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ON PURE REFUTATION FORMULATIONS OF SENTENTIAL LOGICS

In [1] J. Łukasiewicz introduced the following refutation rule: "If $\alpha \to \beta$ is not provable and β is refutable then α is refutable", which employs the notion of provability as well as that of refutability. In this paper we give refutation formulations for a large class of logics using only pure refutation rules, i.e. rules of the form: "If α is refutable then β is refutable". This syntactical way of presenting a logic is similar to the semantic one in that both methods are negative: A formula is a theorem if it cannot be refuted by a syntactical (semantical) device.

Because of the generality of the result we are using sequents rather than formulas.

Let $\mathcal{F}$ be a fixed algebra of formulas, i.e. $\mathcal{F} = (For, f_1, \ldots, f_k)$, where For is the set of all formulas generated from the set Var of sentential variables by the connectives $f_1, \ldots, f_k$. For every $\alpha \in For$, the symbol $S(\alpha)$ will denote the set of all subformulas of α, and for any $X \subseteq For$, $S(X) = \bigcup\{S(\alpha) : \alpha \in X\}$. With every set $Y \subseteq_f For$ ("$X \subseteq_f Y$" stands for "X is a finite subset of Y") we associate a one-to-one mapping g_Y from $S(Y)$ into Var in such a way that $g_Y(p) = p$ for each $p \in S(Y) \cap Var$. For every $\beta \in S(Y), g_Y(\beta)$ will be denoted by p_β. If $\gamma = f(\beta_1, \ldots, \beta_n)$, where f is a connective, then we write $F(\gamma)$ instead of $f(p_{\beta_1}, \ldots, p_{\beta_n})$.

A sequent is a pair (X, α), where $X \cup \{\alpha\} \subset_f For$. A rule is a subset of the set $\{(\Sigma, S) : \Sigma \cup \{S\}$ is a finite set of sequents$\}$. Let Q be a set of sequents and let R be a set of rules. Then a sequent (X, α) is said to be derivable from Q by R iff there is a finite sequence of sequents $S_1, \ldots, S_m$ such that $S_m = (X, \alpha)$ and for each $1 \leq i \leq m$, either $S_i \in Q$ or S_i is obtained from some preceding sequents by a rule of R.

By a sentential logic we mean a structural consequence operation on $\mathcal{F}$, i.e. a function $C : \{X : X \subseteq For\} \to For$ such that for all $X, Y \subseteq For$ the following conditions are satisfied: $X \subseteq C(X); C(C(X)) \subseteq C(X); C(X) \subseteq C(Y)$ whenever $X \subseteq Y$; and $e(C(X)) \subseteq C(e(X))$ for every substitution $e : For \to For$. A sequent (X, α) is said to be provable in a logic C iff $\alpha \in C(X)$.

A matrix is a pair $M = (\mathcal{A}, D)$, where $\mathcal{A}$ is an algebra similar to $\mathcal{F}$ and $D \subseteq A$. We say that a matrix is countable iff its algebra is countable. A set $\mathbf{K}$ of matrices (called a matrix semantics) is countable iff $\mathbf{K}$ is a countable set of countable matrices. Every matrix induces a consequence operation Cn_M defined thus: $\alpha \in Cn_M(X)$ iff for every valuation $v : For \to A$ in M, if $v(X) \subseteq D$ then $v(\alpha) \in D$. If $\mathbf{K}$ is a matrix semantics then $Cn_{\mathbf{K}}$ is defined as follows: $\alpha \in Cn_{\mathbf{K}}(X)$ iff $\alpha \in Cn_M(X)$ for every $M \in \mathbf{K}$. With every countable matrix semantics $\mathbf{K}$ we associate a one-to-one mapping $g_{\mathbf{K}}$ from $Z = \bigcup\{A : (\mathcal{A}, D) \in \mathbf{K}\}$ into Var. For every $x \in Z, g_{\mathbf{K}}(x)$ will be denoted by p_x.

A logic C is said to be equivalential (cf. [2]) iff there is a finite set $E(p, q)$ of formulas in two distinct variables such that the following conditions are satisfied:

(i) $E(\alpha, \alpha) \subseteq C(\emptyset)$,
(ii) $E(\beta, \alpha) \subseteq C(E(\alpha, \beta))$,
(iii) $E(\alpha, \gamma) \subseteq C(E(\alpha, \beta), E(\beta, \gamma))$,
(iv) $\beta \in C(E(\alpha, \beta) \cup \{\alpha\})$,
(v) $E(f(\alpha_1, \ldots, \alpha_n), f(\beta_1, \ldots, \beta_n)) \subseteq C(E(\alpha_1, \beta_1) \cup \ldots \cup E(\alpha_n, \beta_n))$,

where f is a connective and n is the arity of f.

For every countable algebra $\mathcal{A}$ similar to $\mathcal{F}$ and for every finite set $E(p, q)$ of wffs in two variables we define the description of $\mathcal{A}$ in terms of E (cf. [5], [4]) as follows: $D_E(\mathcal{A}) = \{E(f(p_{x_1}, \ldots, p_{x_n}), p_{f(x_1,\ldots,x_n)}) : f$ is a connective, n in the arity of f, $x_1, \ldots, x_n \in A\}$. Moreover for every countable matrix $M = (\mathcal{A}, D)$, we define the following sets of sequents:

$$Q_E(M) = \{(\bigcup X, p_a) : X \subseteq_f D_E(\mathcal{A}), a \in A - D\},$$

$$\overline{Q}_E(M) = \{(\bigcup X \cup Z, p_a) : X \subseteq_f D_E(\mathcal{A}), Z \subseteq_f \{p_d : d \in D\}, a \in A - D\}.$$

Further if $\mathbf{K}$ is a countable matrix semantics then we put

$$Q_E(\mathbf{K}) = \bigcup\{Q_E(M) : M \in \mathbf{K}\},$$

$$\overline{Q}_E(\mathbf{K}) = \bigcup\{\overline{Q}_E(M) : M \in \mathbf{K}\}.$$

Observe that if $\mathbf{K}$ is a recursive matrix semantics then the sets $\overline{Q}_E(\mathbf{K})$, $Q_E(\mathbf{K})$ are recursive.

We will consider the following rules:

$$r_{sb} \quad \frac{(e(X), e(\alpha))}{(X, \alpha)} \qquad (e \text{ is a substitution})$$

$$r_{ex} \quad \frac{(E(\beta,\gamma) \cup X, \alpha)}{(X(\beta/\gamma), \alpha(\beta/\gamma))} \quad \text{where } X \cup \{\alpha, \beta, \gamma\} \subseteq_f For.$$

As usual $\alpha(\beta/\gamma)$ is the formula obtained from α by replacing some occurrences of β with γ. Of course $X(\beta/\gamma) = \{\alpha(\beta/\gamma) : \alpha \in X\}$.

For every sequent (X, α) we define the normal form of (X, α) (cf. [3]) thus:

$$N((X,\alpha)) = (\Delta(X \cup \{\alpha\}) \cup \{p_\gamma : \gamma \in X\}, p_\alpha),$$

where $\Delta(Y) = \bigcup\{E(F(\beta), p_\beta) : \beta \in S(Y) - Var\}$ for every $Y \subseteq_f For$.

LEMMA. *If (X, α) is a sequent then the rule*

$$\frac{N((X,\alpha))}{(X,\alpha)}$$

is derivable from the rule r_{ex}.

PROOF. If $S(X \cup \{\alpha\}) - Var = \emptyset$ the $N((X,\alpha)) = (X,\alpha)$, so we assume that

$$S(X \cup \{\alpha\}) - Var = \{\gamma_1, \ldots, \gamma_m\}$$

for some $1 \leq m$ and $\gamma_i \neq \gamma_j$ for all $1 \leq i \neq j \leq m$. Consider the sequence of sequents: $S_0, S_1, \ldots, S_m$, where $S_0 = N((X,\alpha)), S_i = t_i(/S_{i-1}/^i)$ $(1 \leq i \leq m)$ and

(i) For $0 \leq i \leq m$, t_i is the substitution defined thus: $t_0(p) = p$ $(p \in Var)$, and for $1 \leq i \leq m$,

$$t_i(p) = \begin{cases} t_{i-1} \circ \ldots \circ t_0(f(\gamma_i)) & \text{if } p = p_{\gamma_i} \\ p & \text{otherwise.} \end{cases}$$

(ii) For every sequent (Y, β) and $1 \leq i \leq m$, the symbol $/(Y,\beta)/^i$ denotes the sequent $(Y - E(t_i(p_{\gamma_i}), p_{\gamma_i}), \beta)$.

By an easy induction on i it can be shown that for each $i \geq 0$, S_i has $E(t_i \circ \ldots \circ t_0(F(\gamma_k)), p_{\gamma_k})$ as its part for all $i < k \leq m$. Therefore it is clear that for every $1 \leq i \leq m, S_i$ is obtained from S_{i-1} by deleting $E(t_i(p_{\gamma_i}), p_{\gamma_i})$ and replacing all occurrences of p_{γ_i} with $t_i(p_{\gamma_i})$, which is clearly an application of the rule r_{ex}. We should show that $S_m = (X, \alpha)$ it is easily to see that

$$S_m = t_m \circ \ldots \circ t_1((\{p_\gamma : \gamma \in X\}, p_\alpha)).$$

Now we note the following

Proposition. *For any $0 \leq k \leq m$, if $1 \leq i \leq m$ and $p_\phi \in S(t_k \circ \ldots \circ t_0(F(\gamma_i)))$ then $\phi \in S(\gamma_i) - \{\gamma_i\}$.*

Proof. Indeed, Proposition is true for $k = 0$.
Assume that $k > 0$.

Case 1. $p_{\gamma_k} \notin S(t_{k-1} \circ \ldots \circ t_0(F(\gamma_i)))$. So if

$$p_\phi \in S(t_k \circ \ldots \circ t_0(F(\gamma_i))) = S(t_{k-1} \circ \ldots \circ t_0(F(\gamma_i))),$$

then by ind. hyp. $\phi \in S(\gamma_i) - \{\gamma_i\}$.

Case 2. $p_{\gamma_k} \in S(t_{k-1} \circ \ldots \circ t_0(F(\gamma_i)))$. Then by ind. hyp. $\gamma_k \in S(\gamma_i) - \{\gamma_i\}$. Assume that $p_\phi \in S(t_k \circ \ldots \circ t_0(F(\gamma_i)))$. If $p_\phi \notin S(t_k(p_{\gamma_k}))$ then $p_\phi \in S(t_{k-1} \circ \ldots \circ t_0(F(\gamma_i)))$ and $\phi \in S(\gamma_i) - \{\gamma_i\}$ by ind. hyp. If

$$p_\phi \in S(t_k(p_{\gamma_k})) = S(t_{k-1} \circ \ldots \circ t_0(F(\gamma_i)))$$

then by ind. hyp. $\phi \in S(\gamma_k) \subseteq S(\gamma_i) - \{\gamma_i\}$.

Since, by Proposition, $p_{\gamma_i} \notin S(t_i(p_{\gamma_i}))$, it is easy to show by induction on the complexity of β that for every $\beta \in S(X \cup \{\alpha\})$ we have $t_m \circ \ldots \circ t_0(p_\beta) = \beta$. Hence $S_m = (X, \alpha)$. $\square$

For any logics C, C' we write $C =_f C'$ instead of $C(X) = C'(X)$ for every $X \subseteq_f For$.

Theorem 1. *Let C be an equivalential logic such that $C =_f Cn_{\mathbf{K}}$ for some countable matrix semantics $\mathbf{K}$. Then for every sequent (X, α) we have (X, α) is not provable in C iff (X, α) is derivable from $\overline{Q}_E(\mathbf{K})$ by the rules r_{sb}, r_{ex}.*

PROOF. (←) It is easy to see that $\overline{Q}_E(\mathbf{K}) \subseteq \{(Y, \beta)$ is a sequent: $\beta \notin C(Y)\}$. (E is the set existing by the definition of an equivalential logic.) Moreover the set of unprovable sequents of C is closed under rules r_{sb}, r_{ex}.
(→) Assume that $\alpha \notin C(X)$. Then $v(X) \subseteq D, v(\alpha) \notin D$ for some valuation v in some $(\mathcal{A}, D) \in \mathbf{K}$. Let $\chi = (T \cup \{p_{v(\gamma)} : \gamma \in X\}, p_{v(\alpha)})$, where

$T = \bigcup\{E(f(p_{v(\beta_1)}, \ldots, p_{v(\beta_n)}), p_{f(v(\beta_1),\ldots,v(\beta_n))}) :$ f is a connective, n is the arity of $f, f(\beta_1, \ldots, \beta_n) \in S(X \cup \{\alpha\})\}$.

Obviously $\chi \in \overline{Q}_E(\mathbf{K})$. Now let e be a substitution such that $e(p_\beta) = p_{v(\beta)}$ $(\beta \in S(X \cup \{\alpha\}))$. Then

$$e(N((X, \alpha))) = e((\Delta(X \cup \{\alpha\}) \cup \{p_\gamma : \gamma \in X\}, p_\alpha)) = \chi.$$

Hence $N((X, \alpha))$ is derivable from $\overline{Q}_E(\mathbf{K})$ by r_{sb} from χ. Therefore by Lemma (X, α) is derivable from $\overline{Q}_E(\mathbf{K})$ by r_{sb}, r_{ex}. □

If we identify a logic C with the set $C(\emptyset)$ of its theorems then we have the following

THEOREM 2. *Let C be an equivalential logic such that $C(\emptyset) = Cn_{\mathbf{K}}(\emptyset)$ for some countable matrix semantics $\mathbf{K}$. Then $\alpha \notin C(\emptyset)$ iff the sequent $(\emptyset, \alpha)$ is derivable from $Q_E(\mathbf{K})$ by the rules r_{sb}, r_{ex}.*

PROOF. Similar to that of Theorem 1.

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