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MERTON'S AND KMV MODELS IN CREDIT RISK MANAGEMENT

Introduction

Contemporary economy, hidden in shade of world crisis, tries to renew it's attitude to various aspects of financial risk. A strong tension of bad debts, both in micro and macro scale emphasizes the menace to financial stability not only of particular entities, but also for whole markets and even states. Credit reliability of debtors becomes the main point of interest for growing number of frightened creditors. Owners of credit instruments more often consider them as even riskier then other types of securities. Awareness of shifting main risk determinants stimulates investigating of more effective methods of credit risk management. Over the last decade, a large number of models have been developed to estimate and price credit risk. In most cases, credit models concentrate on one single important issue - default risk. Investigating its characteristic (mostly finding statistical distribution) analytics can transform it into related dilemmas: how to measure it and how to price credit risk. The first one gives a chance of proper distinguishing more risky investments from the saver ones, the second one allows to calculate the value of the debt considering yield margin reflecting risk undertaken.

Many categories of models may be distinguished on the basis of the approach they adopt. Specific ones can be considered as "structural". They treat the firm's liabilities as contingent claim issued against underlying assets. As the market value of a firm liabilities approaches the market value of assets, the probability of default increases. Intuitively perceived estimation of market values can be revealed from book values of assets and liabilities, thus probability of default of firm's liabilities (credit risk) derives from the capital structure of the firm. Well known structural models of credit risk come origin mostly from theoretical Merton's works (1974, p. 449-470), which became theoretically extended and practically implemented by the KMV Corporation.

1. The Merton Model – basic concepts

Following Merton's model for the valuation of corporate securities we usually consider a simplified case of a firm with risky assets valued today by the market at A_0 level. The value at time t in the future A_t is uncertain, due to many external and internal factors (economic risk, business risk, foreign exchange risk, industry risk, etc.). Typically we assume, that the returns on the firm's assets are distributed normally and their behavior can be described with Brownian motion formulation (1).

$$\frac{dA_t}{A_t} = \mu dt + \sigma_A dz \tag{1}$$

Symbol σ_A stands for constant assets volatility and μ for constant drift^{*}. Value *dz* denotes for random value taken from standardized normal distribution (Figure 1).



Fig. 1. Basic concept of Merton's model.

Consequently, the firm's assets value assumed to obey a lognormal diffusion process with a constant volatility is given by (2).

$$A_t = A_{\alpha} e^{\left(r - \frac{\sigma_A^2}{4}\right)t + \sigma_A \cdot r Z_t}$$
(2)

Value A_0 is initial value of the assets specified at t = 0. The expected value of the assets at the time t can be given by (3):

$$E(A_{t}) = A_{\rho} e^{\Theta t}$$
⁽³⁾

"In the presence of perfect markets free of transaction costs, taxes and informational differences between market participants, the value of the firm is indepen-

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^{*} Considering it in detail μ as risk free rate of growth r or expected rate of return μA is one of the future discussed factors distinguishing KMV model from pure Merton's idea.

dent of its capital structure and is simply given by the sum of the debt and equity values." (Ong, 2005, p. 81). That assumption allows to consider situation, that firm has issued two classes of securities: equity and zero-coupon bond. The equity receives no dividends. The bonds represent the firm's debt obligation maturing at time T with principal value D. If at time T the firm's asset value exceeds the promised payment D, the lenders are paid the promised amount and the shareholders receive the residual asset value. If the asset value is lower than the promised payment, the firm defaults, the lenders receive a payment equal to the asset value, and the shareholders get nothing (Hull, Nelken, White, 2004).

Let's assume notation:

- 1. A₀ denotes value of the firm's assets today and A_T on date T.
- 2. E_0 denotes value of the firm's equity today and E_T on date T.

When the debt matures on date *T* provided, there is enough value in the firm to meet this payment $(A_T > D)$, debtholders will receive the full face value *D* due to them and equityholders receive the balance $A_T - D$. However, if the value of the firm's assets on date *T* is insufficient to meet the debtholders¹ claims (i.e. $A_T < D$), the debtholders receive total assets value, and the equityholders receive nothing (Figure 2). Thus, the amount D_T received by the debtholders on date *T* can be expressed with (4):



Fig. 2. Payoff from debt at maturity period T related to value of assets

The key insight in Merton's paper is that (4) can be rewritten as the payoff from an option position, and thus option-pricing techniques can then be brought to bear on the problem of pricing risky debt. Instead of conditional formula (4), payoffs received by the debtholders at the time T may also be expressed with unconditional expression (5):

$$D_T = D - max \left[D - A_{T_t} \mathbf{0} \right] \tag{5}$$

The most significant advantage of that representation (Sundarman, 2001, p. 3) is drawn from the readable interpretation (Figure 3):

- 1. The first component of formula (*D*), represents the payoff from investing in a risk-free zero coupon bond maturing at time *T* with a face value of *D*.
- 2. The second one $(-max[D A_T, 0])$, is the payoff from a short position in a put option on the firm's assets with a strike price of D and a maturity date of T.



Fig. 3. Decomposition of debt value at the time T

Decomposition illustrated in Figure 3. defines a procedure to valuate present value of risky debt, consisting of two steps:

- 1) identifying present value D of the risk-free debt,
- 2) subtracting the present value of the PUT option.

The first step of procedure is straightforward (typically the formula of continuous compounding of interest is used instead of discrete one). The second step is clearly valuing the put option, for which we need an option pricing model. Merton invoked the Black-Scholes model assuming that the firm value A_t follows a lognormal diffusion with constant volatility σ_A , and that the risk-free rate of interest *r* is constant. Under these assumptions, the value of the PUT option may be obtained from the Black-Scholes pricing formula:

$$PUT = De^{-rT}N(-d_2) - A_0N(-d_1)$$
⁽⁶⁾

$$d_{1} = \frac{\ln\left(\frac{A_{0}}{D}\right) + \left(r + \frac{1}{2}\sigma_{A}^{2}\right)T}{\sigma_{A}\sqrt{T}}$$

$$\tag{7}$$

$$d_{2} = \frac{ln\left(\frac{A_{0}}{D}\right) + \left(r - \frac{1}{2}\sigma_{A}^{2}\right)T}{\sigma_{A}\sqrt{T}} = d_{1} - \sigma_{A}\sqrt{T}$$
(8)

where:

- E_0 - market value of the firm's equity (today),

- A_0 market value of the firm's assets (today),
- σ_A volatility of the firm's assets (std.dev. of annualized rate of return),
- σ_E volatility of the firm's equity (std.dev. of annualized rate of return),
- D total amount of the firm's debt,
- T- time to maturity of the firm's debt,
- r risk free interest rate,
- N(*) cumulative normal distribution function.

The value of the put option determines the price differential between today's risky and riskless value of the debt, so the market value of debt D_0 can be identified with the equation (9):

$$D_0 = De^{-rT} - (PUT) \tag{9}$$

A higher value of the *PUT* determines the greater distance between the price of risky and riskless bonds, increasing the interest rate spread. Thus, for example, as volatility of the firm value increases, the spread on risky debt must grow, alongside with the value of the put option. Similarly, as the risk-free interest rate increases, the spread on risky debt must decrease as well.

2. Unobservability of the firm value process

The significant problem appearing while attempting a practical implementation of the Merton's model of debt valuation is, that both: the firm value A_0 and its volatility σ_A are usually unobservable. Although the firm value process and its volatility are themselves directly unobservable, it is possible to use prices of traded securities issued by the firm to identify these quantities implicitly. Lets suppose that the firm is publicly traded with observable equity prices. Let E_0 denote the present value of the firm's equity, and σ_E explains volatility of equity (can easily be estimated from the data on equity prices). Both: E_0 and σ_E can be used to obtain estimates of A_0 and σ_A . The first step in this procedure is to express equity value E_0 itself as an option on the firm value. Equityholders receive only an amount remaining after paying the debtholders on date T (10):

$$E_T = \begin{cases} A_T - D & \text{if} \quad A_T \ge D \\ 0 & \text{otherwise} \end{cases}$$
(10)



Fig. 4. Payoff from equity at debt maturity period T

It is simply the payoff from holding a long position in a call option on the firm's assets with strike point *D* and maturity *T*:

$$E_T = max[0, A_T - D] \tag{11}$$

Following similar procedure as for debt pricing, present value of the firm (equity) can also be defined in accordance to Black-Scholes assumptions. The Merton's model links the market value of the equity and the market value of the assets as follows (12):

$$E_0 = CALL = A_0 N(d_1) - De^{-rT} N(d_2)$$
(12)

Since the equity prices E_0 are observable, we got one equation with the two unknowns A_0 and σ_A . To be able to solve for these quantities, we need a second equation. One can proof, that volatility of equity and asset are related according to expression (13):

$$\sigma_{\overline{e}} = \frac{A_0}{E_0} N(d_1) \sigma_A \tag{13}$$

Once we have risk-free interest rate and the time horizon of the debt, the only unknown quantities are the value of the firm's assets A_0 and its volatility σ_A . Thus now we can solve the two nonlinear simultaneous equation (12) and (13) to determine A_0 and σ_A by the equity value, volatility value and capital structure (Lu, 2008, p. 12).

3. Implied credit spread of risky debt

Merton's model can be used to explain "credit spread", defined as difference between the yield on the risky debt and the risk – free rate. Let's define D_0 as the market price of the debt at time zero. The value of the assets is equal to total

value of the two sources of financing: equity and debt, so that present value of the debt can be expressed with expression (14):

$$D_0 = A_0 - E_0 \tag{14}$$

Replacing E_0 with (12) drives to (15), and finally to (16):

$$D_{0} = A_{0} - A_{0}N(d_{1}) + De^{-rT}N(d_{2}) = A_{0}(1 - N(d_{1})) + De^{-rT}N(d_{2})$$

= $A_{0}N(-d_{1}) + De^{-rT}N(d_{2})$ (15)

$$D_0 = A_0 N(-d_1) + D e^{-rT} N(d_2)$$
(16)

The yield to maturity for the debt can alternatively be defined implicitly by (17):

$$D_0 = De^{-\gamma T} \tag{17}$$

Comparing right sides of equations (16) and (17) we can derive the value of yield rate y (18):

$$De^{-yT} = A_0 N(-d_1) + De^{-vT} N(d_2)$$

$$y = -\frac{1}{T} ln \left(\frac{A_0}{D} N(-d_1) + e^{-vT} N(d_2) \right)$$
(18)

The same result can be gained from the fundamental formula on rate of return with continuous compounding (19):

$$y = \frac{1}{T} ln \left(\frac{D}{D_0} \right) = \frac{1}{T} ln \left(\frac{D}{A_0 N (-d_1) + D e^{-rT} N (d_2)} \right)$$
(19)

The credit spread implied by the Merton model can be finally obtained by reducing the yield rate with the risk – free rate (20):

$$\mathbf{s} = \mathbf{y} - \mathbf{r} \tag{20}$$

In summary, a creditworthiness of a firm (level of credit risk related to its obligations), can be displayed with implied credit spread for its debt which is dependent on three important ingredients: leverage ratio, assets volatility σ_A , and the time to repayment *T* (Debt maturity). Figures 5 to 7 illustrate three examples of credit spread calculation. Black - Scholes - Merton Model Calculating implied credit spread

100,00	A ₀	market value of the firm's assets (today)	
25,00%	S _A	volatility of the firm's assets (std.dev. of annualized rate of return)	
50,00	D	total amount of the firm's notional debt	
5,00%	r	risk free interest rate	
1	T	time to matunity of the firm's debt	
3,097588722 2,847588722	d ₁ d ₂	$d_1 = \frac{\ln\left(\frac{A_0}{D}\right) + \left(r + \frac{1}{2}\sigma_A^2\right)T}{\sigma_A\sqrt{T}} \qquad d_2 = \frac{\ln\left(\frac{V_A}{D}\right) + \left(r - \frac{1}{2}\right)}{\sigma_A\sqrt{T}}$	$\frac{\sigma_A{}^2}{T} = d_1 - \sigma_A \sqrt{T}$
0,01	PUT	differential between today's risky and riskless debt	$PUT = De^{-rT}N(-d_2) - A_0N(-d_1)$
47,55	D ₀	market value of the firm's debt (today)	$\begin{split} D_0 &= De^{-rT} - \text{differential between risky and riskless debt} \\ D_0 &= A_0 N(-d_1) + De^{-rT} N(d_2) \end{split}$
47,55	D ₀	market value of the firm's debt (today)	
5,02%	y	annualized continuous yield of return	$y = \frac{1}{r} ln \left(\frac{D}{D_0} \right)$
0,02%	s	implied credit spread	s = y - r





Fig. 6. Example of large debt contribution, resulting with 2 p.p. implied credit spread

Black - Scholes - Merton Model Calculating implied credit spread					
100,00	Ao	market value of the firm's assets (today)			
50,00%	S _A	volatility of the firm's assets (std.dev. of annualized rate of return)			
50,00	D	total amount of the firm's notional debt			
5,00%	r	risk free interest rate			
1	т	time to maturity of the firm's debt			
1,736294361 1,236294361	dı dz	$d_1 = \frac{ln\left(\frac{A_0}{D}\right) + \left(r + \frac{1}{2}\sigma_A^2\right)T}{\sigma_A\sqrt{T}} \qquad d_2 = \frac{ln\left(\frac{V_A}{D}\right) + \left(r - \frac{1}{2}\sigma_A^2\right)T}{\sigma_A\sqrt{T}}$	$\frac{\tau_A^2}{T} = d_1 - \sigma_A \sqrt{T}$		
1,02	PUT	differential between today's risky and riskless debt	$PUT = De^{-rT}N(-d_2) - A_0N(-d_1)$		
46,54	Do	market value of the firm's debt (today)	$D_0 = De^{-rT} - differential between risky and riskless debt$		
46,54	Do	market value of the firm's debt (today)	$D_0 = A_0 N(-d_1) + De^{-rT} N(d_2)$		
7,17% 2,17%	y s	annualized continuous yield of return implied credit spread	$y = \frac{1}{r} ln \left(\frac{D}{D_0} \right)$ s = y - r		

Fig. 7. Example of "save" structure of capital related to large assets volatility, resulting also with 2 p.p. implied credit spread

Initial model presented at Figure 5 illustrates rather comfortable situations for creditors. Calculated credit spread is close to zero confirming a low level of the credit risk. Changing the structure of financing firm's activity by blowing up debt contribution (80 in place of 50 - Figure 6) increases probability that the market value of the assets drop below the debt value causing default situation. It results with implied credit spread over 2 p.p. greater then previously. Analogical situation could happen for initial value of the debt, but higher volatility of the assets (50% in place of 25% - Figure 7). Also in that case, market value of the firm's debt become lower then initially and consequently, the realized yield of return is higher, compensating increased credit risk.

4. The KMV Model

The practical implementation of Merton's model, has received considerable commercial attention in recent years. One of them is KMV model which in fact is a modified version of the Merton's concept, varying from the original with a few aspects.

According to preceding discussion, in the Merton's model, a nominal value of firm's obligation was considered as a terminal value for firm's assets. "KMV Corporation has observed from a sample of several hundred companies that firms are generally more likely to default when their asset values reach a certain critical level somewhere between the value of total liabilities and the value of short-term debt. Therefore, in practice, using D alone as the threshold in the tail distribution might not be an accurate measure of the actual probability of default. KMV implements an additional step and refers to this critical threshold for defaulting as the Default Point." (Ong, 2005, p. 84). The ambiguity of formal bankruptcy state and the situation, when assets value fall below the value of liabilities makes theory of determining accurate threshold level for default situation quite soft. For KMV model Default Point (DPT) is roughly approximated by the sum of all the Short Term Debt (STD) and half of the Long Term Debt (LTD) (21):

$$DPT = STD + 0,5LTD \tag{21}$$

Another point is, that for practical reasons, before computing the probability of default, the KMV approach implements an intermediate phase of computation of an index called Distance to Default (DD) (Lu, 2008, p. 12). "It is defined as the distance between the expected assets value of the firm at the analysis horizon, (...) and the default point, normalized by standard deviation of the future asset returns." (Ong, 2005, p. 84). Formally it is defined as follows (22).

$$DD = \frac{E(A_T) - DPT}{\sigma}$$
(22)

Following that idea, to derive the probability of default for a particular firm, we must calculate the distance to default first. The probability of default for any time horizon is strongly related to *DD*. The larger *DD*, the smaller *PD* that means the less chance the company will default.



Fig. 8. Distribution of the firm's asset value at maturity of the debt

Calculating *Distance to Default* can be decomposed into two stages, derived from Figure 8:

1) calculating absolute Distance to Default – DD',

2) calculating relative Distance to Default – DD.

Absolute Distance to Default (DD') it is expressed (in percents of expected assets) as a distance between expected assets and Default Point (DPT). It can be displayed as a sum of initial distance and the growth of that distance within the period T (23).

$$DD^* = \ln\left(\frac{A_0}{DFT}\right) + \left(\mu_A - \frac{1}{2}\sigma_A^2\right)T$$
(23)

Not as in pure Merton concept, in KMV model μ_A is no longer risk-free rate but expected rate of the return of the firm's asset and DPT is Default Point instead of nominal value *D* (the face value of the debt). A little confusing is considering in formula (23) expected growth of assets as equal to $(\mu_A - \frac{1}{2}\sigma_A^{-1})$ instead of simply μ_A . While rate of return is normally distributed, consequently future value of investment (or effective yield of return) is distributed lognormaly. Relation between those two distributions is explained with (24), where, as previously, μ_A is the drift rate (expected rate of return) and σ_A is the volatility of the underlying (*The Professional...*, 2004).

$$ln\left(\frac{S_T}{S}\right) \sim N\left(\left(\mu_A - \frac{\sigma_A^2}{2}\right)T, \sigma_A \sqrt{T}\right)$$
(24)

Dividing absolute value DD' with calibrated (according to T – usually annualized) volatility of assets, we can calculate DD in relative terms as a multiplier of standard deviation (25).

$$DD = d_2 = \frac{\ln\left(\frac{A_0}{DFT}\right) + \left(\mu_A - \frac{1}{2}\sigma_A^2\right)T}{\sigma_A T}$$
(25)

It is easy to notice, that such estimation of *Distance to Default* is very similar to d_2 (considering mentioned above replacement of r with μ_A and D with DPT). "The similarity is not an accident and is the result of a relationship between the risk – neutral probability and the actual probability. The actual probability uses the expected return of the assets in the drift term, while the risk-neutral probability uses the risk free rate r." (Ong, 2005, p. 86).

5. Probability of Default vs. Expected Default Frequency

Considering the most simplified situation of normally distributed assets value after period T, according to the definition of default (value of firm's asset falls below the value of DPT), we can estimate the probability of default with formula (26).

$$PD = 1 - N(d_2) = N(-d_2) \tag{26}$$

Because of well known problem of fatter tails in real credit loss distribution, that type of estimation (even with lognormal distribution instead of normal) is underappreciated. In that situation, one more distinguishing feature of KMV model is, that it operates on the historical set of frequencies of default rather then on theoretical normal or log-normal distribution. Consequently, in KMV model Probabilility of Default (PD) is replaced with Expected Default Frequency (EDF). "Using historical information about large sample of firms, including firms that have defaulted, one can track, for each time horizon, the proportion of firms of a given ranking (...) that actually defaulted after one year." (Crouchy, Galai, Mark, 2006, p. 277). An example of that dependence is shown at Figure 9. For DD = 3, Expected Default Frequency is equal to 40 basic points. That means, that according to data base analysis, 0.4% of registered firms with DD = 3 defaulted after one year. At the same time, for DD = 1, EDF grows to 120 b.p. = 1.2%.



Fig. 9. Correlation between Distance to Default and Expected Default Frequency

Simplicity of using EDF vs. DD curve makes concept of KMV model very comprehensive and easy to implement.

Conclusion

As the theoretical idea of Merton's and KMV model seems to be very convincing, the most important question arises, whether the results given by this approach is really any better then the probabilities empirically derived by rating agencies, and related to popular rating grades. Very limited credibility of the rating agencies being accused widely for undermining financial stability of contemporary economies is a strong incentive for searching more transparent and based on clear fundamental assumptions models of risk assessment. Even since using a sound, theoretical foundation, in practice, using discussed models boils down to a subjective estimation of the most input parameters making gained results not fully convincing. Thus, not undermining the credibility neither pure Merton's conception nor KMV model, it should be always recommended to use them with caution and an open mind.

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MODEL MERTONA I KMV W ZARZĄDZANIU RYZYKIEM KREDYTOWYM

Streszczenie

Widmo kryzysu finansowego, w którym są pogrążone współczesne gospodarki, wywiera rosnącą presję na inwestorów poszukujących coraz efektywniejszych narzędzi zarządzania ryzykiem kredytowym. Czynnikiem szczególnie mobilizującym do tych poszukiwań stała się znaczna utrata wiarygodności przez instytucje ratingowe, których oceny były dotychczas głównym wyznacznikiem zdolności kredytowej kredytobiorców. Alternatywą zdają się być modele strukturalne bazujące na fundamentalnych przesłankach odwołujących się głównie do relacji aktywów dłużnika do wielkości jego długu wraz z prognozowaną zmiennością wartości rynkowej aktywów. Do najbardziej znanych modeli tej kategorii należą model Mertona i jego praktyczna implementacja określana mianem modelu KMV. Opracowanie zawiera zarys koncepcji powyższych modeli, ze szczególnym wskazaniem na przesłanki ich wykorzystania.