|^{1/\max\{4p\alpha, 6p\}} \right]^2 \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/2p\alpha}\}
\end{aligned} \tag{183}$$

and

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} [|h_3(x)|^{2p}] \nu(dt, dx) \right]^{1/2p} \\
& \leq 2e^{3c} \left[ \mathcal{C} + C + \frac{\max\{4p\alpha, 6p\}}{\max\{4p\alpha, 6p\}-1} |\mathbb{E} [|\mathcal{M}_N|^{\max\{4p\alpha, 6p\}}]|^{1/\max\{4p\alpha, 6p\}} \right]^3 \\
& \cdot \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/\max\{(4/3)p\alpha, 2p\}}\} [\nu([0, T] \times \mathbb{R}^d)]^{\left[\frac{1}{2p} - \frac{1}{\max\{(4/3)p\alpha, 2p\}}\right]} \\
& \leq 2e^{3c} \left[ \mathcal{C} + C + \frac{\max\{4p\alpha, 6p\}}{\max\{4p\alpha, 6p\}-1} |\mathbb{E} [|\mathcal{M}_N|^{\max\{4p\alpha, 6p\}}]|^{1/\max\{4p\alpha, 6p\}} \right]^3 \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/2p}\}.
\end{aligned} \tag{184}$$

This and (180) show that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} \left[ |[h_2(x)]^\alpha h_3(x)|^p \right] \nu(dt, dx) \right]^{1/p} \\
& \leq \left[ \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} [|h_2(x)|^{2p\alpha}] \nu(dt, dx) \right]^{1/2p\alpha} \right]^\alpha \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} [|h_3(x)|^{2p}] \nu(dt, dx) \right]^{1/2p} \\
& \leq 2^{\alpha+1} e^{(2\alpha+3)c} \left[ \mathcal{C} + C + \frac{\max\{4p\alpha, 6p\}}{\max\{4p\alpha, 6p\}-1} |\mathbb{E} [|\mathcal{M}_N|^{\max\{4p\alpha, 6p\}}]|^{1/\max\{4p\alpha, 6p\}} \right]^{2\alpha+3} \\
& \cdot \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/2p}\} \left[ \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/2p\alpha}\} \right]^\alpha.
\end{aligned} \tag{185}$$

Hence, we obtain that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} \left[ |[h_2(x)]^\alpha h_3(x)|^p \right] \nu(dt, dx) \right]^{1/p} \\
& \leq 2^{\alpha+1} e^{(2\alpha+3)c} \left[ \mathcal{C} + C + \frac{\max\{4p\alpha, 6p\}}{\max\{4p\alpha, 6p\}-1} |\mathbb{E} [|\mathcal{M}_N|^{\max\{4p\alpha, 6p\}}]|^{1/\max\{4p\alpha, 6p\}} \right]^{2\alpha+3} \\
& \cdot \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/p}\}.
\end{aligned} \tag{186}$$

This establishes item (ii). This completes the proof of Lemma 4.10.  $\square$

**Corollary 4.11.** *Let  $M, N, d, \mathfrak{d}, k \in \mathbb{N}$ ,  $p \in [2, \infty)$ ,  $\alpha, c, C, \mathcal{C}, \mathfrak{C} \in [0, \infty)$ ,  $T, \mathfrak{D} \in (0, \infty)$ ,  $B \in \mathbb{R}^{d \times k}$ ,  $\varepsilon \in (0, 1]$ ,  $\mathfrak{f}^1, \mathfrak{f}^0 \in \mathbf{N}$  satisfy that  $\mathcal{I}(\mathfrak{f}^1) = \mathcal{O}(\mathfrak{f}^1) = \mathcal{I}(\mathfrak{f}^0) = d$ ,  $\mathcal{O}(\mathfrak{f}^0) = \mathfrak{d}$ , and*

$$\mathfrak{D} = 2160[\log_2(\varepsilon^{-1}) + 4] - 504, \tag{187}$$

let  $\nu: \mathcal{B}([0, T] \times \mathbb{R}^d) \rightarrow [0, \infty)$  be a finite measure which satisfies that

$$\mathcal{C} = \max \left\{ \left[ \int_{[0,T] \times \mathbb{R}^d} \|x\|^{\max\{4p\alpha, 6p\}} \nu(dt, dx) \right]^{1/\max\{4p\alpha, 6p\}}, 1 \right\}, \tag{188}$$

assume for all  $x, y \in \mathbb{R}^d$  that  $\|(\mathcal{R}_\tau(\mathfrak{f}^1))(x)\| \leq C + c\|x\|$  and

$$\|(\mathcal{R}_\tau(\mathfrak{f}^0))(x) - (\mathcal{R}_\tau(\mathfrak{f}^0))(y)\| \leq \mathfrak{C}(1 + \|x\|^\alpha + \|y\|^\alpha)\|x - y\|, \tag{189}$$

let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$  be a filtered probability space which satisfies the usual conditions<sup>1</sup>, let  $W^m: [0, T] \times \Omega \rightarrow \mathbb{R}^k$ ,  $m \in \{1, 2, \dots, M\}$ , be standard  $(\mathbb{F}_t)_{t \in [0, T]}$ -Brownian motions, and let

<sup>1</sup>Note that we say that a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$  satisfies the usual conditions if and only if it holds for all  $t \in [0, T]$  that  $\{A \in \mathcal{F}: \mathbb{P}(A) = 0\} \subseteq \mathbb{F}_t = (\bigcap_{s \in (t, T]} \mathbb{F}_s)$ ; cf., e.g., Liu & Röckner [38, Definition 2.1.11].

$Y^m = (Y_t^{m,x}(\omega))_{(t,x,\omega) \in [0,T] \times \mathbb{R}^d \times \Omega} : [0, T] \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ ,  $m \in \{1, 2, \dots, M\}$ , satisfy for all  $m \in \{1, 2, \dots, M\}$ ,  $x \in \mathbb{R}^d$ ,  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in [\frac{nT}{N}, \frac{(n+1)T}{N}]$  that  $Y_0^{m,x} = x$  and

$$Y_t^{m,x} = Y_{\frac{nT}{N}}^{m,x} + \left(\frac{tN}{T} - n\right) \left[ \frac{T}{N} (\mathcal{R}_\tau(\mathbf{f}^1))(Y_{\frac{nT}{N}}^{m,x}) + B(W_{\frac{(n+1)T}{N}}^m - W_{\frac{nT}{N}}^m) \right] \quad (190)$$

(cf. Definitions 2.1, 4.1, 4.3, and 4.4). Then there exists  $(\Psi_\omega)_{\omega \in \Omega} \subseteq \mathbf{N}$  such that

(i) it holds for all  $\omega \in \Omega$  that  $\mathcal{R}_\tau(\Psi_\omega) \in C(\mathbb{R}^{d+1}, \mathbb{R}^d)$ ,

(ii) it holds for all  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that  $\Omega \ni \omega \mapsto (\mathcal{R}_\tau(\Psi_\omega))(t, x) \in \mathbb{R}^d$  is measurable,

(iii) it holds that

$$\begin{aligned} & \left[ \int_{[0,T] \times \mathbb{R}^d} \int_{\Omega} \left\| (\mathcal{R}_\tau(\Psi_\omega))(t, x) - \frac{1}{M} \left[ \sum_{m=1}^M (\mathcal{R}_\tau(\mathbf{f}^0))(Y_t^{m,x}(\omega)) \right] \right\|^p \mathbb{P}(d\omega) \nu(dt, dx) \right]^{1/p} \\ & \leq [2p \max\{C, \mathcal{C}\} \max\{T, 1\} \max\{\alpha, 1\}]^{2\alpha+3} \left[ 1 + \sqrt{\text{Trace}(B^*B)} \right]^{2\alpha+3} \\ & \cdot \varepsilon d^{(\alpha+1)/2} \mathfrak{C} e^{(2\alpha+3)cT} 2^{2\alpha+4} 3^\alpha \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/p}\}, \end{aligned} \quad (191)$$

and

(iv) it holds for all  $\omega \in \Omega$  that

$$\mathcal{P}(\Psi_\omega) \leq 2M^2 \mathcal{P}(\mathbf{f}^0) + 9M^2 N^6 d^{16} [2\mathcal{L}(\mathbf{f}^1) + \mathfrak{D} + (24 + 6\mathcal{L}(\mathbf{f}^1) + [4 + \mathcal{P}(\mathbf{f}^1)]^2)^2]. \quad (192)$$

*Proof of Corollary 4.11.* Throughout this proof let  $h_{m,r} : \mathbb{R}^d \times \Omega \rightarrow [0, \infty)$ ,  $m \in \{1, 2, \dots, M\}$ ,  $r \in \mathbb{R}$ , satisfy for all  $m \in \{1, 2, \dots, M\}$ ,  $r \in \mathbb{R}$ ,  $x \in \mathbb{R}^d$  that

$$h_{m,r}(x) = 1 + \left[ \|x\| + CT + \max_{n \in \{0, 1, \dots, N\}} \|BW_{\frac{nT}{N}}^m\| \right]^r \exp(rcT), \quad (193)$$

let  $(\Psi_\omega)_{\omega \in \Omega} \subseteq \mathbf{N}$  satisfy that

(I) it holds for all  $\omega \in \Omega$  that  $\mathcal{R}_\tau(\Psi_\omega) \in C(\mathbb{R}^{d+1}, \mathbb{R}^d)$ ,

(II) it holds for all  $\omega \in \Omega$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} & \left\| (\mathcal{R}_\tau(\Psi_\omega))(t, x) - \frac{1}{M} \left[ \sum_{m=1}^M (\mathcal{R}_\tau(\mathbf{f}^0))(Y_t^{m,x}(\omega)) \right] \right\| \\ & \leq \frac{2\varepsilon \mathfrak{C} \sqrt{d}}{M} \left[ \sum_{m=1}^M [1 + 2d^{\alpha/2} 6^\alpha |h_{m,2}(x, \omega)|^\alpha] h_{m,3}(x, \omega) \right], \end{aligned} \quad (194)$$

(III) it holds for all  $\omega \in \Omega$  that

$$\mathcal{P}(\Psi_\omega) \leq 2M^2 \mathcal{P}(\mathbf{f}^0) + 9M^2 N^6 d^{16} [2\mathcal{L}(\mathbf{f}^1) + \mathfrak{D} + (24 + 6\mathcal{L}(\mathbf{f}^1) + [4 + \mathcal{P}(\mathbf{f}^1)]^2)^2], \quad (195)$$

and

(IV) it holds for all  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that  $\Omega \ni \omega \mapsto (\mathcal{R}_\tau(\Psi_\omega))(t, x) \in \mathbb{R}^d$  is measurable

(cf. Proposition 4.9), let  $Z = (Z_t^{x,y})_{(t,x,y) \in [0,T] \times \mathbb{R}^d \times (\mathbb{R}^d)^N} : [0, T] \times \mathbb{R}^d \times (\mathbb{R}^d)^N \rightarrow \mathbb{R}^d$  satisfy for all  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in [\frac{nT}{N}, \frac{(n+1)T}{N}]$ ,  $x \in \mathbb{R}^d$ ,  $y = (y_1, y_2, \dots, y_N) \in (\mathbb{R}^d)^N$  that  $Z_0^{x,y} = x$  and

$$Z_t^{x,y} = Z_{\frac{nT}{N}}^{x,y} + \left(\frac{tN}{T} - n\right) \left[\frac{T}{N} (\mathcal{R}_t(\mathbf{f}^1)) \left(Z_{\frac{nT}{N}}^{x,y}\right) + y_{n+1}\right], \quad (196)$$

and let  $\mathcal{W}^m : [0, T] \times \mathbb{R}^d \times \Omega \rightarrow [0, T] \times \mathbb{R}^d \times (\mathbb{R}^d)^N$ ,  $m \in \{1, 2, \dots, M\}$ , satisfy for all  $m \in \{1, 2, \dots, M\}$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that

$$\mathcal{W}^m(t, x) = \left(t, x, B\left(W_{\frac{T}{N}}^m - W_0^m\right), B\left(W_{\frac{2T}{N}}^m - W_{\frac{T}{N}}^m\right), \dots, B\left(W_{\frac{NT}{N}}^m - W_{\frac{(N-1)T}{N}}^m\right)\right). \quad (197)$$

Note that (I), (IV), and Beck et al. [2, Lemma 2.4] demonstrate that  $[0, T] \times \mathbb{R}^d \times \Omega \ni (t, x, \omega) \mapsto (\mathcal{R}_t(\Psi_\omega))(t, x) \in \mathbb{R}^d$  is measurable. In addition, observe that (190), (196), and (197) ensure that for all  $m \in \{1, 2, \dots, M\}$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  it holds that

$$Y_t^{m,x} = (Z \circ \mathcal{W}^m)(t, x). \quad (198)$$

Next note that Grohs et al. [23, Lemma 3.8] (applied with  $N \leftarrow N$ ,  $d \leftarrow d$ ,  $\mu \leftarrow \mathcal{R}_t(\mathbf{f}^1)$ ,  $T \leftarrow T$ ,  $(\{-1, 0, 1, \dots, N+1\} \ni n \mapsto t_n \in \mathbb{R}) \leftarrow (\{-1, 0, 1, \dots, N+1\} \ni n \mapsto \frac{nT}{N} \in \mathbb{R})$ ,  $Y \leftarrow Z$  in the notation of Grohs et al. [23, Lemma 3.8]) proves that  $Z \in C([0, T] \times \mathbb{R}^d \times (\mathbb{R}^d)^N, \mathbb{R}^d)$ . Combining this with (198) and the fact that for all  $m \in \{1, 2, \dots, M\}$  it holds that  $\mathcal{W}^m$  is measurable shows that for all  $m \in \{1, 2, \dots, M\}$  it holds that  $Y^m$  is measurable. The fact that  $\mathcal{R}_t(\mathbf{f}^0) \in C(\mathbb{R}^d, \mathbb{R}^d)$  hence ensures that

$$[0, T] \times \mathbb{R}^d \times \Omega \ni (t, x, \omega) \mapsto \frac{1}{M} \left[ \sum_{m=1}^M (\mathcal{R}_t(\mathbf{f}^0))(Y_t^{m,x}(\omega)) \right] \in \mathbb{R}^d \quad (199)$$

is measurable. Combining this with (194) and the fact that  $[0, T] \times \mathbb{R}^d \times \Omega \ni (t, x, \omega) \mapsto (\mathcal{R}_t(\Psi_\omega))(t, x) \in \mathbb{R}^d$  is measurable proves that

$$\begin{aligned} & \left[ \int_{[0,T] \times \mathbb{R}^d} \int_{\Omega} \left\| (\mathcal{R}_t(\Psi_\omega))(t, x) - \frac{1}{M} \left[ \sum_{m=1}^M (\mathcal{R}_t(\mathbf{f}^0))(Y_t^{m,x}(\omega)) \right] \right\|^p \mathbb{P}(d\omega) \nu(dt, dx) \right]^{1/p} \\ & \leq \frac{2\varepsilon \mathfrak{C} \sqrt{d}}{M} \left[ \int_{[0,T] \times \mathbb{R}^d} \int_{\Omega} \left| \sum_{m=1}^M [1 + 2d^{\alpha/2} 6^\alpha |h_{m,2}(x, \omega)|^\alpha] h_{m,3}(x, \omega) \right|^p \mathbb{P}(d\omega) \nu(dt, dx) \right]^{1/p}. \end{aligned} \quad (200)$$

The triangle inequality therefore implies that

$$\begin{aligned} & \left[ \int_{[0,T] \times \mathbb{R}^d} \int_{\Omega} \left\| (\mathcal{R}_t(\Psi_\omega))(t, x) - \frac{1}{M} \left[ \sum_{m=1}^M (\mathcal{R}_t(\mathbf{f}^0))(Y_t^{m,x}(\omega)) \right] \right\|^p \mathbb{P}(d\omega) \nu(dt, dx) \right]^{1/p} \\ & \leq \frac{2\varepsilon \mathfrak{C} \sqrt{d}}{M} \sum_{m=1}^M \left[ \int_{[0,T] \times \mathbb{R}^d} \int_{\Omega} |h_{m,3}(x, \omega)|^p \mathbb{P}(d\omega) \nu(dt, dx) \right]^{1/p} \\ & + \frac{4d^{\alpha/2} 6^\alpha \varepsilon \mathfrak{C} \sqrt{d}}{M} \sum_{m=1}^M \left[ \int_{[0,T] \times \mathbb{R}^d} \int_{\Omega} [|h_{m,2}(x, \omega)|^\alpha h_{m,3}(x, \omega)]^p \mathbb{P}(d\omega) \nu(dt, dx) \right]^{1/p}. \end{aligned} \quad (201)$$

Next note that (193), Lemma 4.10, and the fact that for all  $m \in \{1, 2, \dots, M\}$  it holds that  $(\|BW_{\frac{nT}{N}}^m\|)_{n \in \{0, 1, \dots, N\}}$  is a nonnegative  $(\mathbb{F}_{\frac{nT}{N}})_{n \in \{0, 1, \dots, N\}}$ -submartingale demonstrate that for all  $m \in \{1, 2, \dots, M\}$  it holds that

$$\begin{aligned} & \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E}[|h_{m,3}(x)|^p] \nu(dt, dx) \right]^{1/p} \\ & \leq 2e^{3cT} \left[ \mathcal{C} + CT + \frac{3p}{3p-1} |\mathbb{E}[\|BW_T^m\|^{3p}]|^{1/3p} \right]^3 \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/p}\} \end{aligned} \quad (202)$$

and

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} [ | [h_{m,2}(x)]^\alpha h_{m,3}(x) |^p ] \nu(dt, dx) \right]^{1/p} \\
& \leq \left[ \mathcal{C} + CT + \frac{\max\{4p\alpha, 6p\}}{\max\{4p\alpha, 6p\}-1} |\mathbb{E} [ \| BW_T^m \|_{\max\{4p\alpha, 6p\}} ] |^{1/\max\{4p\alpha, 6p\}} \right]^{2\alpha+3} \\
& \quad \cdot 2^{\alpha+1} e^{(2\alpha+3)cT} \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/p}\}.
\end{aligned} \tag{203}$$

Moreover, observe that Lemma 2.2, the fact that for all  $m \in \{1, 2, \dots, M\}$  it holds that  $BW_T^m$  is a Gaussian random variable, and the fact that for all  $m \in \{1, 2, \dots, M\}$  it holds that  $\text{Cov}(BW_T^m) = TB^*B$  ensure that for all  $m \in \{1, 2, \dots, M\}$ ,  $q \in [2, \infty)$  it holds that

$$\begin{aligned}
\frac{q}{q-1} |\mathbb{E} [ \| BW_T^m \|_q ]|^{1/q} & \leq \frac{q}{q-1} \sqrt{\max\{1, q-1\} T \text{Trace}(B^*B)} \\
& = \frac{q}{q-1} \sqrt{(q-1) T \text{Trace}(B^*B)} \\
& = \frac{q}{\sqrt{q-1}} \sqrt{T \text{Trace}(B^*B)}.
\end{aligned} \tag{204}$$

Combining this with (202) and (203) assures that

$$\begin{aligned}
& \frac{1}{M} \sum_{m=1}^M \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} [ | h_{m,3}(x) |^p ] \nu(dt, dx) \right]^{1/p} \\
& \leq 2e^{3cT} \left[ \mathcal{C} + CT + \frac{3p}{\sqrt{3p-1}} \sqrt{T \text{Trace}(B^*B)} \right]^3 \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/p}\}
\end{aligned} \tag{205}$$

and

$$\begin{aligned}
& \frac{1}{M} \sum_{m=1}^M \left[ \int_{[0,T] \times \mathbb{R}^d} \mathbb{E} [ | [h_{m,2}(x)]^\alpha h_{m,3}(x) |^p ] \nu(dt, dx) \right]^{1/p} \\
& \leq \left[ \mathcal{C} + CT + \frac{\max\{4p\alpha, 6p\}}{\sqrt{\max\{4p\alpha, 6p\}-1}} \sqrt{T \text{Trace}(B^*B)} \right]^{2\alpha+3} \\
& \quad \cdot 2^{\alpha+1} e^{(2\alpha+3)cT} \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/p}\}.
\end{aligned} \tag{206}$$

This and (201) demonstrate that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} \int_{\Omega} \left\| (\mathcal{R}_\tau(\Psi_\omega))(t, x) - \frac{1}{M} \left[ \sum_{m=1}^M (\mathcal{R}_\tau(\mathfrak{f}^0))(Y_t^{m,x}(\omega)) \right] \right\|^p \mathbb{P}(d\omega) \nu(dt, dx) \right]^{1/p} \\
& \leq 4\varepsilon \mathfrak{C} e^{3cT} \sqrt{d} \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/p}\} \left[ \mathcal{C} + CT + \frac{3p}{\sqrt{3p-1}} \sqrt{T \text{Trace}(B^*B)} \right]^3 \\
& \quad + \left[ \mathcal{C} + CT + \frac{\max\{4p\alpha, 6p\}}{\sqrt{\max\{4p\alpha, 6p\}-1}} \sqrt{T \text{Trace}(B^*B)} \right]^{2\alpha+3} \\
& \quad \cdot 4\varepsilon \mathfrak{C} 6^\alpha d^{\alpha/2} 2^{\alpha+1} e^{(2\alpha+3)cT} \sqrt{d} \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/p}\}.
\end{aligned} \tag{207}$$

The fact that  $[2, \infty) \ni x \mapsto x/\sqrt{x-1} \in \mathbb{R}$  is non-decreasing hence implies that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} \int_{\Omega} \left\| (\mathcal{R}_\tau(\Psi_\omega))(t, x) - \frac{1}{M} \left[ \sum_{m=1}^M (\mathcal{R}_\tau(\mathfrak{f}^0))(Y_t^{m,x}(\omega)) \right] \right\|^p \mathbb{P}(d\omega) \nu(dt, dx) \right]^{1/p} \\
& \leq \left[ \mathcal{C} + CT + \frac{\max\{4p\alpha, 6p\}}{\sqrt{\max\{4p\alpha, 6p\}-1}} \sqrt{T \text{Trace}(B^*B)} \right]^{2\alpha+3} \\
& \quad \cdot \left[ 4\varepsilon \mathfrak{C} e^{3cT} \sqrt{d} + 4\varepsilon \mathfrak{C} 6^\alpha d^{\alpha/2} 2^{\alpha+1} e^{(2\alpha+3)cT} \sqrt{d} \right] \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/p}\} \\
& \leq |\max\{C, \mathcal{C}\}|^{2\alpha+3} |\max\{T, 1\}|^{2\alpha+3} \left[ 2 + \frac{\max\{4p\alpha, 6p\}}{\sqrt{\max\{4p\alpha, 6p\}-1}} \sqrt{\text{Trace}(B^*B)} \right]^{2\alpha+3} \\
& \quad \cdot \varepsilon d^{(\alpha+1)/2} \mathfrak{C} e^{(2\alpha+3)cT} [4 + 6^\alpha 2^{\alpha+1} 4] \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/p}\}.
\end{aligned} \tag{208}$$

The fact that  $\sqrt{\max\{4p\alpha, 6p\} - 1} \geq \sqrt{6p - 1} \geq \sqrt{11} \geq 3$  therefore ensures that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} \int_{\Omega} \left\| (\mathcal{R}_{\tau}(\Psi_{\omega}))(t, x) - \frac{1}{M} \left[ \sum_{m=1}^M (\mathcal{R}_{\tau}(f^0))(Y_t^{m,x}(\omega)) \right] \right\|^p \mathbb{P}(d\omega) \nu(dt, dx) \right]^{1/p} \\
& \leq |\max\{C, \mathcal{C}\}|^{2\alpha+3} |\max\{T, 1\}|^{2\alpha+3} \left[ 2 + \max\{(4/3)p\alpha, 2p\} \sqrt{\text{Trace}(B^*B)} \right]^{2\alpha+3} \\
& \cdot \varepsilon d^{(\alpha+1)/2} \mathfrak{C} e^{(2\alpha+3)cT} \left[ 4 + 2^{2\alpha+3} 3^{\alpha} \right] \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/p}\} \\
& \leq |\max\{C, \mathcal{C}\}|^{2\alpha+3} |\max\{T, 1\}|^{2\alpha+3} [2p \max\{\alpha, 1\}]^{2\alpha+3} \left[ 1 + \sqrt{\text{Trace}(B^*B)} \right]^{2\alpha+3} \\
& \cdot \varepsilon d^{(\alpha+1)/2} \mathfrak{C} e^{(2\alpha+3)cT} 2^{2\alpha+4} 3^{\alpha} \max\{1, [\nu([0, T] \times \mathbb{R}^d)]^{1/p}\}.
\end{aligned} \tag{209}$$

Combining this with (I), (III), and (IV) establishes items (i)–(iv). This completes the proof of Corollary 4.11.  $\square$

## 4.5 Approximation error estimates for DNNs

**Proposition 4.12.** *Let  $T, \kappa \in (0, \infty)$ ,  $\eta \in [1, \infty)$ ,  $p \in [2, \infty)$ , let  $A_d = (a_{d,i,j})_{(i,j) \in \{1,2,\dots,d\}^2} \in \mathbb{R}^{d \times d}$ ,  $d \in \mathbb{N}$ , be symmetric positive semidefinite matrices, let  $\nu_d: \mathcal{B}([0, T] \times \mathbb{R}^d) \rightarrow [0, \infty)$ ,  $d \in \mathbb{N}$ , be finite measures which satisfy for all  $d \in \mathbb{N}$  that*

$$\left[ \int_{[0,T] \times \mathbb{R}^d} \|x\|^{2p \max\{2\kappa, 3\}} \nu_d(dt, dx) \right]^{1/p} \leq \eta d^m, \tag{210}$$

let  $f_d^m: \mathbb{R}^d \rightarrow \mathbb{R}^{md-m+1}$ ,  $m \in \{0, 1\}$ ,  $d \in \mathbb{N}$ , be functions, let  $(f_{d,\varepsilon}^m)_{(m,d,\varepsilon) \in \{0,1\} \times \mathbb{N} \times (0,1]} \subseteq \mathbf{N}$ , assume for all  $m \in \{0, 1\}$ ,  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$ ,  $x, y \in \mathbb{R}^d$  that

$$\mathcal{R}_{\tau}(f_{d,\varepsilon}^0) \in C(\mathbb{R}^d, \mathbb{R}), \quad \mathcal{R}_{\tau}(f_{d,\varepsilon}^1) \in C(\mathbb{R}^d, \mathbb{R}^d), \quad \mathcal{P}(f_{d,\varepsilon}^m) \leq \kappa d^{\kappa} \varepsilon^{-\kappa}, \tag{211}$$

$$\|f_d^1(x) - f_d^1(y)\| \leq \kappa \|x - y\|, \quad \|(\mathcal{R}_{\tau}(f_{d,\varepsilon}^1))(x)\| \leq \kappa (d^{\kappa} + \|x\|), \tag{212}$$

$$|(\mathcal{R}_{\tau}(f_{d,\varepsilon}^0))(x) - (\mathcal{R}_{\tau}(f_{d,\varepsilon}^0))(y)| \leq \kappa d^{\kappa} (1 + \|x\|^{\kappa} + \|y\|^{\kappa}) \|x - y\|, \tag{213}$$

$$\|f_d^m(x) - (\mathcal{R}_{\tau}(f_{d,\varepsilon}^m))(x)\| \leq \varepsilon \kappa d^{\kappa} (1 + \|x\|^{\kappa}), \tag{214}$$

$$\text{and} \quad |f_d^0(x)| + \text{Trace}(A_d) \leq \kappa d^{\kappa} (1 + \|x\|^{\kappa}), \tag{215}$$

and for every  $d \in \mathbb{N}$  let  $u_d \in \{v \in C([0, T] \times \mathbb{R}^d, \mathbb{R}) : \inf_{q \in (0, \infty)} \sup_{(t,y) \in [0,T] \times \mathbb{R}^d} \frac{|v(t,y)|}{1+\|y\|^q} < \infty\}$  be a viscosity solution of

$$\left( \frac{\partial}{\partial t} u_d \right)(t, x) = \left( \frac{\partial}{\partial x} u_d \right)(t, x) f_d^1(x) + \sum_{i,j=1}^d a_{d,i,j} \left( \frac{\partial^2}{\partial x_i \partial x_j} u_d \right)(t, x) \tag{216}$$

with  $u_d(0, x) = f_d^0(x)$  for  $(t, x) \in (0, T) \times \mathbb{R}^d$  (cf. Definitions 2.1, 4.1, 4.3, and 4.4). Then there exist  $\mathcal{C} \in \mathbb{R}$  and  $(\mathbf{u}_{d,N,M,\delta})_{(d,N,M,\delta) \in \mathbb{N}^3 \times (0,1]} \subseteq \mathbf{N}$  such that

(i) it holds for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$  that  $\mathcal{R}_{\tau}(\mathbf{u}_{d,N,M,\delta}) \in C(\mathbb{R}^{d+1}, \mathbb{R})$ ,

(ii) it holds for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$  that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} |u_d(y) - (\mathcal{R}_{\tau}(\mathbf{u}_{d,N,M,\delta}))(y)|^p \nu_d(dy) \right]^{1/p} \\
& \leq \mathcal{C} \left[ \max\{1, \nu_d([0, T] \times \mathbb{R}^d)\} \right]^{1/p} \\
& \cdot \left[ \frac{d^{\kappa(\kappa+4) + \max\{\eta, \kappa(2\kappa+1)\}}}{N^{1/2}} + \frac{d^{\kappa + \max\{\eta, \kappa^2\}}}{M^{1/2}} + \delta d^{(2\kappa+3) \max\{\eta, \kappa\} + \kappa^2 + (7\kappa+1)/2} \right],
\end{aligned} \tag{217}$$

and

(iii) it holds for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$  that

$$\mathcal{P}(\mathbf{u}_{d,N,M,\delta}) \leq \mathcal{C} M^2 N^{6+4\kappa} [\log_2(\delta^{-1}) + 1]^2 d^{16+8\kappa}. \quad (218)$$

*Proof of Proposition 4.12.* Throughout this proof let  $\mathfrak{D}_\delta \in \mathbb{R}$ ,  $\delta \in (0, 1]$ , satisfy for all  $\delta \in (0, 1]$  that

$$\mathfrak{D}_\delta = 2160[\log_2(\delta^{-1}) + 4] - 504, \quad (219)$$

let  $C_d \in \mathbb{R}$ ,  $d \in \mathbb{N}$ , satisfy for all  $d \in \mathbb{N}$  that

$$C_d = \max \left\{ \left[ \int_{[0,T] \times \mathbb{R}^d} \|x\|^{\max\{4p\kappa, 6p\}} \nu_d(dt, dx) \right]^{1/\max\{4p\kappa, 6p\}}, 1 \right\}, \quad (220)$$

let  $\mathcal{C}_1 \in (0, \infty)$  satisfy that

$$\mathcal{C}_1 = [2p \max\{\eta, \kappa\} \max\{T, 1\} \max\{\kappa, 1\}]^{2\kappa+3} [1 + (2\kappa)^{1/2}]^{2\kappa+3} \kappa e^{(2\kappa+3)\kappa T} 2^{2\kappa+4} 3^\kappa, \quad (221)$$

let  $\mathcal{C}_2 \in (0, \infty)$  satisfy that

$$\mathcal{C}_2 = 2^{57} [\max\{\kappa, 1\}]^8 [\max\{T^{-\kappa/2}, 1\}]^8, \quad (222)$$

let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$  be a filtered probability space which satisfies the usual conditions, let  $W^{d,m} : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ ,  $d, m \in \mathbb{N}$ , be independent standard  $(\mathbb{F}_t)_{t \in [0, T]}$ -Brownian motions, let  $Y^{N,d,m,x} : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ ,  $x \in \mathbb{R}^d$ ,  $N, d, m \in \mathbb{N}$ , be stochastic processes which satisfy for all  $N, d, m \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$ ,  $n \in \{0, 1, \dots, N-1\}$ ,  $t \in [\frac{nT}{N}, \frac{(n+1)T}{N}]$  that  $Y_0^{N,d,m,x} = x$  and

$$Y_t^{N,d,m,x} = Y_{\frac{nT}{N}}^{N,d,m,x} + \left(\frac{tN}{T} - n\right) \left[ \frac{T}{N} (\mathcal{R}_\tau(\mathfrak{f}_{d, \min\{(T/N)^{1/2}, 1\}}^1)) (Y_{\frac{nT}{N}}^{N,d,m,x}) + \sqrt{2A_d} (W_{\frac{(n+1)T}{N}}^{d,m} - W_{\frac{nT}{N}}^{d,m}) \right], \quad (223)$$

let  $(\psi_{d,N,M,\delta,\omega})_{(d,N,M,\delta,\omega) \in \mathbb{N}^3 \times (0,1] \times \Omega} \subseteq \mathbf{N}$  satisfy that

(I) it holds for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$ ,  $\omega \in \Omega$  that  $\mathcal{R}_\tau(\psi_{d,N,M,\delta,\omega}) \in C(\mathbb{R}^{d+1}, \mathbb{R})$ ,

(II) it holds for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that  $\Omega \ni \omega \mapsto (\mathcal{R}_\tau(\psi_{d,N,M,\delta,\omega}))(t, x) \in \mathbb{R}$  is measurable,

(III) it holds for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$  that

$$\begin{aligned} & \left[ \int_{[0,T] \times \mathbb{R}^d} \int_{\Omega} \left| (\mathcal{R}_\tau(\psi_{d,N,M,\delta,\omega}))(t, x) \right. \right. \\ & \quad \left. \left. - \frac{1}{M} \left[ \sum_{m=1}^M (\mathcal{R}_\tau(\mathfrak{f}_{d, \min\{(T/N)^{1/2}, 1\}}^0)) (Y_t^{N,d,m,x}(\omega)) \right] \right|^p \mathbb{P}(d\omega) \nu_d(dt, dx) \right]^{1/p} \\ & \leq [2p \max\{C_d, \kappa d^\kappa\} \max\{T, 1\} \max\{\kappa, 1\}]^{2\kappa+3} [1 + \sqrt{\text{Trace}(2A_d)}]^{2\kappa+3} \\ & \quad \cdot \delta d^{(\kappa+1)/2} \kappa d^\kappa e^{(2\kappa+3)\kappa T} 2^{2\kappa+4} 3^\kappa \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\}, \end{aligned} \quad (224)$$

and

(IV) it holds for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$ ,  $\omega \in \Omega$  that

$$\begin{aligned} \mathcal{P}(\psi_{d,N,M,\delta,\omega}) & \leq 2M^2 \mathcal{P}(\mathfrak{f}_{d, \min\{(T/N)^{1/2}, 1\}}^0) + 9M^2 N^6 d^{16} \left[ 2\mathcal{L}(\mathfrak{f}_{d, \min\{(T/N)^{1/2}, 1\}}^1) + \mathfrak{D}_\delta \right. \\ & \quad \left. + (24 + 6\mathcal{L}(\mathfrak{f}_{d, \min\{(T/N)^{1/2}, 1\}}^1) + [4 + \mathcal{P}(\mathfrak{f}_{d, \min\{(T/N)^{1/2}, 1\}}^1)]^2)^2 \right]^2 \end{aligned} \quad (225)$$

(cf. Corollary 4.11 (applied with  $M \leftarrow M, N \leftarrow N, d \leftarrow d, \mathfrak{d} \leftarrow 1, k \leftarrow d, p \leftarrow p, \alpha \leftarrow \kappa, c \leftarrow \kappa, C \leftarrow \kappa d^\kappa, \mathcal{C} \leftarrow C_d, \mathfrak{C} \leftarrow \kappa d^\kappa, T \leftarrow T, \mathfrak{D} \leftarrow \mathfrak{D}_\delta, B \leftarrow \sqrt{2A_d}, \varepsilon \leftarrow \delta, \mathfrak{f}^1 \leftarrow \mathfrak{f}_{d, \min\{(T/N)^{1/2}, 1\}}^1, \mathfrak{f}^0 \leftarrow \mathfrak{f}_{d, \min\{(T/N)^{1/2}, 1\}}^0, \nu \leftarrow \nu_d, (\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$ ,  $W^m \leftarrow W^{d, m}, Y^m \leftarrow (Y^{N, d, m, x})_{x \in \mathbb{R}^d}$  for  $d, N, M \in \mathbb{N}, m \in \{1, 2, \dots, M\}, \delta \in (0, 1]$  in the notation of Corollary 4.11)), let  $Z_{d, N, M, \delta}: \Omega \rightarrow [0, \infty]$ ,  $d, N, M \in \mathbb{N}, \delta \in (0, 1]$ , satisfy for all  $d, N, M \in \mathbb{N}, \delta \in (0, 1], \omega \in \Omega$  that

$$Z_{d, N, M, \delta}(\omega) = \int_{[0, T] \times \mathbb{R}^d} |u_d(y) - (\mathcal{R}_\tau(\psi_{d, N, M, \delta, \omega}))(y)|^p \nu_d(dy) \quad (226)$$

(cf. (I)), and let  $\mathcal{C}_3 \in (0, \infty)$  satisfy that for all  $d, N, M \in \mathbb{N}$  it holds that

$$\begin{aligned} & \left( \mathbb{E} \left[ \int_{[0, T] \times \mathbb{R}^d} |u_d(t, x) - \frac{1}{M} \left[ \sum_{m=1}^M (\mathcal{R}_\tau(\mathfrak{f}_{d, \min\{(T/N)^{1/2}, 1\}}^0)) (Y_t^{N, d, m, x}) \right]|^p \nu_d(dt, dx) \right] \right)^{1/p} \\ & \leq \mathcal{C}_3 \left[ \frac{d^{\kappa(\kappa+4) + \max\{\eta, \kappa(2\kappa+1)\}}}{N^{1/2}} + \frac{d^{\kappa + \max\{\eta, \kappa^2\}}}{M^{1/2}} \right] \left[ \max\{1, \nu_d([0, T] \times \mathbb{R}^d)\} \right]^{1/p} \end{aligned} \quad (227)$$

(cf. item (ii) in Proposition 3.2 (applied with  $T \leftarrow T, \kappa \leftarrow \kappa, \eta \leftarrow \eta, p \leftarrow p, A_d \leftarrow A_d, \nu_d \leftarrow \nu_d, f_d^0 \leftarrow f_d^0, f_d^1 \leftarrow f_d^1, F_{d, \varepsilon}^0 \leftarrow \mathcal{R}_\tau(f_{d, \varepsilon}^0), F_{d, \varepsilon}^1 \leftarrow \mathcal{R}_\tau(f_{d, \varepsilon}^1), (\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P}), W^{d, m} \leftarrow W^{d, m}, Y^{N, d, m, x} \leftarrow Y^{N, d, m, x}$  for  $d, N, M \in \mathbb{N}, m \in \{1, 2, \dots, M\}, \varepsilon \in (0, 1], x \in \mathbb{R}^d$  in the notation of Proposition 3.2)). Observe that (210) and (220) demonstrate that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned} \max\{C_d, \kappa d^\kappa\} &= \max \left\{ \left[ \int_{[0, T] \times \mathbb{R}^d} \|x\|^{2p \max\{2\kappa, 3\}} \nu_d(dt, dx) \right]^{1/2p \max\{2\kappa, 3\}}, 1, \kappa d^\kappa \right\} \\ &\leq \max\{(\eta d^\eta)^{1/(2 \max\{2\kappa, 3\})}, 1, \kappa d^\kappa\} \leq \max\{\eta d^\eta, 1, \kappa d^\kappa\} \\ &\leq \max\{\eta, 1, \kappa\} d^{\max\{\eta, \kappa\}} = \max\{\eta, \kappa\} d^{\max\{\eta, \kappa\}}. \end{aligned} \quad (228)$$

Next note that (215) proves that

$$1 + \sqrt{\text{Trace}(2A_d)} \leq 1 + (2\kappa d^\kappa)^{1/2} \leq d^{\kappa/2} [1 + (2\kappa)^{1/2}]. \quad (229)$$

Combining this with (224) and (228) ensures that for all  $d, N, M \in \mathbb{N}, \delta \in (0, 1]$  it holds that

$$\begin{aligned} & \left[ \int_{[0, T] \times \mathbb{R}^d} \int_{\Omega} \left| (\mathcal{R}_\tau(\psi_{d, N, M, \delta, \omega}))(t, x) \right. \right. \\ & \quad \left. \left. - \frac{1}{M} \left[ \sum_{m=1}^M (\mathcal{R}_\tau(\mathfrak{f}_{d, \min\{(T/N)^{1/2}, 1\}}^0)) (Y_t^{N, d, m, x}(\omega)) \right] \right|^p \mathbb{P}(d\omega) \nu_d(dt, dx) \right]^{1/p} \\ & \leq [2p \max\{\eta, \kappa\} d^{\max\{\eta, \kappa\}} \max\{T, 1\} \max\{\kappa, 1\}]^{2\kappa+3} [d^{\kappa/2} [1 + (2\kappa)^{1/2}]]^{2\kappa+3} \\ & \quad \cdot \delta d^{(\kappa+1)/2} \kappa d^\kappa e^{(2\kappa+3)\kappa T} 2^{2\kappa+4} 3^\kappa \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\} \\ & \leq \delta [2p \max\{\eta, \kappa\} \max\{T, 1\} \max\{\kappa, 1\}]^{2\kappa+3} [1 + (2\kappa)^{1/2}]^{2\kappa+3} \kappa e^{(2\kappa+3)\kappa T} 2^{2\kappa+4} 3^\kappa \\ & \quad \cdot d^{(2\kappa+3) \max\{\eta, \kappa\} + \kappa(\kappa+2) + \kappa + (\kappa+1)/2} \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\}. \end{aligned} \quad (230)$$

This and (221) imply that for all  $d, N, M \in \mathbb{N}, \delta \in (0, 1]$  it holds that

$$\begin{aligned} & \left[ \int_{[0, T] \times \mathbb{R}^d} \int_{\Omega} \left| (\mathcal{R}_\tau(\psi_{d, N, M, \delta, \omega}))(t, x) \right. \right. \\ & \quad \left. \left. - \frac{1}{M} \left[ \sum_{m=1}^M (\mathcal{R}_\tau(\mathfrak{f}_{d, \min\{(T/N)^{1/2}, 1\}}^0)) (Y_t^{N, d, m, x}(\omega)) \right] \right|^p \mathbb{P}(d\omega) \nu_d(dt, dx) \right]^{1/p} \\ & \leq \mathcal{C}_1 \delta d^{(2\kappa+3) \max\{\eta, \kappa\} + \kappa(\kappa+2) + \kappa + (\kappa+1)/2} \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\}. \end{aligned} \quad (231)$$



Furthermore, observe that (211) shows that for all  $d, N \in \mathbb{N}$ ,  $m \in \{0, 1\}$  it holds that

$$\begin{aligned} \mathcal{L}(\mathfrak{f}_{d, \min\{(T/N)^{1/2}, 1\}}^m) &\leq \mathcal{P}(\mathfrak{f}_{d, \min\{(T/N)^{1/2}, 1\}}^m) \leq \kappa d^\kappa [\min\{(T/N)^{1/2}, 1\}]^{-\kappa} \\ &= \kappa d^\kappa N^{\kappa/2} [\min\{T^{1/2}, N^{1/2}\}]^{-\kappa} \leq \kappa d^\kappa N^{\kappa/2} [\min\{T^{1/2}, 1\}]^{-\kappa} \\ &= \kappa d^\kappa N^{\kappa/2} \max\{T^{-\kappa/2}, 1\}. \end{aligned} \quad (232)$$

Hence, we obtain that for all  $d, N \in \mathbb{N}$ ,  $\delta \in (0, 1]$  it holds that

$$\begin{aligned} &24 + 6\mathcal{L}(\mathfrak{f}_{d, \min\{(T/N)^{1/2}, 1\}}^1) + [4 + \mathcal{P}(\mathfrak{f}_{d, \min\{(T/N)^{1/2}, 1\}}^1)]^2 \\ &\leq 24 + 6\mathcal{P}(\mathfrak{f}_{d, \min\{(T/N)^{1/2}, 1\}}^1) + 25[\mathcal{P}(\mathfrak{f}_{d, \min\{(T/N)^{1/2}, 1\}}^1)]^2 \\ &\leq 24 + 31[\mathcal{P}(\mathfrak{f}_{d, \min\{(T/N)^{1/2}, 1\}}^1)]^2 \\ &\leq 55[\mathcal{P}(\mathfrak{f}_{d, \min\{(T/N)^{1/2}, 1\}}^1)]^2 \leq 2^6 \kappa^2 d^{2\kappa} N^\kappa |\max\{T^{-\kappa/2}, 1\}|^2. \end{aligned} \quad (233)$$

This, (IV), and (232) establish that for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$ ,  $\omega \in \Omega$  it holds that

$$\begin{aligned} \mathcal{P}(\psi_{d, N, M, \delta, \omega}) &\leq 2M^2 \kappa d^\kappa N^{\kappa/2} \max\{T^{-\kappa/2}, 1\} \\ &+ 9M^2 N^6 d^{16} [2\kappa d^\kappa N^{\kappa/2} \max\{T^{-\kappa/2}, 1\} + \mathfrak{D}_\delta + 2^{12} \kappa^4 d^{4\kappa} N^{2\kappa} |\max\{T^{-\kappa/2}, 1\}|^4]^2 \\ &\leq 2M^2 \kappa d^\kappa N^{\kappa/2} \max\{T^{-\kappa/2}, 1\} \\ &+ 9M^2 N^6 d^{16} [(2^{12} + 3)\mathfrak{D}_\delta |\max\{\kappa, 1\}|^4 d^{4\kappa} N^{2\kappa} |\max\{T^{-\kappa/2}, 1\}|^4]^2 \\ &\leq 2M^2 \kappa d^\kappa N^{\kappa/2} \max\{T^{-\kappa/2}, 1\} \\ &+ 2^{26} 3^2 M^2 N^6 d^{16} [\mathfrak{D}_\delta |\max\{\kappa, 1\}|^4 d^{4\kappa} N^{2\kappa} |\max\{T^{-\kappa/2}, 1\}|^4]^2. \end{aligned} \quad (234)$$

Moreover, observe that (219) proves for all  $\delta \in (0, 1]$  that

$$\mathfrak{D}_\delta = 2160 \log_2(\delta^{-1}) + 8136 \leq 8136[\log_2(\delta^{-1}) + 1] \leq 2^{13}[\log_2(\delta^{-1}) + 1]. \quad (235)$$

Combining this with (234) ensures that for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$ ,  $\omega \in \Omega$  it holds that

$$\begin{aligned} \mathcal{P}(\psi_{d, N, M, \delta, \omega}) &\leq 2M^2 \kappa d^\kappa N^{\kappa/2} \max\{T^{-\kappa/2}, 1\} \\ &+ 2^{26} 3^2 M^2 N^{6+4\kappa} d^{16+8\kappa} (\mathfrak{D}_\delta)^2 [\max\{\kappa, 1\}]^8 [\max\{T^{-\kappa/2}, 1\}]^8 \\ &\leq 2^{27} 3^2 M^2 N^{6+4\kappa} d^{16+8\kappa} (\mathfrak{D}_\delta)^2 [\max\{\kappa, 1\}]^8 [\max\{T^{-\kappa/2}, 1\}]^8 \\ &\leq 2^{31} M^2 N^{6+4\kappa} d^{16+8\kappa} (\mathfrak{D}_\delta)^2 [\max\{\kappa, 1\}]^8 [\max\{T^{-\kappa/2}, 1\}]^8 \\ &\leq 2^{57} [\max\{\kappa, 1\}]^8 [\max\{T^{-\kappa/2}, 1\}]^8 M^2 N^{6+4\kappa} [\log_2(\delta^{-1}) + 1]^2 d^{16+8\kappa}. \end{aligned} \quad (236)$$

This and (222) prove that for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$ ,  $\omega \in \Omega$  it holds that

$$\mathcal{P}(\psi_{d, N, M, \delta, \omega}) \leq \mathcal{C}_2 M^2 N^{6+4\kappa} [\log_2(\delta^{-1}) + 1]^2 d^{16+8\kappa}. \quad (237)$$

Next note that (I), (II), and, e.g., Beck et al. [2, Lemma 2.4] show that for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$  it holds that  $[0, T] \times \mathbb{R}^d \times \Omega \ni (t, x, \omega) \mapsto (\mathcal{R}_\tau(\psi_{d, N, M, \delta, \omega}))(t, x) \in \mathbb{R}$  is measurable. The triangle inequality and Fubini's theorem hence establish that for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$

it holds that

$$\begin{aligned}
& \left[ \int_{\Omega} \int_{[0,T] \times \mathbb{R}^d} |u_d(y) - (\mathcal{R}_{\tau}(\psi_{d,N,M,\delta,\omega}))(y)|^p \nu_d(dy) \mathbb{P}(d\omega) \right]^{1/p} \\
& \leq \left( \mathbb{E} \left[ \int_{[0,T] \times \mathbb{R}^d} \left| u_d(t,x) - \frac{1}{M} \left[ \sum_{m=1}^M (\mathcal{R}_{\tau}(\mathfrak{f}_{d,\min\{(T/N)^{1/2},1\}}^0))(Y_t^{N,d,m,x}) \right] \right|^p \nu_d(dt,dx) \right] \right)^{1/p} \\
& + \left[ \int_{[0,T] \times \mathbb{R}^d} \int_{\Omega} |(\mathcal{R}_{\tau}(\psi_{d,N,M,\delta,\omega}))(t,x) \right. \\
& \left. - \frac{1}{M} \left[ \sum_{m=1}^M (\mathcal{R}_{\tau}(\mathfrak{f}_{d,\min\{(T/N)^{1/2},1\}}^0))(Y_t^{N,d,m,x}(\omega)) \right] \right|^p \mathbb{P}(d\omega) \nu_d(dt,dx) \right]^{1/p}.
\end{aligned} \tag{238}$$

Combining this with (227) and (231) ensures that for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$  it holds that

$$\begin{aligned}
& \left[ \int_{\Omega} \int_{[0,T] \times \mathbb{R}^d} |u_d(y) - (\mathcal{R}_{\tau}(\psi_{d,N,M,\delta,\omega}))(y)|^p \nu_d(dy) \mathbb{P}(d\omega) \right]^{1/p} \\
& \leq \mathcal{C}_3 \left[ \frac{d^{\kappa(\kappa+4)+\max\{\eta,\kappa(2\kappa+1)\}}}{N^{1/2}} + \frac{d^{\kappa+\max\{\eta,\kappa^2\}}}{M^{1/2}} \right] [\max\{1, \nu_d([0, T] \times \mathbb{R}^d)\}]^{1/p} \\
& + \mathcal{C}_1 \delta d^{(2\kappa+3)\max\{\eta,\kappa\}+\kappa(\kappa+2)+\kappa+(\kappa+1)/2} \max\{1, [\nu_d([0, T] \times \mathbb{R}^d)]^{1/p}\}.
\end{aligned} \tag{239}$$

Hence, we obtain that for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$  it holds that

$$\begin{aligned}
& \left[ \int_{\Omega} \int_{[0,T] \times \mathbb{R}^d} |u_d(y) - (\mathcal{R}_{\tau}(\psi_{d,N,M,\delta,\omega}))(y)|^p \nu_d(dy) \mathbb{P}(d\omega) \right]^{1/p} \\
& \leq \max\{\mathcal{C}_1, \mathcal{C}_3\} [\max\{1, \nu_d([0, T] \times \mathbb{R}^d)\}]^{1/p} \\
& \cdot \left[ \frac{d^{\kappa(\kappa+4)+\max\{\eta,\kappa(2\kappa+1)\}}}{N^{1/2}} + \frac{d^{\kappa+\max\{\eta,\kappa^2\}}}{M^{1/2}} + \delta d^{(2\kappa+3)\max\{\eta,\kappa\}+\kappa(\kappa+2)+\kappa+(\kappa+1)/2} \right].
\end{aligned} \tag{240}$$

Next note that (I), (II), and, e.g., Beck et al. [2, Lemma 2.4] demonstrate that for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$  it holds that  $[0, T] \times \mathbb{R}^d \times \Omega \ni (t, x, \omega) \mapsto (\mathcal{R}_{\tau}(\psi_{d,N,M,\delta,\omega}))(t, x) \in \mathbb{R}$  is measurable. Combining this with Fubini's theorem, (226), (240), and the fact that  $\forall d \in \mathbb{N}: u_d \in C([0, T] \times \mathbb{R}^d, \mathbb{R})$  proves that

A) it holds for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$  that  $Z_{d,N,M,\delta}$  is a random variable and

B) it holds for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$  that

$$\begin{aligned}
& \mathbb{E}[|Z_{d,N,M,\delta}|] \leq [\max\{\mathcal{C}_1, \mathcal{C}_3\}]^p \max\{1, \nu_d([0, T] \times \mathbb{R}^d)\} \\
& \cdot \left[ \frac{d^{\kappa(\kappa+4)+\max\{\eta,\kappa(2\kappa+1)\}}}{N^{1/2}} + \frac{d^{\kappa+\max\{\eta,\kappa^2\}}}{M^{1/2}} + \delta d^{(2\kappa+3)\max\{\eta,\kappa\}+\kappa(\kappa+2)+\kappa+(\kappa+1)/2} \right]^p.
\end{aligned} \tag{241}$$

This and, e.g., [35, Proposition 2.3] prove that there exist  $\mathfrak{w}_{d,N,M,\delta} \in \Omega$ ,  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$ ,

which satisfy that for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$  it holds that

$$\begin{aligned}
& \int_{[0, T] \times \mathbb{R}^d} |u_d(y) - (\mathcal{R}_\tau(\psi_{d, N, M, \delta, \mathbf{w}_{d, N, M, \delta}}))(y)|^p \nu_d(dy) = Z_{d, N, M, \delta}(\mathbf{w}_{d, N, M, \delta}) \\
& \leq [\max\{\mathcal{C}_1, \mathcal{C}_3\}]^p \max\{1, \nu_d([0, T] \times \mathbb{R}^d)\} \\
& \cdot \left[ \frac{d^{\kappa(\kappa+4) + \max\{\eta, \kappa(2\kappa+1)\}}}{N^{1/2}} + \frac{d^{\kappa + \max\{\eta, \kappa^2\}}}{M^{1/2}} + \delta d^{(2\kappa+3) \max\{\eta, \kappa\} + \kappa(\kappa+2) + \kappa + (\kappa+1)/2} \right]^p \\
& = [\max\{\mathcal{C}_1, \mathcal{C}_3\}]^p \max\{1, \nu_d([0, T] \times \mathbb{R}^d)\} \\
& \cdot \left[ \frac{d^{\kappa(\kappa+4) + \max\{\eta, \kappa(2\kappa+1)\}}}{N^{1/2}} + \frac{d^{\kappa + \max\{\eta, \kappa^2\}}}{M^{1/2}} + \delta d^{(2\kappa+3) \max\{\eta, \kappa\} + \kappa^2 + (7\kappa+1)/2} \right]^p.
\end{aligned} \tag{242}$$

Combining this, (I), and (237) establishes items (i)–(iii). This completes the proof of Proposition 4.12.  $\square$

## 4.6 Cost estimates for DNNs

**Theorem 4.13.** *Let  $T, \kappa, \eta, c \in (0, \infty)$ ,  $p \in [2, \infty)$  satisfy that*

$$c = 18 + 12\kappa + 4 \max\{\eta, \kappa^2\} + 4\eta + [2\kappa(\kappa + 4) + 2 \max\{\eta, \kappa(2\kappa + 1)\} + 2\eta](6 + 4\kappa), \tag{243}$$

let  $A_d = (a_{d, i, j})_{(i, j) \in \{1, 2, \dots, d\}^2} \in \mathbb{R}^{d \times d}$ ,  $d \in \mathbb{N}$ , be symmetric positive semidefinite matrices, let  $\nu_d: \mathcal{B}([0, T] \times \mathbb{R}^d) \rightarrow [0, \infty)$ ,  $d \in \mathbb{N}$ , be finite measures which satisfy for all  $d \in \mathbb{N}$  that

$$\left[ \max\left\{1, \nu_d([0, T] \times \mathbb{R}^d), \int_{[0, T] \times \mathbb{R}^d} \|x\|^{2p \max\{2\kappa, 3\}} \nu_d(dt, dx)\right\} \right]^{1/p} \leq \eta d^n, \tag{244}$$

let  $f_d^m: \mathbb{R}^d \rightarrow \mathbb{R}^{md-m+1}$ ,  $m \in \{0, 1\}$ ,  $d \in \mathbb{N}$ , be functions, let  $(\mathfrak{f}_{d, \varepsilon}^m)_{(m, d, \varepsilon) \in \{0, 1\} \times \mathbb{N} \times (0, 1]} \subseteq \mathbf{N}$ , assume for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$ ,  $m \in \{0, 1\}$ ,  $x, y \in \mathbb{R}^d$  that

$$\mathcal{R}_\tau(\mathfrak{f}_{d, \varepsilon}^0) \in C(\mathbb{R}^d, \mathbb{R}), \quad \mathcal{R}_\tau(\mathfrak{f}_{d, \varepsilon}^1) \in C(\mathbb{R}^d, \mathbb{R}^d), \quad \mathcal{P}(\mathfrak{f}_{d, \varepsilon}^m) \leq \kappa d^\kappa \varepsilon^{-\kappa}, \tag{245}$$

$$\|f_d^1(x) - f_d^1(y)\| \leq \kappa \|x - y\|, \quad \|(\mathcal{R}_\tau(\mathfrak{f}_{d, \varepsilon}^1))(x)\| \leq \kappa(d^\kappa + \|x\|), \tag{246}$$

$$\|(\mathcal{R}_\tau(\mathfrak{f}_{d, \varepsilon}^0))(x) - (\mathcal{R}_\tau(\mathfrak{f}_{d, \varepsilon}^0))(y)\| \leq \kappa d^\kappa (1 + \|x\|^\kappa + \|y\|^\kappa) \|x - y\|, \tag{247}$$

$$\|f_d^m(x) - (\mathcal{R}_\tau(\mathfrak{f}_{d, \varepsilon}^m))(x)\| \leq \varepsilon \kappa d^\kappa (1 + \|x\|^\kappa), \tag{248}$$

$$\text{and } |f_d^0(x)| + \text{Trace}(A_d) \leq \kappa d^\kappa (1 + \|x\|^\kappa), \tag{249}$$

and for every  $d \in \mathbb{N}$  let  $u_d \in \{v \in C([0, T] \times \mathbb{R}^d, \mathbb{R}): \inf_{q \in (0, \infty)} \sup_{(t, y) \in [0, T] \times \mathbb{R}^d} \frac{|v(t, y)|}{1 + \|y\|^q} < \infty\}$  be a viscosity solution of

$$\left(\frac{\partial}{\partial t} u_d\right)(t, x) = \left(\frac{\partial}{\partial x} u_d\right)(t, x) f_d^1(x) + \sum_{i, j=1}^d a_{d, i, j} \left(\frac{\partial^2}{\partial x_i \partial x_j} u_d\right)(t, x) \tag{250}$$

with  $u_d(0, x) = f_d^0(x)$  for  $(t, x) \in (0, T) \times \mathbb{R}^d$  (cf. Definitions 2.1, 4.1, 4.3, and 4.4). Then there exist  $\mathfrak{C} \in \mathbb{R}$  and  $(\mathbf{u}_{d, \varepsilon})_{d \in \mathbb{N}, \varepsilon \in (0, 1]} \subseteq \mathbf{N}$  such that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  it holds that  $\mathcal{R}_\tau(\mathbf{u}_{d, \varepsilon}) \in C(\mathbb{R}^{d+1}, \mathbb{R})$ ,  $\mathcal{P}(\mathbf{u}_{d, \varepsilon}) \leq \mathfrak{C} \varepsilon^{-(18+8\kappa)} d^c$ , and

$$\left[ \int_{[0, T] \times \mathbb{R}^d} |u_d(y) - (\mathcal{R}_\tau(\mathbf{u}_{d, \varepsilon}))(y)|^p \nu_d(dy) \right]^{1/p} \leq \varepsilon. \tag{251}$$

*Proof of Theorem 4.13.* Note that (244) implies that

$$\left[ \max \left\{ 1, \nu_1([0, T] \times \mathbb{R}^d), \int_{[0, T] \times \mathbb{R}^d} \|x\|^{2p \max\{2\kappa, 3\}} \nu_1(dt, dx) \right\} \right]^{1/p} \leq \eta. \quad (252)$$

This proves that  $\eta \in [1, \infty)$ . Proposition 4.12 hence ensures that there exist  $\mathcal{C}_1 \in (0, \infty)$  and  $(\Phi_{d, N, M, \delta})_{(d, N, M, \delta) \in \mathbb{N}^3 \times (0, 1]} \subseteq \mathbf{N}$  which satisfy that

(I) it holds for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$  that  $\mathcal{R}_\tau(\Phi_{d, N, M, \delta}) \in C(\mathbb{R}^{d+1}, \mathbb{R})$ ,

(II) it holds for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$  that

$$\begin{aligned} & \left[ \int_{[0, T] \times \mathbb{R}^d} |u_d(y) - (\mathcal{R}_\tau(\Phi_{d, N, M, \delta}))(y)|^p \nu_d(dy) \right]^{1/p} \\ & \leq \mathcal{C}_1 \left[ \max\{1, \nu_d([0, T] \times \mathbb{R}^d)\} \right]^{1/p} \\ & \cdot \left[ \frac{d^{\kappa(\kappa+4) + \max\{\eta, \kappa(2\kappa+1)\}}}{N^{1/2}} + \frac{d^{\kappa + \max\{\eta, \kappa^2\}}}{M^{1/2}} + \delta d^{(2\kappa+3) \max\{\eta, \kappa\} + \kappa^2 + (7\kappa+1)/2} \right], \end{aligned} \quad (253)$$

and

(III) it holds for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$  that

$$\mathcal{P}(\Phi_{d, N, M, \delta}) \leq \mathcal{C}_1 M^2 N^{6+4\kappa} [\log_2(\delta^{-1}) + 1]^2 d^{16+8\kappa}. \quad (254)$$

Next let  $\mathcal{C}_2, \mathfrak{C} \in (0, \infty)$  satisfy that

$$\mathcal{C}_2 = \max\{0, \log_2(3\mathcal{C}_1\eta)\} + \frac{1}{\ln(2)} + \frac{1}{\ln(2)} \left[ 2(\kappa + 2) \max\{\eta, \kappa\} + \kappa^2 + (7\kappa+1)/2 \right] \quad (255)$$

and

$$\mathfrak{C} = \mathcal{C}_1 2^{8+4\kappa} (3\mathcal{C}_1\eta)^{16+8\kappa} [\mathcal{C}_2 + 1]^2, \quad (256)$$

let  $\mathcal{D}_{d, \varepsilon} \in (0, 1]$ ,  $\varepsilon \in (0, 1]$ ,  $d \in \mathbb{N}$ , satisfy for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  that

$$\mathcal{D}_{d, \varepsilon} = \min\{1, (3\mathcal{C}_1\eta)^{-1} \varepsilon d^{-2(\kappa+2) \max\{\eta, \kappa\} - \kappa^2 - (7\kappa+1)/2}\}, \quad (257)$$

let  $\mathfrak{M}_{d, \varepsilon} \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$ ,  $d \in \mathbb{N}$ , satisfy for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  that

$$\mathfrak{M}_{d, \varepsilon} = \min[\mathbb{N} \cap [(3\mathcal{C}_1\eta)^2 \varepsilon^{-2} d^{2\kappa+2 \max\{\eta, \kappa^2\} + 2\eta}, \infty)], \quad (258)$$

let  $\mathfrak{N}_{d, \varepsilon} \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$ ,  $d \in \mathbb{N}$ , satisfy for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  that

$$\mathfrak{N}_{d, \varepsilon} = \min[\mathbb{N} \cap [(3\mathcal{C}_1\eta)^2 \varepsilon^{-2} d^{2\kappa(\kappa+4) + 2 \max\{\eta, \kappa(2\kappa+1)\} + 2\eta}, \infty)], \quad (259)$$

and let  $\mathbf{u}_{d, \varepsilon} \in \mathbf{N}$ ,  $\varepsilon \in (0, 1]$ ,  $d \in \mathbb{N}$ , satisfy for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  that

$$\mathbf{u}_{d, \varepsilon} = \Phi_{d, \mathfrak{M}_{d, \varepsilon}, \mathfrak{N}_{d, \varepsilon}, \mathcal{D}_{d, \varepsilon}}. \quad (260)$$

Observe that (244) and (253) ensure that for all  $d, N, M \in \mathbb{N}$ ,  $\delta \in (0, 1]$  it holds that

$$\begin{aligned} & \left[ \int_{[0, T] \times \mathbb{R}^d} |u_d(y) - (\mathcal{R}_\tau(\Phi_{d, N, M, \delta}))(y)|^p \nu_d(dy) \right]^{1/p} \\ & \leq \mathcal{C}_1 \eta \left[ \frac{d^{\kappa(\kappa+4) + \max\{\eta, \kappa(2\kappa+1)\} + \eta}}{N^{1/2}} + \frac{d^{\kappa + \max\{\eta, \kappa^2\} + \eta}}{M^{1/2}} + \delta d^{2(\kappa+2) \max\{\eta, \kappa\} + \kappa^2 + (7\kappa+1)/2} \right]. \end{aligned} \quad (261)$$

Combining this, (260), (257), (258), and (259) implies that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  it holds that

$$\begin{aligned}
& \left[ \int_{[0,T] \times \mathbb{R}^d} |u_d(y) - (\mathcal{R}_\tau(\mathbf{u}_{d,\varepsilon}))(y)|^p \nu_d(dy) \right]^{1/p} \\
&= \left[ \int_{[0,T] \times \mathbb{R}^d} |u_d(y) - (\mathcal{R}_\tau(\Phi_{d,\mathfrak{N}_{d,\varepsilon},\mathfrak{M}_{d,\varepsilon},\mathcal{D}_{d,\varepsilon}}))(y)|^p \nu_d(dy) \right]^{1/p} \\
&\leq \mathcal{C}_1 \eta \left[ \frac{d^{\kappa(\kappa+4)+\max\{\eta,\kappa(2\kappa+1)\}+\eta}}{|\mathfrak{N}_{d,\varepsilon}|^{1/2}} + \frac{d^{\kappa+\max\{\eta,\kappa^2\}+\eta}}{|\mathfrak{M}_{d,\varepsilon}|^{1/2}} + \mathcal{D}_{d,\varepsilon} d^{2(\kappa+2)\max\{\eta,\kappa\}+\kappa^2+(7\kappa+1)/2} \right] \\
&\leq \mathcal{C}_1 \eta \left[ \frac{d^{\kappa(\kappa+4)+\max\{\eta,\kappa(2\kappa+1)\}+\eta}}{|(3\mathcal{C}_1\eta)^2 \varepsilon^{-2} d^{2\kappa(\kappa+4)+2\max\{\eta,\kappa(2\kappa+1)\}+2\eta}|^{1/2}} + \frac{d^{\kappa+\max\{\eta,\kappa^2\}+\eta}}{|(3\mathcal{C}_1\eta)^2 \varepsilon^{-2} d^{2\kappa+2\max\{\eta,\kappa^2\}+2\eta}|^{1/2}} \right] \\
&+ (3\mathcal{C}_1\eta)^{-1} \varepsilon d^{-2(\kappa+2)\max\{\eta,\kappa\}-\kappa^2-(7\kappa+1)/2} d^{2(\kappa+2)\max\{\eta,\kappa\}+\kappa^2+(7\kappa+1)/2} \\
&= \mathcal{C}_1 \eta \left( \frac{1}{3\mathcal{C}_1\eta\varepsilon^{-1}} + \frac{1}{3\mathcal{C}_1\eta\varepsilon^{-1}} + \frac{\varepsilon}{3\mathcal{C}_1\eta} \right) = \varepsilon.
\end{aligned} \tag{262}$$

In addition, observe that (255), (257), and the fact that  $\forall x \in [1, \infty): \log_2(x) = \ln(x)/\ln(2) \leq x/\ln(2)$  demonstrate that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  it holds that

$$\begin{aligned}
& \log_2((\mathcal{D}_{d,\varepsilon})^{-1}) = \max\{0, \log_2(3\mathcal{C}_1\eta \varepsilon^{-1} d^{2(\kappa+2)\max\{\eta,\kappa\}+\kappa^2+(7\kappa+1)/2})\} \\
&\leq \max\{0, \log_2(3\mathcal{C}_1\eta) + \log_2(\varepsilon^{-1}) + [2(\kappa+2)\max\{\eta,\kappa\} + \kappa^2 + (7\kappa+1)/2] \log_2(d)\} \\
&\leq \max\{0, \log_2(3\mathcal{C}_1\eta)\} + \frac{1}{\ln(2)} \varepsilon^{-1} + \frac{1}{\ln(2)} [2(\kappa+2)\max\{\eta,\kappa\} + \kappa^2 + (7\kappa+1)/2] d \\
&\leq \varepsilon^{-1} d (\max\{0, \log_2(3\mathcal{C}_1\eta)\} + \frac{1}{\ln(2)} + \frac{1}{\ln(2)} [2(\kappa+2)\max\{\eta,\kappa\} + \kappa^2 + (7\kappa+1)/2]) \\
&= \varepsilon^{-1} d \mathcal{C}_2.
\end{aligned} \tag{263}$$

Combining this with (III), (258), (259), and (260) proves that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  it holds that

$$\begin{aligned}
\mathcal{P}(\mathbf{u}_{d,\varepsilon}) &= \mathcal{P}(\Phi_{d,\mathfrak{N}_{d,\varepsilon},\mathfrak{M}_{d,\varepsilon},\mathcal{D}_{d,\varepsilon}}) \leq \mathcal{C}_1 (\mathfrak{M}_{d,\varepsilon})^2 d^{16+8\kappa} (\mathfrak{N}_{d,\varepsilon})^{6+4\kappa} [\log_2((\mathcal{D}_{d,\varepsilon})^{-1}) + 1]^2 \\
&\leq \mathcal{C}_1 [(3\mathcal{C}_1\eta)^2 \varepsilon^{-2} d^{2\kappa+2\max\{\eta,\kappa^2\}+2\eta} + 1]^2 d^{16+8\kappa} [\varepsilon^{-1} d \mathcal{C}_2 + 1]^2 \\
&\cdot [(3\mathcal{C}_1\eta)^2 \varepsilon^{-2} d^{2\kappa(\kappa+4)+2\max\{\eta,\kappa(2\kappa+1)\}+2\eta} + 1]^{6+4\kappa} \\
&\leq \mathcal{C}_1 2^{8+4\kappa} (3\mathcal{C}_1\eta)^4 \varepsilon^{-4} d^{4\kappa+4\max\{\eta,\kappa^2\}+4\eta} d^{16+8\kappa} \varepsilon^{-2} d^2 [\mathcal{C}_2 + 1]^2 \\
&\cdot (3\mathcal{C}_1\eta)^{12+8\kappa} \varepsilon^{-(12+8\kappa)} d^{[2\kappa(\kappa+4)+2\max\{\eta,\kappa(2\kappa+1)\}+2\eta](6+4\kappa)}.
\end{aligned} \tag{264}$$

This, (243), and (256) ensure that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  it holds that

$$\begin{aligned}
\mathcal{P}(\mathbf{u}_{d,\varepsilon}) &\leq \mathcal{C}_1 2^{8+4\kappa} (3\mathcal{C}_1\eta)^{16+8\kappa} \varepsilon^{-(18+8\kappa)} [\mathcal{C}_2 + 1]^2 \\
&\cdot d^{2+4\kappa+4\max\{\eta,\kappa^2\}+4\eta+16+8\kappa+[2\kappa(\kappa+4)+2\max\{\eta,\kappa(2\kappa+1)\}+2\eta](6+4\kappa)} \\
&= \mathfrak{C}_\varepsilon^{-(18+8\kappa)} d^{18+12\kappa+4\max\{\eta,\kappa^2\}+4\eta+[2\kappa(\kappa+4)+2\max\{\eta,\kappa(2\kappa+1)\}+2\eta](6+4\kappa)} \\
&= \mathfrak{C}_\varepsilon^{-(18+8\kappa)} d^c.
\end{aligned} \tag{265}$$

Combining this, (I), and (262) establishes (251). This completes the proof of Theorem 4.13.  $\square$

**Corollary 4.14.** *Let  $T, \kappa, \eta \in (0, \infty)$ ,  $p \in [2, \infty)$ , let  $A_d = (a_{d,i,j})_{(i,j) \in \{1,2,\dots,d\}^2} \in \mathbb{R}^{d \times d}$ ,  $d \in \mathbb{N}$ , be symmetric positive semidefinite matrices, let  $\nu_d: \mathcal{B}([0, T] \times \mathbb{R}^d) \rightarrow [0, \infty)$ ,  $d \in \mathbb{N}$ , be finite measures which satisfy for all  $d \in \mathbb{N}$  that*

$$\left[ \max\left\{1, \nu_d([0, T] \times \mathbb{R}^d), \int_{[0,T] \times \mathbb{R}^d} \|x\|^{2p \max\{6\kappa, 2\kappa+2, 3\}} \nu_d(dt, dx)\right\} \right]^{1/p} \leq \eta d^n, \tag{266}$$

let  $f_d^m: \mathbb{R}^d \rightarrow \mathbb{R}^{md-m+1}$ ,  $m \in \{0, 1\}$ ,  $d \in \mathbb{N}$ , be functions, let  $(f_{d,\varepsilon}^m)_{(m,d,\varepsilon) \in \{0,1\} \times \mathbb{N} \times (0,1]} \subseteq \mathbf{N}$ , assume for all  $m \in \{0, 1\}$ ,  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$ ,  $i \in \{1, 2, \dots, d\}$ ,  $x, y \in \mathbb{R}^d$  that

$$\mathcal{R}_\tau(f_{d,\varepsilon}^0) \in C(\mathbb{R}^d, \mathbb{R}), \quad \mathcal{R}_\tau(f_{d,\varepsilon}^1) \in C(\mathbb{R}^d, \mathbb{R}^d), \quad \mathcal{P}(f_{d,\varepsilon}^m) \leq \kappa d^\kappa \varepsilon^{-\kappa}, \quad (267)$$

$$\|f_d^1(x) - f_d^1(y)\| \leq \kappa \|x - y\|, \quad \|(\mathcal{R}_\tau(f_{d,\varepsilon}^1))(x)\| \leq \kappa(d^\kappa + \|x\|), \quad (268)$$

$$\varepsilon |f_d^0(x)| + \varepsilon |a_{d,i,i}| + \|f_d^m(x) - (\mathcal{R}_\tau(f_{d,\varepsilon}^m))(x)\| \leq \varepsilon \kappa d^\kappa (1 + \|x\|^\kappa), \quad (269)$$

$$\text{and} \quad |(\mathcal{R}_\tau(f_{d,\varepsilon}^0))(x) - (\mathcal{R}_\tau(f_{d,\varepsilon}^0))(y)| \leq \kappa d^\kappa (1 + \|x\|^\kappa + \|y\|^\kappa) \|x - y\|, \quad (270)$$

and for every  $d \in \mathbb{N}$  let  $u_d \in \{v \in C([0, T] \times \mathbb{R}^d, \mathbb{R}) : \inf_{q \in (0, \infty)} \sup_{(t,y) \in [0,T] \times \mathbb{R}^d} \frac{|v(t,y)|}{1+|y|^q} < \infty\}$  be a viscosity solution of

$$\left(\frac{\partial}{\partial t} u_d\right)(t, x) = \left(\frac{\partial}{\partial x} u_d\right)(t, x) f_d^1(x) + \sum_{i,j=1}^d a_{d,i,j} \left(\frac{\partial^2}{\partial x_i \partial x_j} u_d\right)(t, x) \quad (271)$$

with  $u_d(0, x) = f_d^0(x)$  for  $(t, x) \in (0, T) \times \mathbb{R}^d$  (cf. Definitions 2.1, 4.1, 4.3, and 4.4). Then there exist  $\mathfrak{C} \in \mathbb{R}$  and  $(\mathbf{u}_{d,\varepsilon})_{d \in \mathbb{N}, \varepsilon \in (0,1]} \subseteq \mathbf{N}$  such that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  it holds that  $\mathcal{R}_\tau(\mathbf{u}_{d,\varepsilon}) \in C(\mathbb{R}^{d+1}, \mathbb{R})$ ,  $\mathcal{P}(\mathbf{u}_{d,\varepsilon}) \leq \mathfrak{C} \varepsilon^{-\mathfrak{C}} d^\mathfrak{C}$ , and

$$\left[ \int_{[0,T] \times \mathbb{R}^d} |u_d(y) - (\mathcal{R}_\tau(\mathbf{u}_{d,\varepsilon}))(y)|^p \nu_d(dy) \right]^{1/p} \leq \varepsilon. \quad (272)$$

*Proof of Corollary 4.14.* Throughout this proof let  $\iota = \max\{3\kappa, \kappa + 1\}$ . Observe that (268) and the fact that  $\iota \geq \kappa$  prove that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$ ,  $x, y \in \mathbb{R}^d$  it holds that

$$\|f_d^1(x) - f_d^1(y)\| \leq \iota \|x - y\| \quad \text{and} \quad \|(\mathcal{R}_\tau(f_{d,\varepsilon}^1))(x)\| \leq \iota(d^\iota + \|x\|). \quad (273)$$

Next note that (270) and the fact that  $\iota \geq 3\kappa$  ensure that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$ ,  $x, y \in \mathbb{R}^d$  it holds that

$$\begin{aligned} |(\mathcal{R}_\tau(f_{d,\varepsilon}^0))(x) - (\mathcal{R}_\tau(f_{d,\varepsilon}^0))(y)| &\leq \kappa d^\iota (3 + \|x\|^\iota + \|y\|^\iota) \|x - y\| \\ &\leq \iota d^\iota (1 + \|x\|^\iota + \|y\|^\iota) \|x - y\|. \end{aligned} \quad (274)$$

In addition, observe that (269) and the fact that  $\iota \geq 2\kappa$  imply that for all  $m \in \{0, 1\}$ ,  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$ ,  $x \in \mathbb{R}^d$  it holds that

$$\begin{aligned} \|f_d^m(x) - (\mathcal{R}_\tau(f_{d,\varepsilon}^m))(x)\| &\leq \varepsilon \kappa d^\kappa (1 + \|x\|^\kappa) \\ &\leq \varepsilon \kappa d^\iota (2 + \|x\|^\iota) \leq \varepsilon \iota d^\iota (1 + \|x\|^\iota). \end{aligned} \quad (275)$$

Moreover, note that (269) and the fact that  $\iota \geq \max\{2\kappa, \kappa + 1\}$  demonstrate that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$ ,  $x \in \mathbb{R}^d$  it holds that

$$\begin{aligned} |f_d^0(x)| + \text{Trace}(A_d) &\leq \kappa d^{\kappa+1} (1 + \|x\|^\kappa) \leq \kappa d^{\kappa+1} (2 + \|x\|^\iota) \\ &\leq 2\kappa d^{\kappa+1} (1 + \|x\|^\iota) \leq \iota d^\iota (1 + \|x\|^\iota). \end{aligned} \quad (276)$$

Furthermore, observe that (267) implies that for all  $m \in \{0, 1\}$ ,  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  it holds that  $\mathcal{P}(f_{d,\varepsilon}^m) \leq \iota d^\iota \varepsilon^{-\iota}$ . Combining this, (266), (273), (274), (275), (276), and Theorem 4.13 establishes that there exist  $\mathfrak{C} \in \mathbb{R}$  and  $(\mathbf{u}_{d,\varepsilon})_{d \in \mathbb{N}, \varepsilon \in (0,1]} \subseteq \mathbf{N}$  which satisfy that

(I) it holds for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  that  $\mathcal{R}_\tau(\mathbf{u}_{d,\varepsilon}) \in C(\mathbb{R}^{d+1}, \mathbb{R})$ ,

(II) it holds for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  that

$$\left[ \int_{[0,T] \times \mathbb{R}^d} |u_d(y) - (\mathcal{R}_\tau(\mathbf{u}_{d,\varepsilon}))(y)|^p \nu_d(dy) \right]^{1/p} \leq \varepsilon, \quad (277)$$

and

(III) it holds for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  that

$$\mathcal{P}(\mathbf{u}_{d,\varepsilon}) \leq C\varepsilon^{-(18+8\iota)} d^{18+12\iota+4\max\{\eta,\iota^2\}+4\eta+[2\iota(\iota+4)+2\max\{\eta,\iota(2\iota+1)\]+2\eta](6+4\iota)}. \quad (278)$$

This proves that for all  $\mathfrak{C} \in [\max\{\mathcal{C}, 18 + 12\iota + 4\max\{\eta, \iota^2\} + 4\eta + [2\iota(\iota + 4) + 2\max\{\eta, \iota(2\iota + 1)\} + 2\eta](6 + 4\iota)\}, \infty)$ ,  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  it holds that  $\mathcal{P}(\mathbf{u}_{d,\varepsilon}) \leq \mathfrak{C}\varepsilon^{-\mathfrak{C}}d^\mathfrak{C}$ . Combining this, (I), and (II) establishes (272). This completes the proof of Corollary 4.14.  $\square$

**Corollary 4.15.** *Let  $T, \kappa \in (0, \infty)$ ,  $\alpha \in \mathbb{R}$ ,  $\beta \in (\alpha, \infty)$ , let  $A_d = (a_{d,i,j})_{(i,j) \in \{1,2,\dots,d\}^2} \in \mathbb{R}^{d \times d}$ ,  $d \in \mathbb{N}$ , be symmetric positive semidefinite matrices, let  $f_d^m: \mathbb{R}^d \rightarrow \mathbb{R}^{md-m+1}$ ,  $m \in \{0, 1\}$ ,  $d \in \mathbb{N}$ , be functions, let  $(\mathfrak{f}_{d,\varepsilon}^m)_{(m,d,\varepsilon) \in \{0,1\} \times \mathbb{N} \times (0,1]} \subseteq \mathbf{N}$ , assume for all  $m \in \{0, 1\}$ ,  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$ ,  $i \in \{1, 2, \dots, d\}$ ,  $x, y \in \mathbb{R}^d$  that*

$$\mathcal{R}_\tau(\mathfrak{f}_{d,\varepsilon}^0) \in C(\mathbb{R}^d, \mathbb{R}), \quad \mathcal{R}_\tau(\mathfrak{f}_{d,\varepsilon}^1) \in C(\mathbb{R}^d, \mathbb{R}^d), \quad \mathcal{P}(\mathfrak{f}_{d,\varepsilon}^m) \leq \kappa d^\kappa \varepsilon^{-\kappa}, \quad (279)$$

$$\|f_d^1(x) - f_d^1(y)\| \leq \kappa \|x - y\|, \quad \|(\mathcal{R}_\tau(\mathfrak{f}_{d,\varepsilon}^1))(x)\| \leq \kappa(d^\kappa + \|x\|), \quad (280)$$

$$\varepsilon |f_d^0(x)| + \varepsilon |a_{d,i,i}| + \|f_d^m(x) - (\mathcal{R}_\tau(\mathfrak{f}_{d,\varepsilon}^m))(x)\| \leq \varepsilon \kappa d^\kappa (1 + \|x\|^\kappa), \quad (281)$$

$$\text{and} \quad |(\mathcal{R}_\tau(\mathfrak{f}_{d,\varepsilon}^0))(x) - (\mathcal{R}_\tau(\mathfrak{f}_{d,\varepsilon}^0))(y)| \leq \kappa d^\kappa (1 + \|x\|^\kappa + \|y\|^\kappa) \|x - y\|, \quad (282)$$

and for every  $d \in \mathbb{N}$  let  $u_d \in \{v \in C([0, T] \times \mathbb{R}^d, \mathbb{R}) : \inf_{q \in (0, \infty)} \sup_{(t,y) \in [0,T] \times \mathbb{R}^d} \frac{|v(t,y)|}{1+\|y\|^q} < \infty\}$  be a viscosity solution of

$$\left(\frac{\partial}{\partial t} u_d\right)(t, x) = \left(\frac{\partial}{\partial x} u_d\right)(t, x) f_d^1(x) + \sum_{i,j=1}^d a_{d,i,j} \left(\frac{\partial^2}{\partial x_i \partial x_j} u_d\right)(t, x) \quad (283)$$

with  $u_d(0, x) = f_d^0(x)$  for  $(t, x) \in (0, T) \times \mathbb{R}^d$  (cf. Definitions 2.1, 4.1, 4.3, and 4.4). Then for every  $p \in (0, \infty)$  there exist  $c \in \mathbb{R}$  and  $(\mathbf{u}_{d,\varepsilon})_{d \in \mathbb{N}, \varepsilon \in (0,1]} \subseteq \mathbf{N}$  such that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  it holds that  $\mathcal{R}_\tau(\mathbf{u}_{d,\varepsilon}) \in C(\mathbb{R}^{d+1}, \mathbb{R})$ ,  $\mathcal{P}(\mathbf{u}_{d,\varepsilon}) \leq c\varepsilon^{-c}d^c$ , and

$$\left[ \int_{[0,T] \times [\alpha, \beta]^d} \frac{|u_d(y) - (\mathcal{R}_\tau(\mathbf{u}_{d,\varepsilon}))(y)|^p}{|\beta - \alpha|^d} dy \right]^{1/p} \leq \varepsilon. \quad (284)$$

*Proof of Corollary 4.15.* Throughout this proof let  $p, q, \eta \in (0, \infty)$  satisfy that  $q = \max\{p, 2\}$  and  $\eta = \max\{6\kappa, 2\kappa+2, 3\} + [\max\{1, T\}]^{1/q} \max\{1, |\alpha|^{2\max\{6\kappa, 2\kappa+2, 3\}}, |\beta|^{2\max\{6\kappa, 2\kappa+2, 3\}}\}$ , for every  $d \in \mathbb{N}$  let  $\mu_d: \mathcal{B}([0, T] \times \mathbb{R}^d) \rightarrow [0, \infty]$  be the Lebesgue-Borel measure on  $[0, T] \times \mathbb{R}^d$ , for every  $d \in \mathbb{N}$  let  $\nu_d: \mathcal{B}([0, T] \times \mathbb{R}^d) \rightarrow [0, \infty]$  be the measure which satisfies for all  $d \in \mathbb{N}$ ,  $B_1 \in \mathcal{B}([0, T])$ ,  $B_2 \in \mathcal{B}(\mathbb{R}^d)$  that

$$\nu_d(B_1 \times B_2) = \frac{\mu_d(B_1 \times ([\alpha, \beta]^d \cap B_2))}{|\beta - \alpha|^d}, \quad (285)$$

let  $\delta: (0, 1] \rightarrow (0, 1]$  satisfy for all  $\varepsilon \in (0, 1]$  that  $\delta(\varepsilon) = \varepsilon[\max\{T, 1\}]^{1/q-1/p}$ , and let  $r: (0, \infty) \rightarrow (0, \infty)$  satisfy for all  $z \in (0, \infty)$  that  $r(z) = z[\max\{T, 1\}]^{z(1/p-1/q)}$ . Observe that (285), Fubini's theorem, and, e.g., Grohs et al. [22, Lemma 3.15] prove that for all  $d \in \mathbb{N}$  it holds that

$$\begin{aligned} & \int_{[0,T] \times \mathbb{R}^d} \|x\|^{2q \max\{6\kappa, 2\kappa+2, 3\}} \nu_d(dt, dx) = \int_{[0,T] \times [\alpha, \beta]^d} \frac{\|x\|^{2q \max\{6\kappa, 2\kappa+2, 3\}}}{|\beta - \alpha|^d} \mu_d(dt, dx) \\ & = T \int_{[\alpha, \beta]^d} \frac{\|x\|^{2q \max\{6\kappa, 2\kappa+2, 3\}}}{|\beta - \alpha|^d} dx \\ & \leq T d^{q \max\{6\kappa, 2\kappa+2, 3\}} \max\{|\alpha|^{2q \max\{6\kappa, 2\kappa+2, 3\}}, |\beta|^{2q \max\{6\kappa, 2\kappa+2, 3\}}\}. \end{aligned} \quad (286)$$

Therefore, we obtain for all  $d \in \mathbb{N}$  that

$$\begin{aligned} & \left[ \max \left\{ 1, \nu_d([0, T] \times \mathbb{R}^d), \int_{[0, T] \times \mathbb{R}^d} \|x\|^{2q \max\{6\kappa, 2\kappa+2, 3\}} \nu_d(dt, dx) \right\} \right]^{1/q} \\ & \leq \left[ \max \{1, T, T d^{q \max\{6\kappa, 2\kappa+2, 3\}} \max\{|\alpha|^{2q \max\{6\kappa, 2\kappa+2, 3\}}, |\beta|^{2q \max\{6\kappa, 2\kappa+2, 3\}}\} \} \right]^{1/q} \\ & \leq d^{\max\{6\kappa, 2\kappa+2, 3\}} [\max\{1, T\}]^{1/q} \max\{1, |\alpha|^{2 \max\{6\kappa, 2\kappa+2, 3\}}, |\beta|^{2 \max\{6\kappa, 2\kappa+2, 3\}}\} \leq \eta d^m. \end{aligned} \quad (287)$$

Corollary 4.14 hence ensures that there exist  $\mathfrak{C} \in (0, \infty)$  and  $(\Phi_{d,\varepsilon})_{d \in \mathbb{N}, \varepsilon \in (0,1]} \subseteq \mathbf{N}$  which satisfy for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  that  $\mathcal{R}_\tau(\Phi_{d,\varepsilon}) \in C(\mathbb{R}^{d+1}, \mathbb{R})$ ,  $\mathcal{P}(\Phi_{d,\varepsilon}) \leq \mathfrak{C} \varepsilon^{-\mathfrak{C}} d^{\mathfrak{C}}$ , and

$$\left[ \int_{[0, T] \times \mathbb{R}^d} |u_d(y) - (\mathcal{R}_\tau(\Phi_{d,\varepsilon}))(y)|^q \nu_d(dy) \right]^{1/q} \leq \varepsilon. \quad (288)$$

Combining this with (285) and Hölder's inequality proves that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  it holds that

$$\begin{aligned} & \left[ \int_{[0, T] \times [\alpha, \beta]^d} \frac{|u_d(y) - (\mathcal{R}_\tau(\Phi_{d,\delta(\varepsilon)}))(y)|^p}{|\beta - \alpha|^d} dy \right]^{1/p} \\ & \leq \left[ \int_{[0, T] \times [\alpha, \beta]^d} \frac{|u_d(y) - (\mathcal{R}_\tau(\Phi_{d,\delta(\varepsilon)}))(y)|^q}{|\beta - \alpha|^d} dy \right]^{1/q} T^{1/p-1/q} \\ & = \left[ \int_{[0, T] \times \mathbb{R}^d} |u_d(y) - (\mathcal{R}_\tau(\Phi_{d,\delta(\varepsilon)}))(y)|^q \nu_d(dy) \right]^{1/q} T^{1/p-1/q} \\ & \leq \delta(\varepsilon) [\max\{T, 1\}]^{1/p-1/q} = \varepsilon [\max\{T, 1\}]^{1/q-1/p} [\max\{T, 1\}]^{1/p-1/q} = \varepsilon. \end{aligned} \quad (289)$$

In addition, observe that for all  $z \in (0, \infty)$  it holds that

$$z \leq z [\max\{T, 1\}]^{z(1/p-1/q)} = r(z). \quad (290)$$

The fact that  $\forall d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$ :  $\mathcal{R}_\tau(\Phi_{d,\varepsilon}) \in C(\mathbb{R}^{d+1}, \mathbb{R})$  and the fact that  $\forall d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$ :  $\mathcal{P}(\Phi_{d,\varepsilon}) \leq \mathfrak{C} \varepsilon^{-\mathfrak{C}} d^{\mathfrak{C}}$  hence show that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  it holds that  $\mathcal{R}_\tau(\Phi_{d,\delta(\varepsilon)}) \in C(\mathbb{R}^{d+1}, \mathbb{R})$  and

$$\begin{aligned} \mathcal{P}(\Phi_{d,\delta(\varepsilon)}) & \leq \mathfrak{C} [\delta(\varepsilon)]^{-\mathfrak{C}} d^{\mathfrak{C}} = \mathfrak{C} [\max\{T, 1\}]^{-\mathfrak{C}(1/q-1/p)} \varepsilon^{-\mathfrak{C}} d^{\mathfrak{C}} \\ & = \mathfrak{C} [\max\{T, 1\}]^{\mathfrak{C}(1/p-1/q)} \varepsilon^{-\mathfrak{C}} d^{\mathfrak{C}} = r(\mathfrak{C}) \varepsilon^{-\mathfrak{C}} d^{\mathfrak{C}} \leq r(\mathfrak{C}) \varepsilon^{-r(\mathfrak{C})} d^{r(\mathfrak{C})}. \end{aligned} \quad (291)$$

This and (289) establish that there exist  $c \in \mathbb{R}$  and  $(\mathbf{u}_{d,\varepsilon})_{d \in \mathbb{N}, \varepsilon \in (0,1]} \subseteq \mathbf{N}$  such that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  it holds that  $\mathcal{R}_\tau(\mathbf{u}_{d,\varepsilon}) \in C(\mathbb{R}^{d+1}, \mathbb{R})$ ,  $\mathcal{P}(\mathbf{u}_{d,\varepsilon}) \leq c \varepsilon^{-c} d^c$ , and

$$\left[ \int_{[0, T] \times [\alpha, \beta]^d} \frac{|u_d(y) - (\mathcal{R}_\tau(\mathbf{u}_{d,\varepsilon}))(y)|^p}{|\beta - \alpha|^d} dy \right]^{1/p} \leq \varepsilon. \quad (292)$$

This completes the proof of Corollary 4.15.  $\square$

**Corollary 4.16.** *Let  $f_d^m: \mathbb{R}^d \rightarrow \mathbb{R}^{md-m+1}$ ,  $m \in \{0, 1\}$ ,  $d \in \mathbb{N}$ , be functions, let  $T, \kappa, p \in (0, \infty)$ ,  $a \in \mathbb{R}$ ,  $b \in (a, \infty)$ ,  $(f_{d,\varepsilon}^m)_{(m,d,\varepsilon) \in \{0,1\} \times \mathbb{N} \times (0,1]} \subseteq \mathbf{N}$ , assume for all  $m \in \{0, 1\}$ ,  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$ ,  $x, y \in \mathbb{R}^d$  that*

$$\mathcal{R}_\tau(f_{d,\varepsilon}^0) \in C(\mathbb{R}^d, \mathbb{R}), \quad \mathcal{R}_\tau(f_{d,\varepsilon}^1) \in C(\mathbb{R}^d, \mathbb{R}^d), \quad \mathcal{P}(f_{d,\varepsilon}^m) \leq \kappa d^\kappa \varepsilon^{-\kappa}, \quad (293)$$

$$\|f_d^1(x) - f_d^1(y)\| \leq \kappa \|x - y\|, \quad \|(\mathcal{R}_\tau(f_{d,\varepsilon}^1))(x)\| \leq \kappa (d^\kappa + \|x\|), \quad (294)$$

$$|(\mathcal{R}_\tau(f_{d,\varepsilon}^0))(x) - (\mathcal{R}_\tau(f_{d,\varepsilon}^0))(y)| \leq \kappa d^\kappa (1 + \|x\|^\kappa + \|y\|^\kappa) \|x - y\|, \quad (295)$$

$$\text{and} \quad \varepsilon |f_d^0(x)| + \|f_d^m(x) - (\mathcal{R}_\tau(f_{d,\varepsilon}^m))(x)\| \leq \varepsilon \kappa d^\kappa (1 + \|x\|^\kappa), \quad (296)$$



and for every  $d \in \mathbb{N}$  let  $u_d \in \{v \in C([0, T] \times \mathbb{R}^d, \mathbb{R}) : \inf_{q \in (0, \infty)} \sup_{(t, y) \in [0, T] \times \mathbb{R}^d} \frac{|v(t, y)|}{1 + \|y\|^q} < \infty\}$  be a viscosity solution of

$$\left(\frac{\partial}{\partial t} u_d\right)(t, x) = (\Delta_x u_d)(t, x) + \left(\frac{\partial}{\partial x} u_d\right)(t, x) f_d^1(x) \quad (297)$$

with  $u_d(0, x) = f_d^0(x)$  for  $(t, x) \in (0, T) \times \mathbb{R}^d$  (cf. Definitions 2.1, 4.1, 4.3, and 4.4). Then there exist  $c \in \mathbb{R}$  and  $(\mathbf{u}_{d, \varepsilon})_{(d, \varepsilon) \in \mathbb{N} \times (0, 1]} \subseteq \mathbf{N}$  such that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  it holds that  $\mathcal{R}_\tau(\mathbf{u}_{d, \varepsilon}) \in C(\mathbb{R}^{d+1}, \mathbb{R})$ ,  $\mathcal{P}(\mathbf{u}_{d, \varepsilon}) \leq c\varepsilon^{-c}d^c$ , and

$$\left[ \int_{[0, T] \times [a, b]^d} \frac{|u_d(y) - (\mathcal{R}_\tau(\mathbf{u}_{d, \varepsilon}))(y)|^p}{|b - a|^d} dy \right]^{1/p} \leq \varepsilon. \quad (298)$$

*Proof of Corollary 4.16.* Throughout this proof let  $\iota = \max\{3\kappa, 2(\kappa + 1)\}$ . Observe that (294) and the fact that  $\iota \geq \kappa$  prove that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$ ,  $x, y \in \mathbb{R}^d$  it holds that

$$\|f_d^1(x) - f_d^1(y)\| \leq \iota \|x - y\| \quad \text{and} \quad \|(\mathcal{R}_\tau(f_{d, \varepsilon}^1))(x)\| \leq \iota(d^\iota + \|x\|). \quad (299)$$

Next note that (295) and the fact that  $\iota \geq 3\kappa$  ensure that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$ ,  $x, y \in \mathbb{R}^d$  it holds that

$$\begin{aligned} |(\mathcal{R}_\tau(f_{d, \varepsilon}^0))(x) - (\mathcal{R}_\tau(f_{d, \varepsilon}^0))(y)| &\leq \kappa d^\iota (3 + \|x\|^\iota + \|y\|^\iota) \|x - y\| \\ &\leq \iota d^\iota (1 + \|x\|^\iota + \|y\|^\iota) \|x - y\|. \end{aligned} \quad (300)$$

In addition, observe that (296) and the fact that  $\iota \geq 2(\kappa + 1)$  show that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$ ,  $m \in \{0, 1\}$ ,  $x \in \mathbb{R}^d$  it holds that

$$\begin{aligned} \varepsilon |f_d^0(x)| + \varepsilon + \|f_d^m(x) - (\mathcal{R}_\tau(f_{d, \varepsilon}^m))(x)\| &\leq \varepsilon \kappa d^\kappa (1 + \|x\|^\kappa) + \varepsilon \\ &\leq \varepsilon (\kappa + 1) d^\kappa (1 + \|x\|^\kappa) \leq \varepsilon (\kappa + 1) d^\kappa (2 + \|x\|^\iota) \leq \varepsilon \iota d^\iota (1 + \|x\|^\iota). \end{aligned} \quad (301)$$

Combining this, (299), (300), and Corollary 4.15 implies that there exist  $c \in \mathbb{R}$  and  $(\mathbf{u}_{d, \varepsilon})_{(d, \varepsilon) \in \mathbb{N} \times (0, 1]} \subseteq \mathbf{N}$  such that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  it holds that  $\mathcal{R}_\tau(\mathbf{u}_{d, \varepsilon}) \in C(\mathbb{R}^{d+1}, \mathbb{R})$ ,  $\mathcal{P}(\mathbf{u}_{d, \varepsilon}) \leq c\varepsilon^{-c}d^c$ , and

$$\left[ \int_{[0, T] \times [a, b]^d} \frac{|u_d(y) - (\mathcal{R}_\tau(\mathbf{u}_{d, \varepsilon}))(y)|^p}{|b - a|^d} dy \right]^{1/p} \leq \varepsilon. \quad (302)$$

This completes the proof of Corollary 4.16. □

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