The Finite Cell Method: High order simulation of complex structures without meshing

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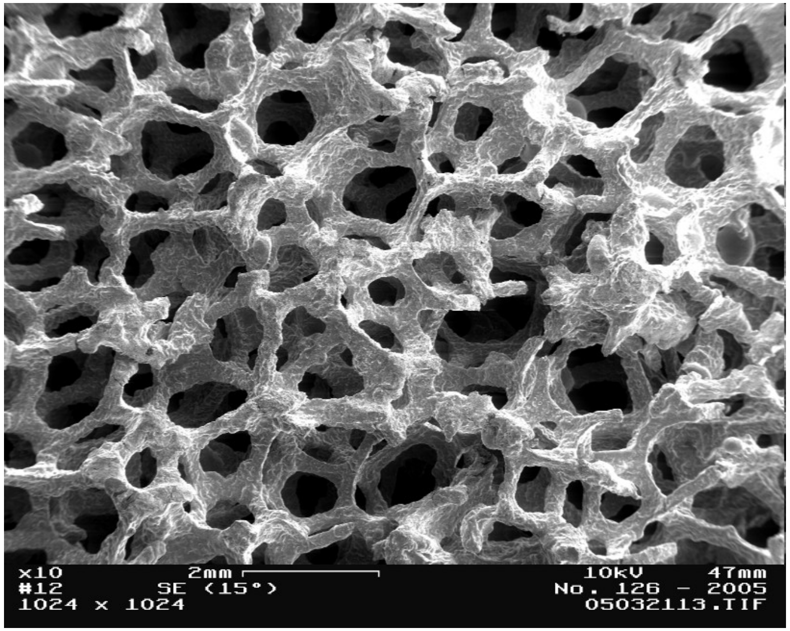
This work has partially been supported from the Excellence Initiative by the German Government, the German Science Foundation (DFG) and SIEMENS AG

Metal foams



From: Wikipedia

Goal: Virtual material design

Obtain mechanical properties from process parameters of foam generation

Step 1: Simulation of foam generation

(Lattice Boltzmann Method)

Step 2: (Microscopic) structural simulation

to obtain mechanical properties

(Finite Cell Method)

Overview

- Motivation
- Foam generation by the Lattice Boltzmann method
- High order finite cell method: Basic principles
- Accuracy considerations
- Aplication to porous structures
- Summary

Lattice-Boltzmann simulation

$$\nabla \cdot \vec{u} = 0$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\nabla)\vec{u} = -\tfrac{1}{\rho_0}\nabla P + \nu \nabla^2 \vec{u}$$

Partial differential equations (Navier–Stokes)

Discretization ↓

Multi–scale analysis ↑

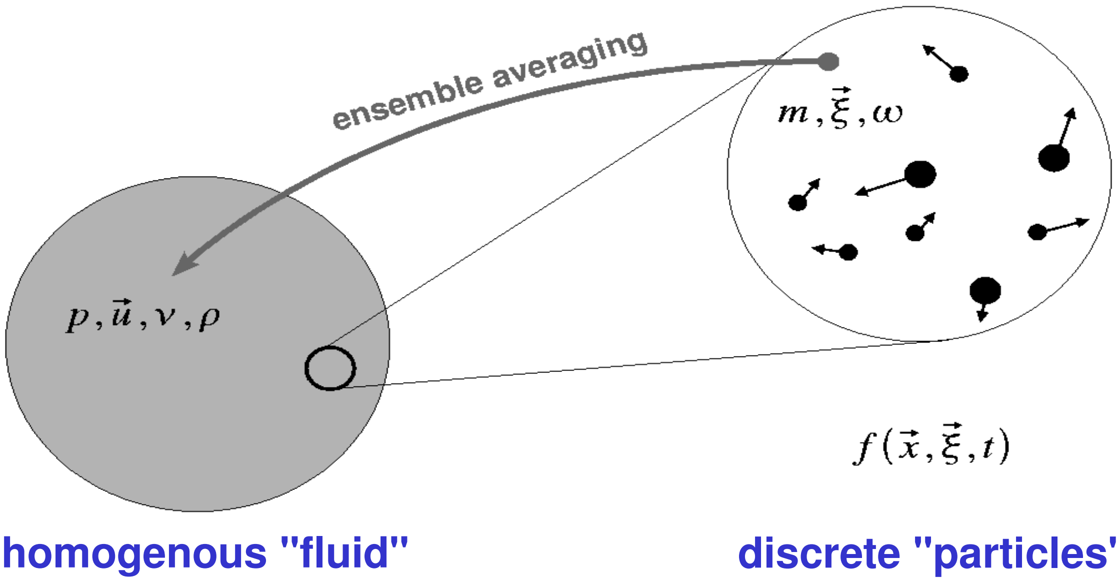
e.g. Finite Difference / Finite Volume equations

Discrete model (LGCA or LBM)

The Boltzmann equation:

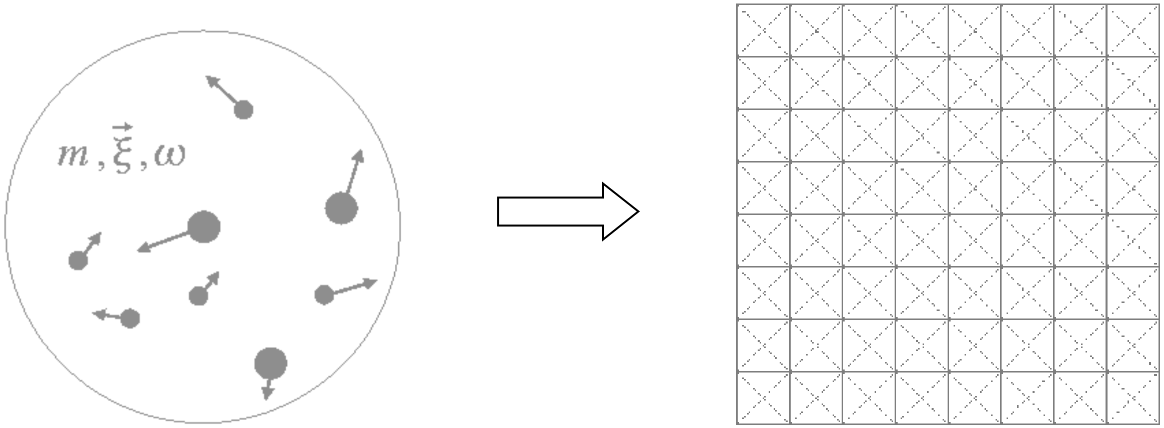
$$\frac{\partial f}{\partial t} + \vec{\xi}\,\frac{\partial f}{\partial \vec{x}} = Q(f,f)$$

- t : time
- $\vec{x}$: spatial coordinate
- $\vec{\xi}$: particle velocity
- f : Probability to find a particle with velocity $\vec{\xi}$ at time t and position $\vec{x}$
- Q : Collision operator



- Simulate multi–particle system behaviour through **artificial microworlds of particles** residing on a computational lattice

continuum



lattice

- Approximate Boltzmann equation on a lattice with FD-method

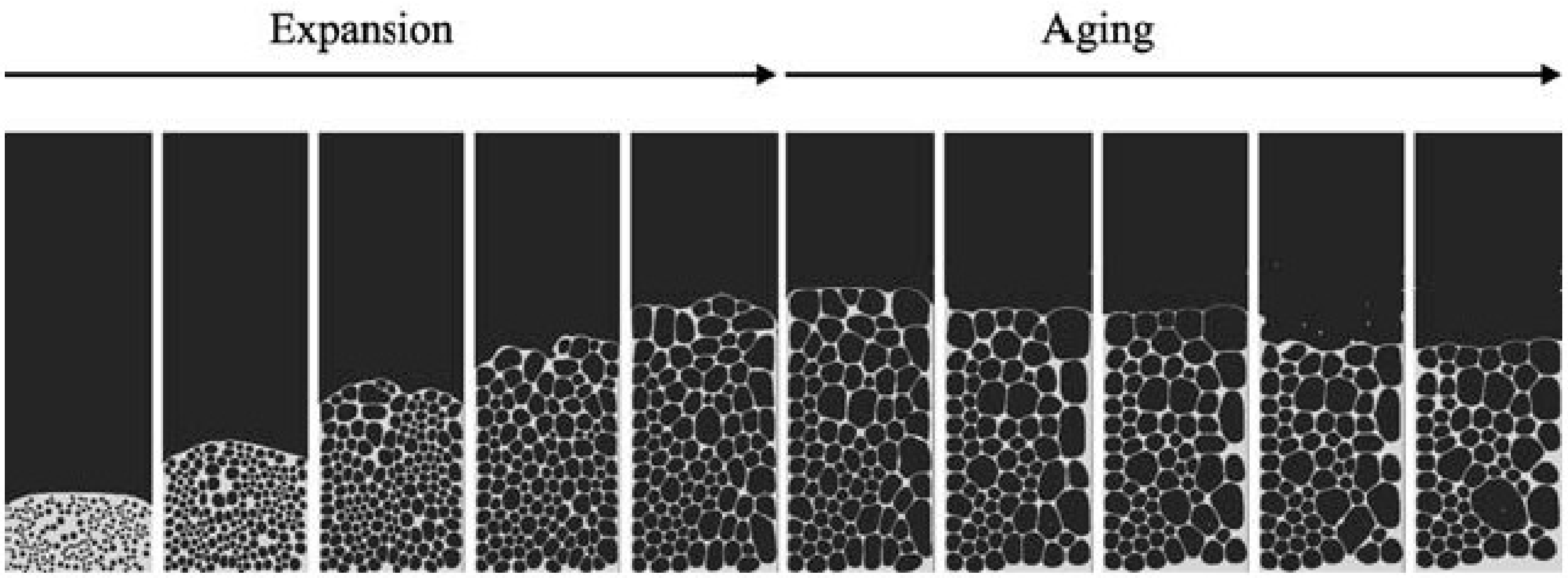
$$f_i(\vec{x}+\vec{e}_i\delta t, t+\delta t) - f_i(\vec{x},t) = -\frac{1}{\tau}\left(f_i(\vec{x},t) - f_i^{eq}(\vec{x},t)\right) + F_i$$

Extension to free surface flow

C. Körner, M. Thies, T. Hofmann, N. Thürey, and U. Rüde:

Journal of Statistical Physics, Vol 121, 2005

- Two phases: Gas and liquid
- Surface capturing method
- Volume of fluid approach



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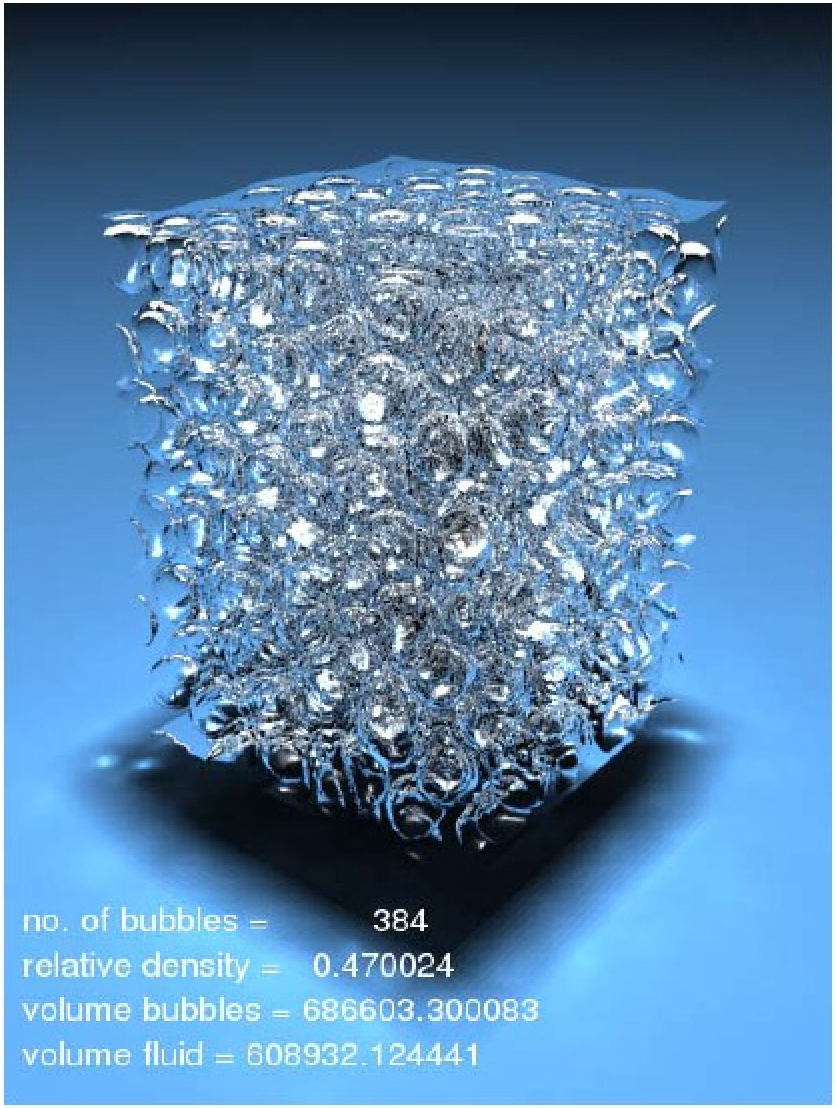

Idea:

- ‚Virtual' material tests to predict macroscopic mechanical properties (e.g. tension and shear test)

Problem:

- Mesh generation!

Our approach:

- Finite Cell Method for mesh-free structural simulation

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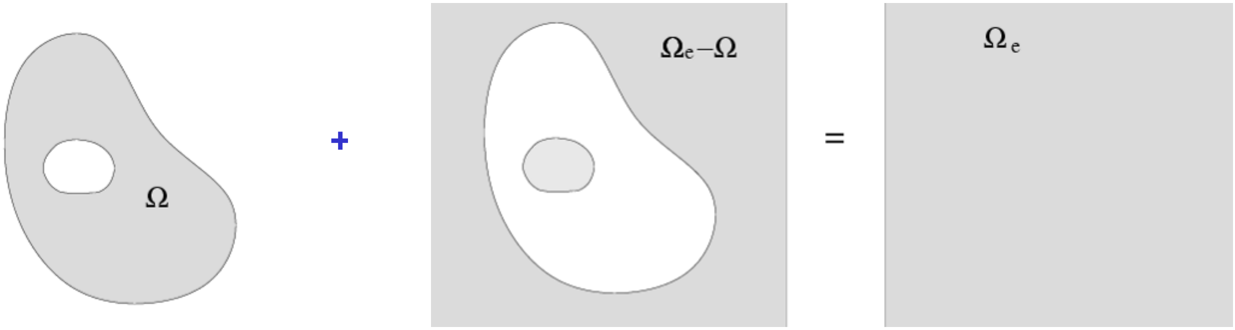
Finite Cell Method (FCM): The basic formulation



$$\mathcal{B}(\mathbf{u},\mathbf{v}) = \mathcal{F}(\mathbf{v})$$

$$\mathcal{B}(\mathbf{u},\mathbf{v}) = \int_{\Omega} [\mathbf{L}\,\mathbf{v}]^T \mathbf{C}\, [\mathbf{L}\,\mathbf{u}]\, d\Omega$$

$$\mathcal{F}(\mathbf{v}) = \int_{\Omega} \mathbf{v}^T \mathbf{f}\, dx\, dy + \int_{\Gamma_N} \mathbf{v}^T \mathbf{t}_N\, d\Gamma$$

exact solution: $\mathbf{u}_{ex}$

$$\mathcal{B}_e(\mathbf{u},\mathbf{v}) = \mathcal{F}_e(\mathbf{v})$$

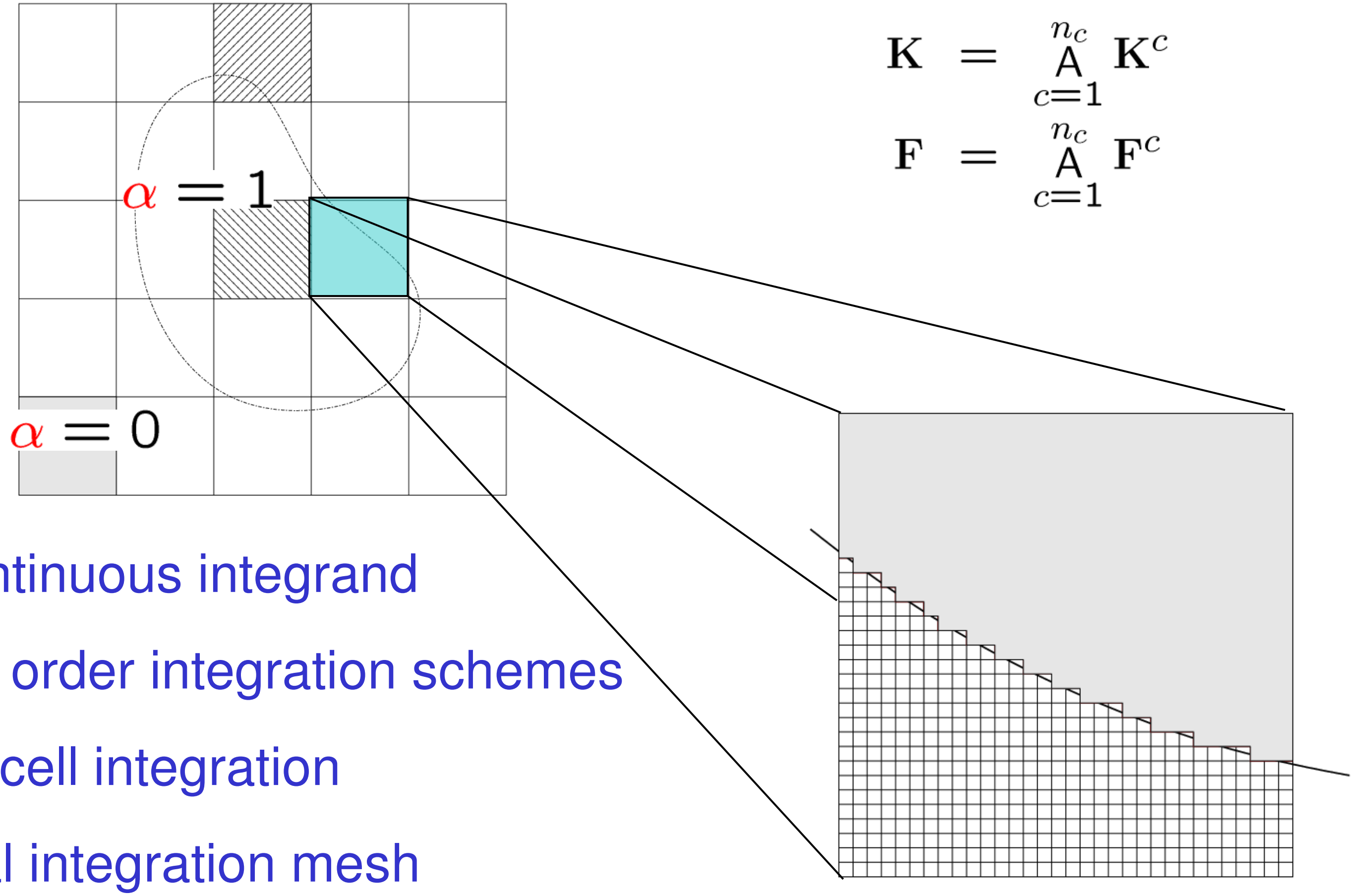
$$\mathcal{B}_e(\mathbf{u},\mathbf{v}) = \int_{\Omega_e} [\mathbf{L}\,\mathbf{v}]^T \mathbf{C}_e\, [\mathbf{L}\,\mathbf{u}]\, d\Omega$$

$$\mathbf{C}_e = \begin{cases} \mathbf{C} & \text{in } \Omega \\ \mathbf{C}^2 & \text{in } \Omega_e \setminus \Omega \end{cases}$$

exact solution: $\mathbf{u}_{e,ex}$

'zero extension': $\mathcal{B}_e(\mathbf{u},\mathbf{v}) = \int_{\Omega_e} [\mathbf{L}\,\mathbf{v}]^T \alpha\mathbf{C}\, [\mathbf{L}\,\mathbf{u}]\, d\Omega \qquad \alpha = \begin{cases} 1.0 \text{ in } \Omega \\ 0.0 \text{ in } \Omega_e \setminus \Omega \end{cases}$

- A ‘cell’ is a finite element on the *extended domain* which needs yet *not conform* with the *boundary of the original domain*
- Discretization by finite element shape functions on ‘cells’



$$\mathbf{K} = \mathop{\mathsf{A}}_{c=1}^{n_c} \mathbf{K}^c$$

$$\mathbf{F} = \mathop{\mathsf{A}}_{c=1}^{n_c} \mathbf{F}^c$$

Discontinuous integrand

- High order integration schemes
- Sub-cell integration
- Local integration mesh

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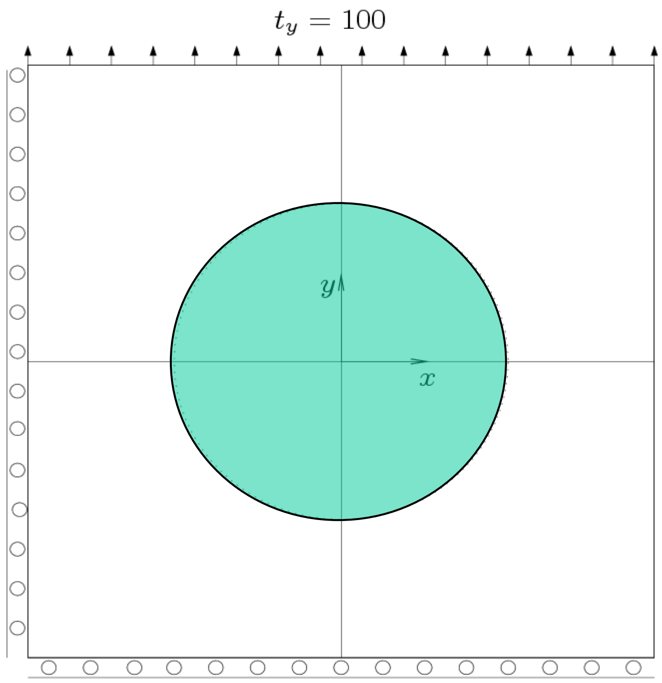
Assumptions:

- The structure consists of (homogeneous or inhomogeneous) material with (uniformly or randomly) distributed voids on the microscale
- Singularities do *not* play a dominant role on the microscale

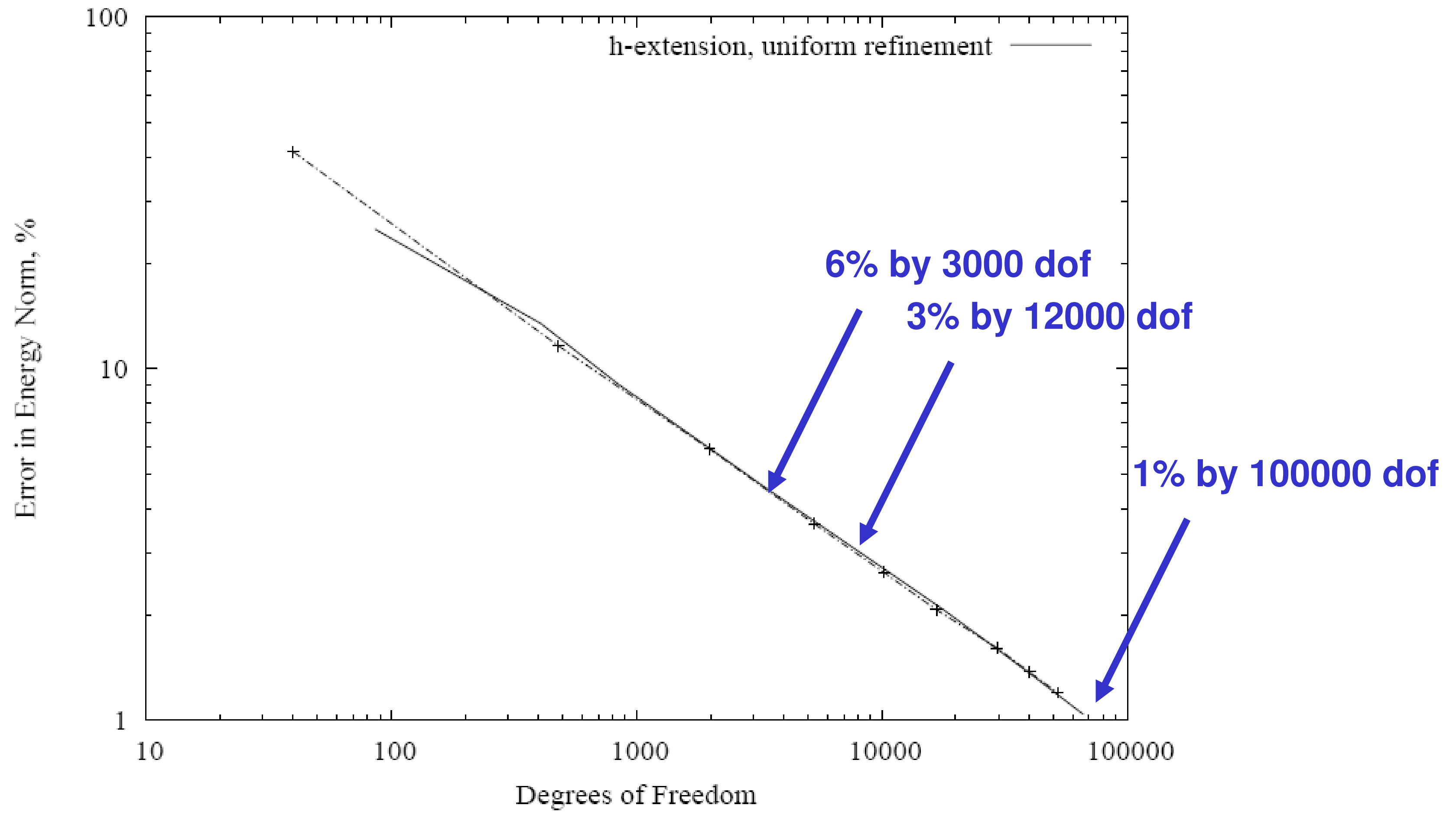
Question:

- What is, for a given desired energy norm error, the ratio of degrees of freedom between FCM and classical FEM (h-extension) ?

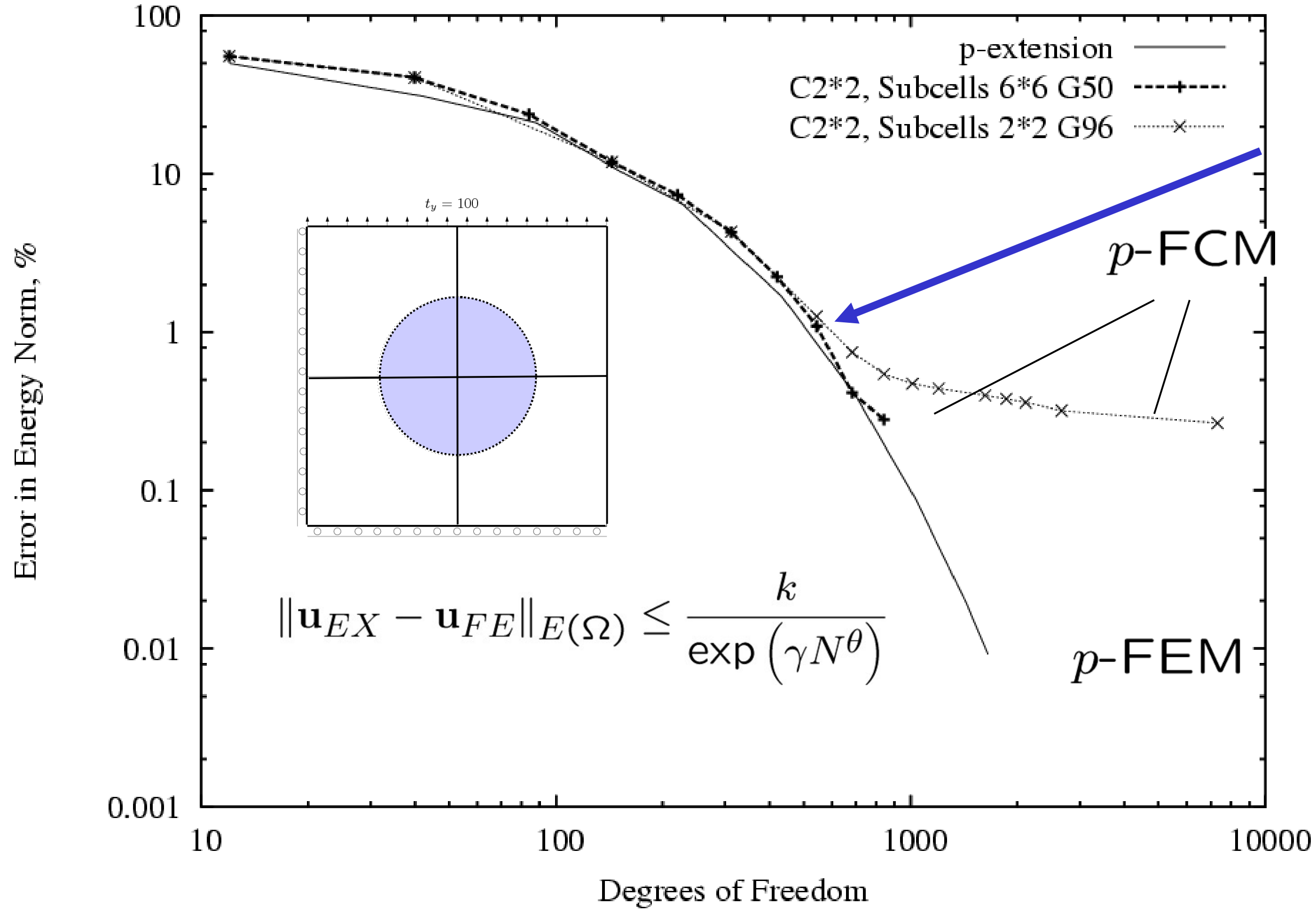
Model problem for investigating the energy error per unit pore



h-extension

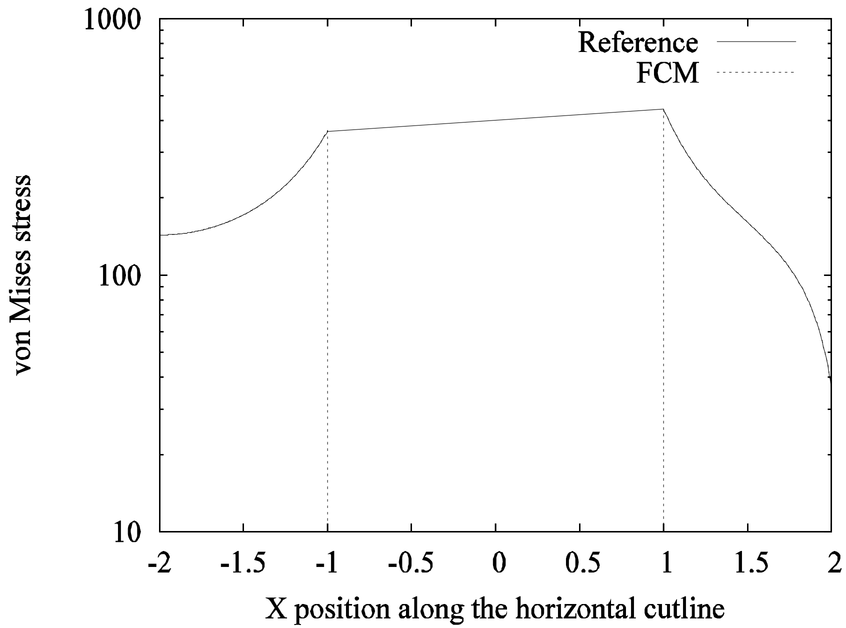

100
10
1
10
100
1000
10000
100000
Error in Energy Norm, %
Degrees of Freedom
h-extension, uniform refinement
6% by 3000 dof
3% by 12000 dof
1% by 100000 dof

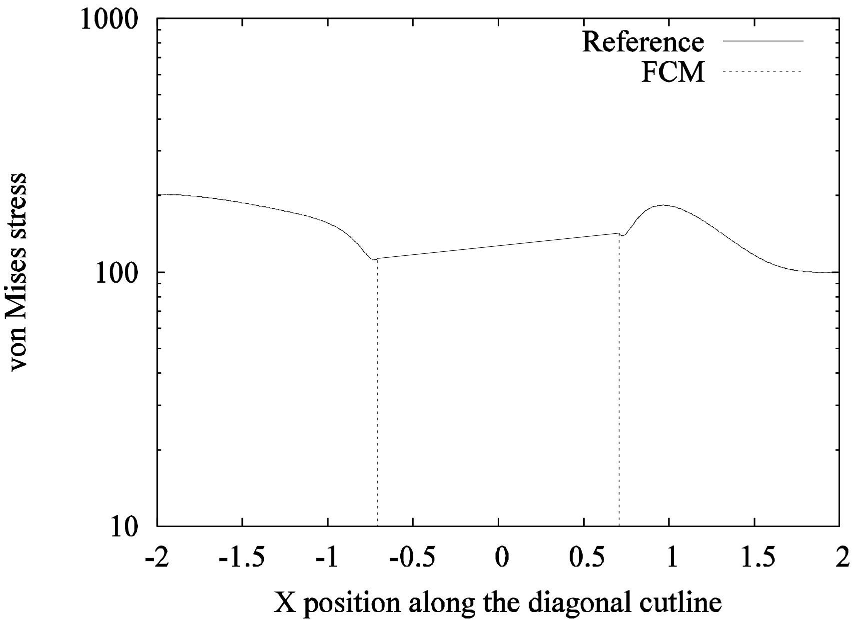
Finite Cell Method: 4 cells, *p*-extension



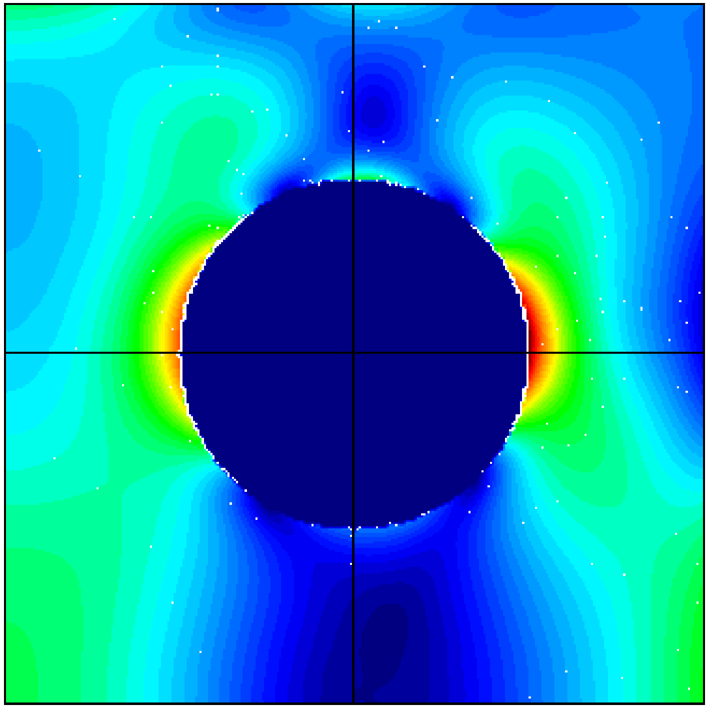
1% by 500 dof

von Mises stress along the cut-lines

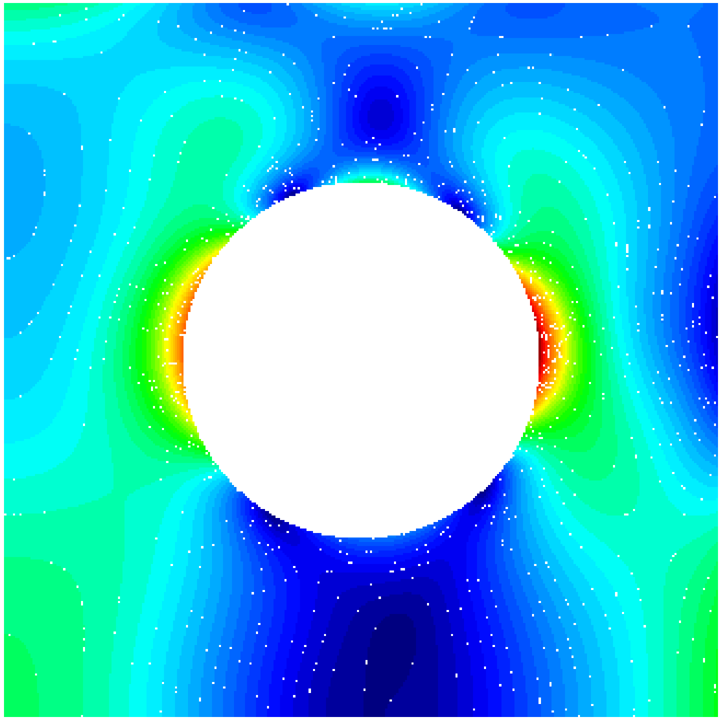




von Mises contours



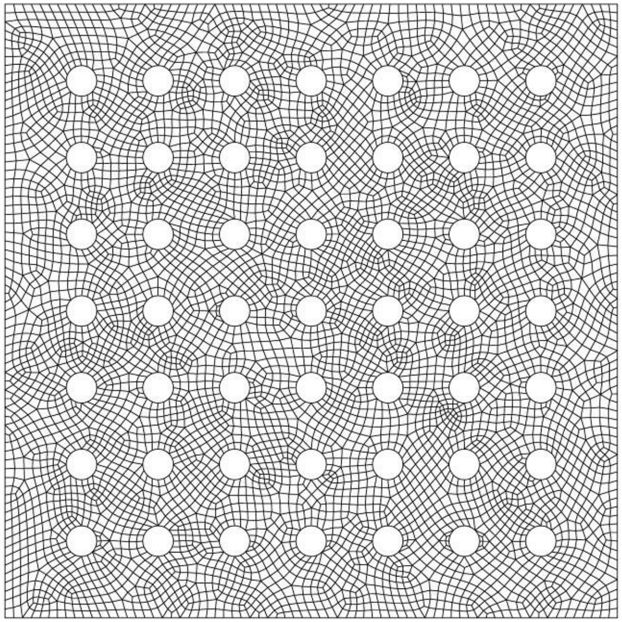
Finite Cell method



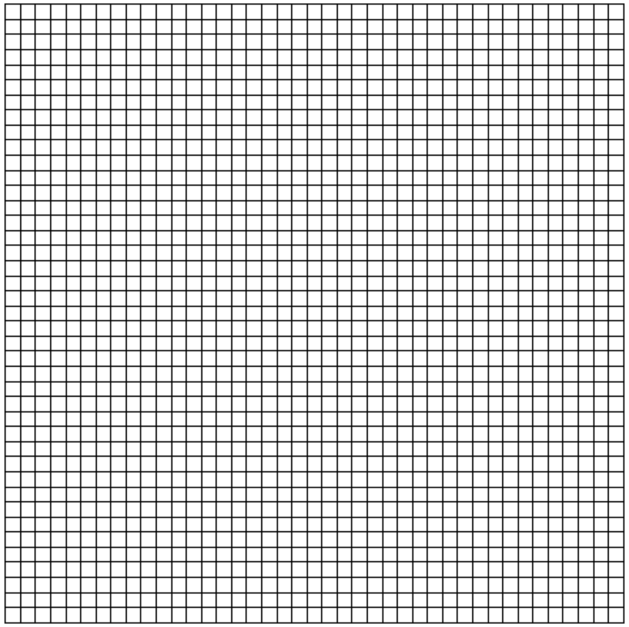
Reference Solution (p-FEM)

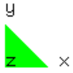
Tensile plate with many inclusions

Finite element mesh

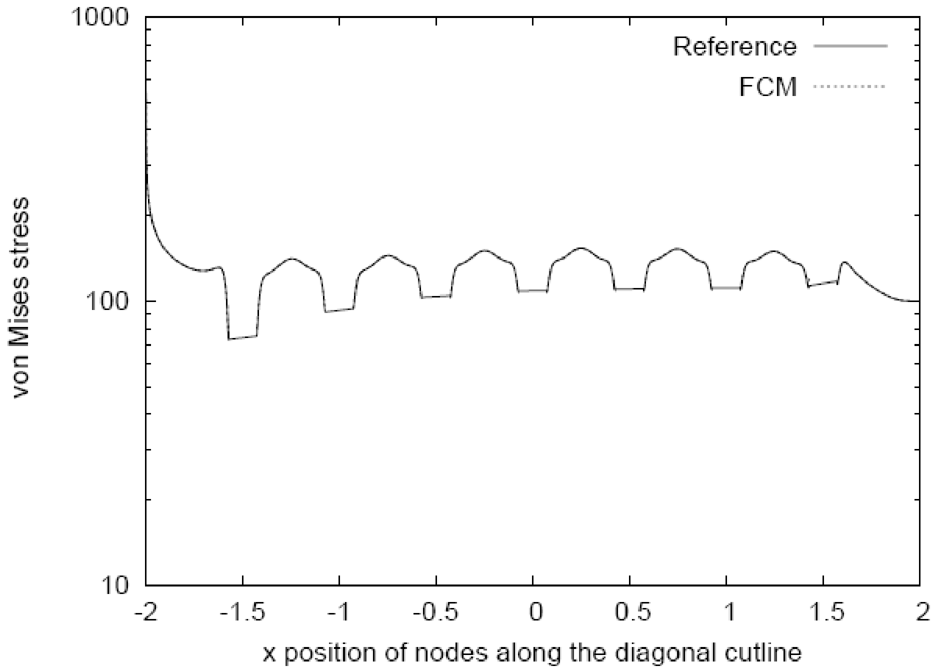


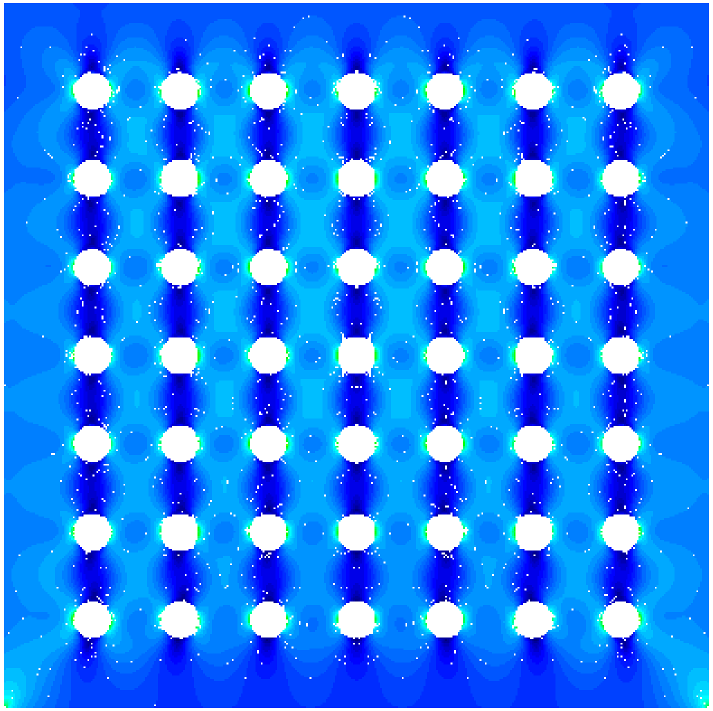
Finite cell mesh





Tensile plate with many inclusions





Error in von Mises stresses nowhere larger than 1%

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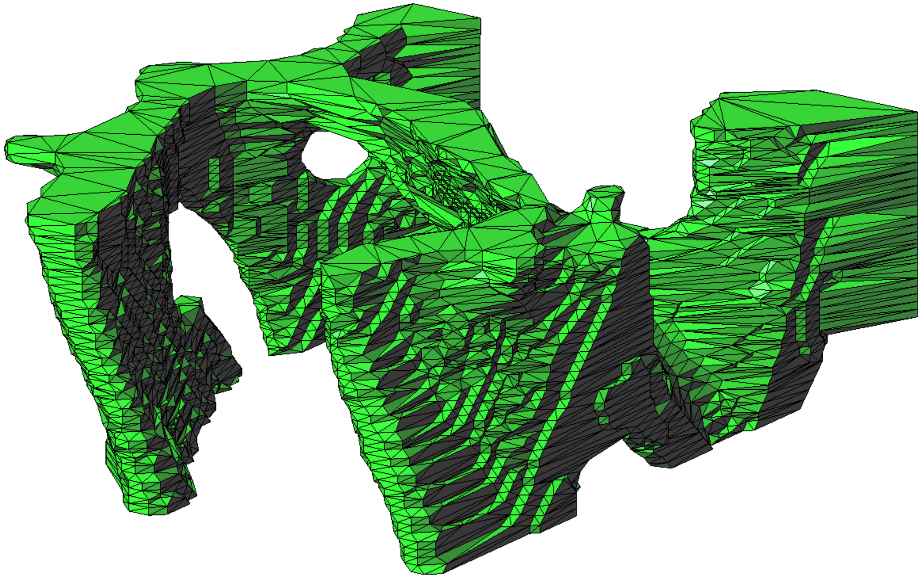
A 3D-example: X-ray MicroCT examination of human bone



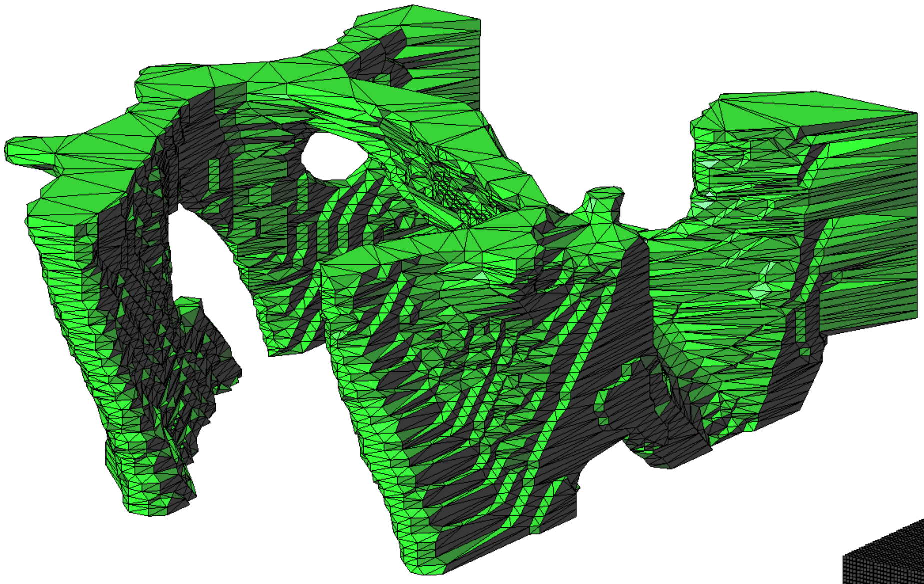
Human bone biopsy, taken from the femoral neck

Geometry:

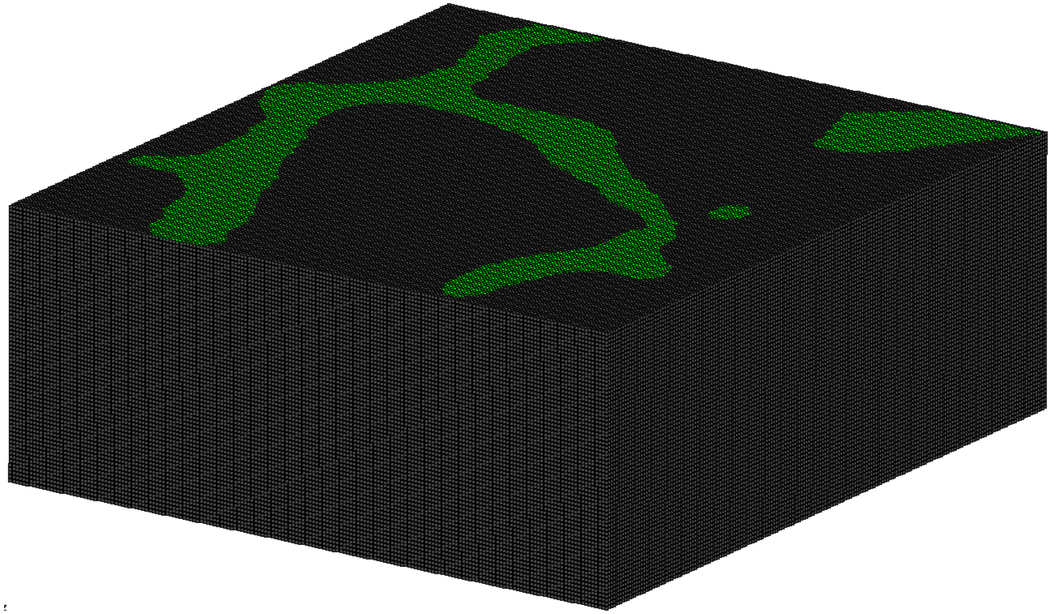
$$10\text{mm} \times 6\text{mm} \times 5\text{mm}$$

Baruffaldi, Perilli, Bn_326, from: *The BEL Repository*, http: //www.techno.ior.it/VRLAB
Instituti Ortopedici Rizzoli, Bologna, Italy

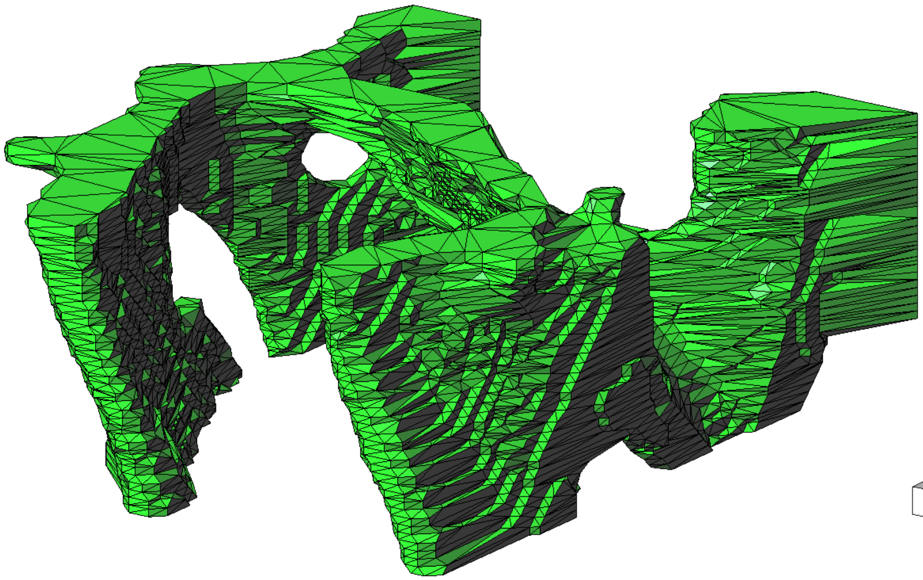
Voxel model



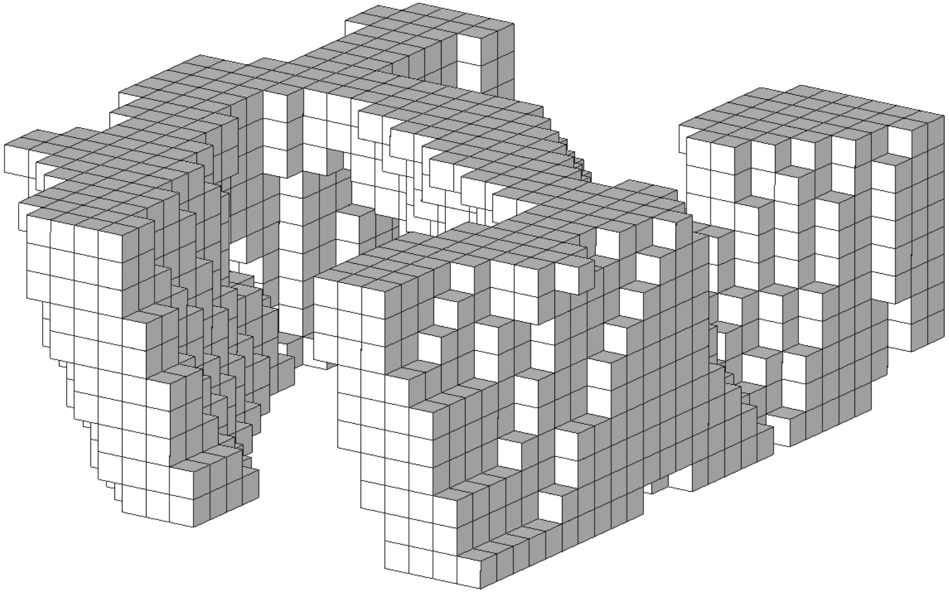
190 x 176 x 71 voxel



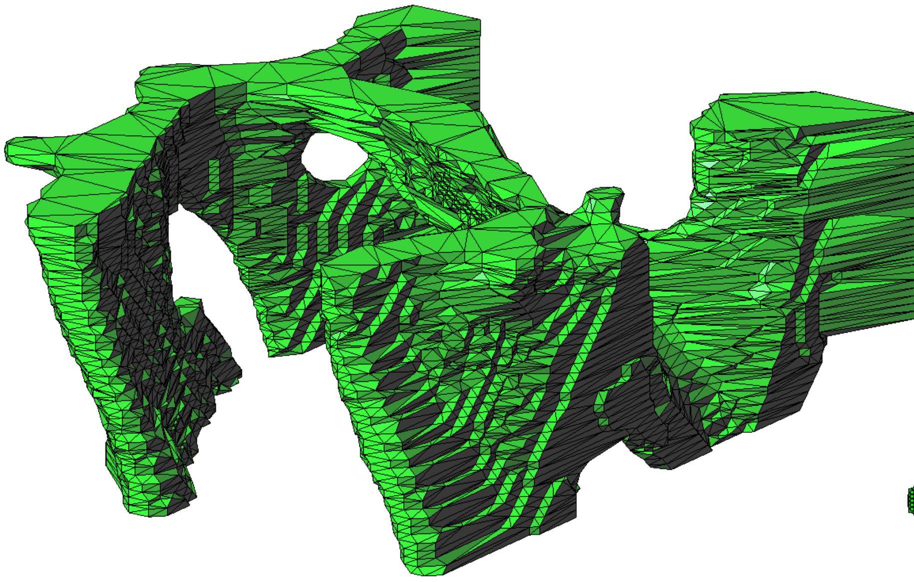
Finite Cells



2412 hexahedrals (cells)



Voxel model



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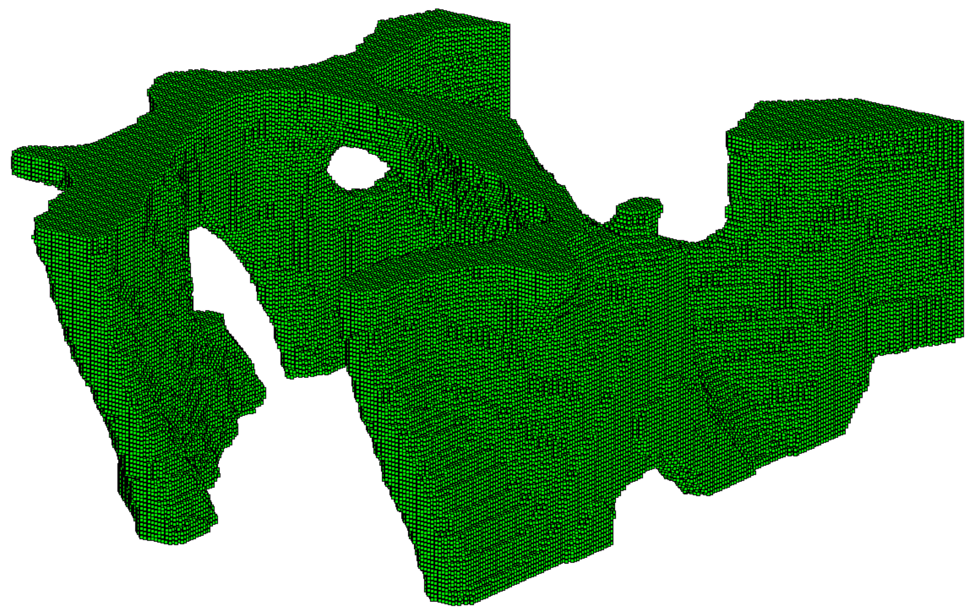
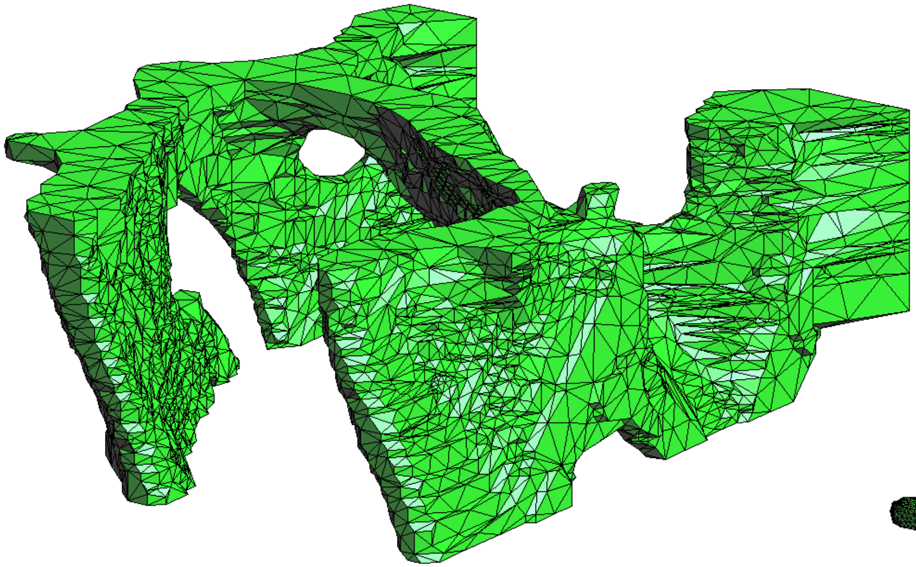
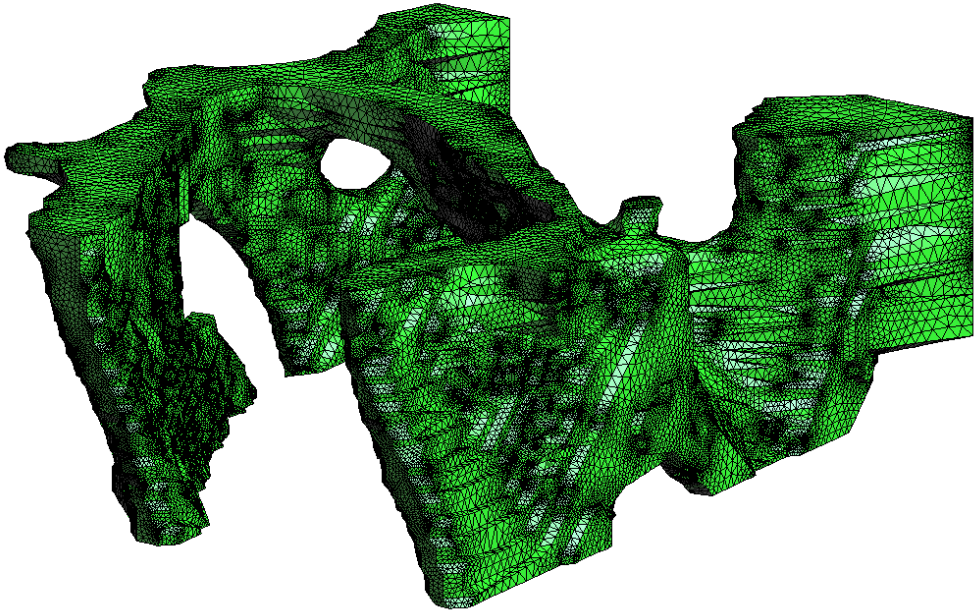


Reference solution based on the FEM (Abaqus, Netgen)



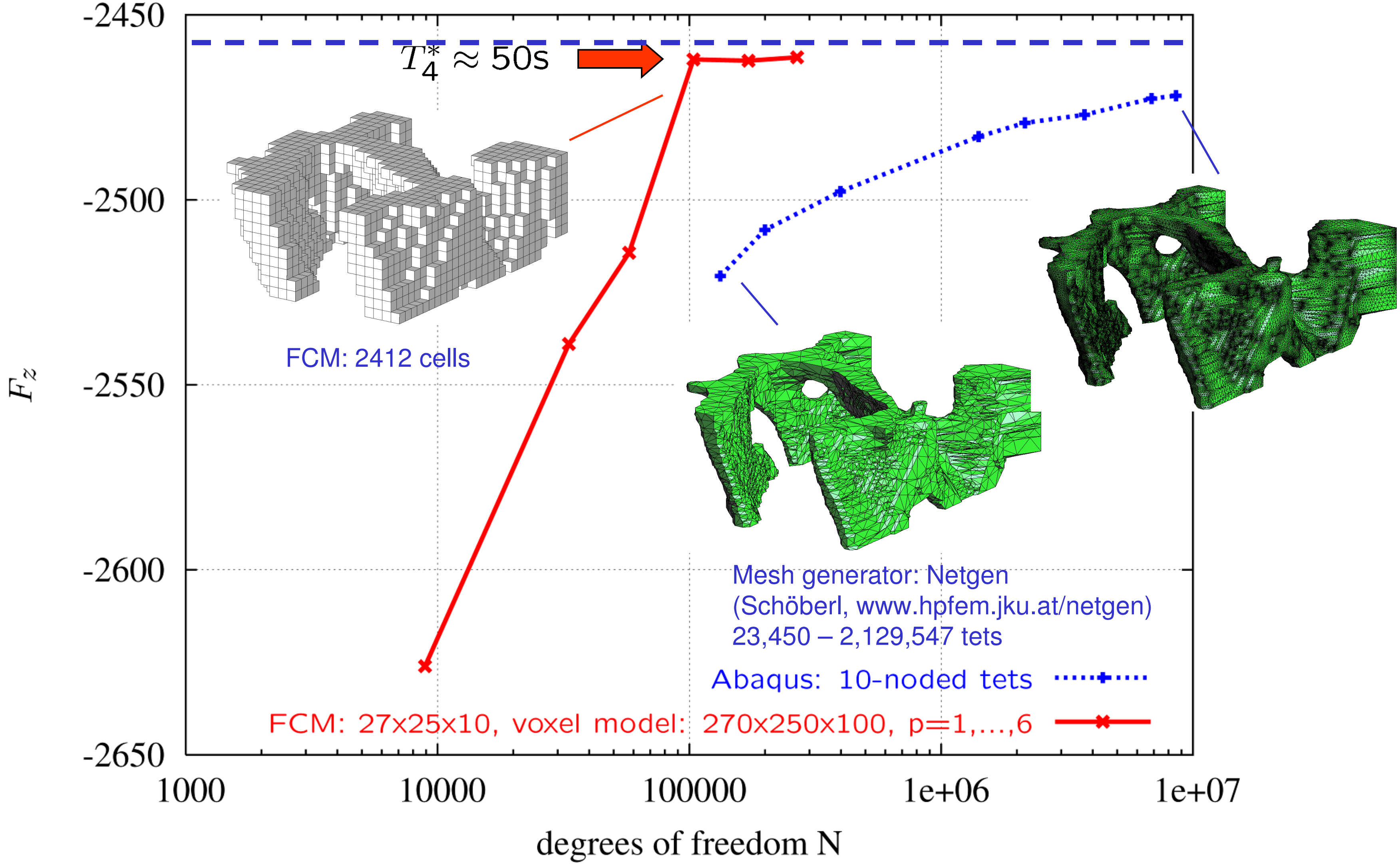


Netgen
(Schöberl, www.hpfem.jku.at/netgen)

23,450 – 2,129,547

quadratic tetrahedral
(10-noded) elements

*Intel® Xeon® E5405 (2,00 GHz, 1.333 MHz, 2 x 6 MB, Quad-Core)

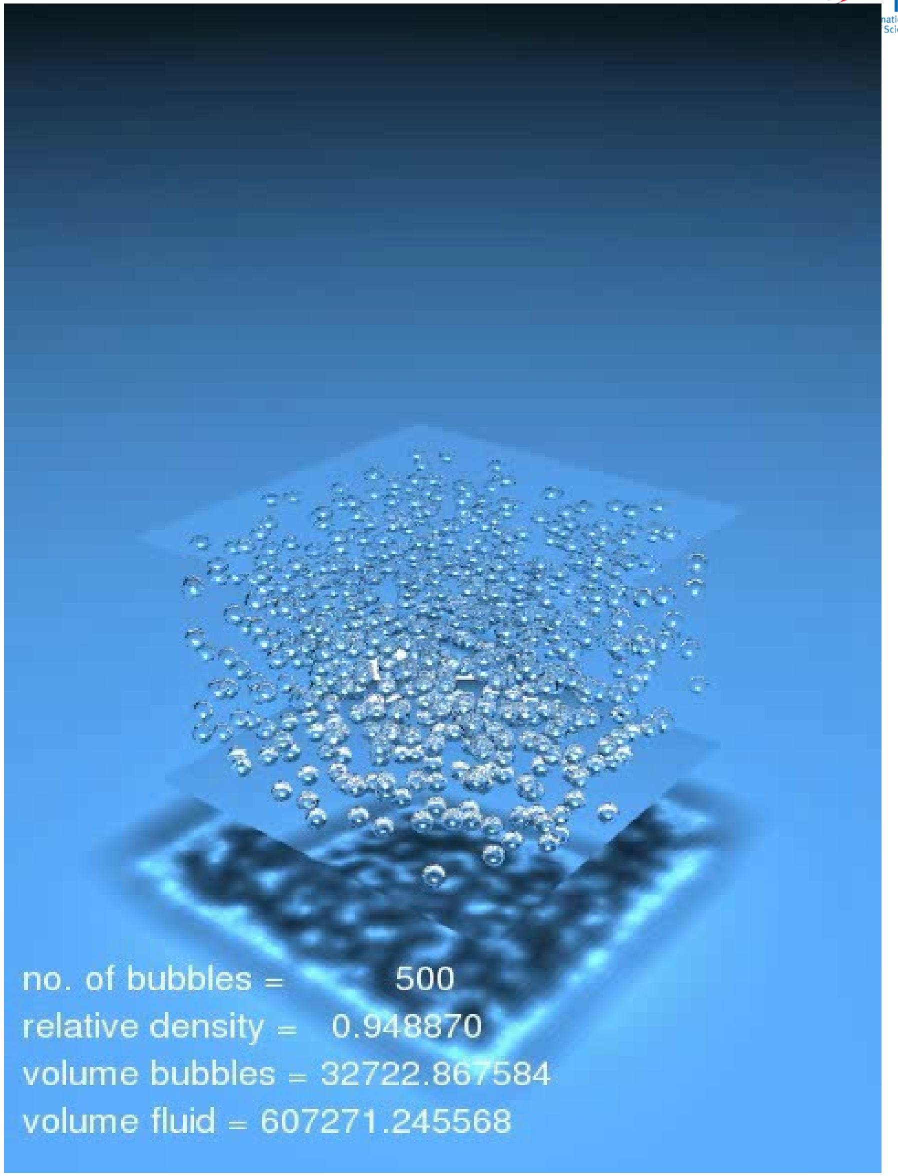


Foam generation with LBM

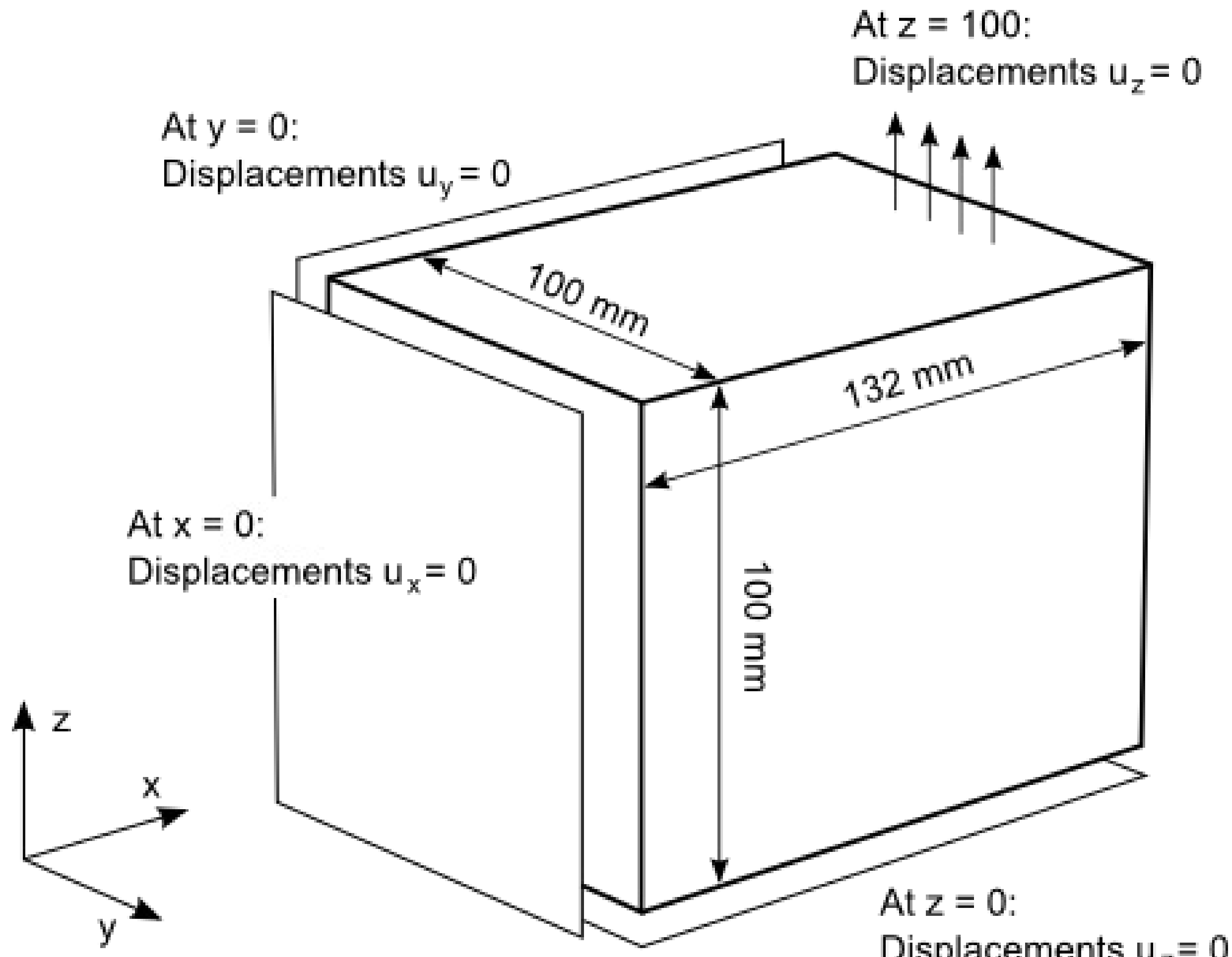
From:

C. Körner, M. Thies, T. Hofmann, N. Thürey, and U. Rüde:

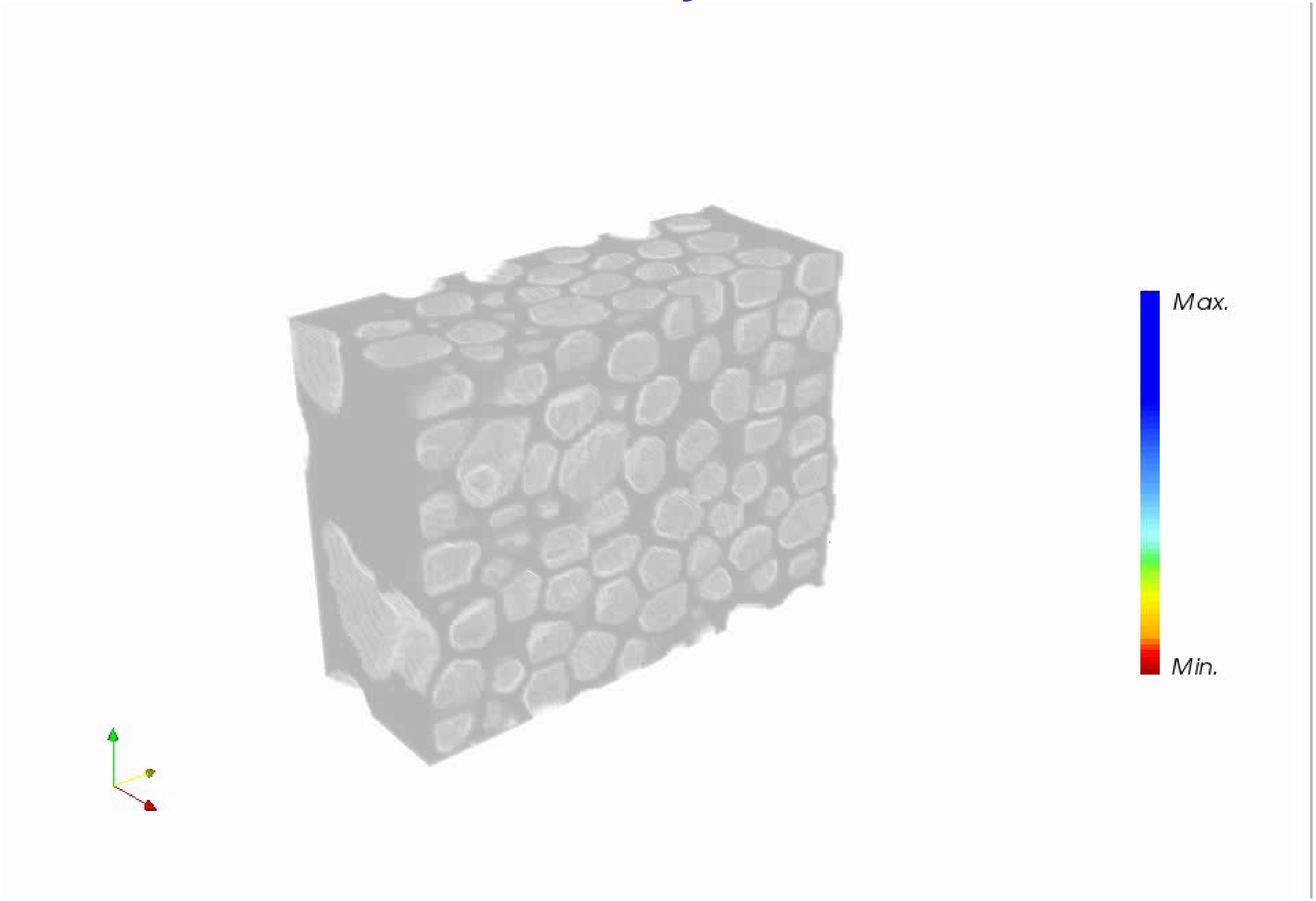
Journal of Statistical Physics, Vol 121, 2005



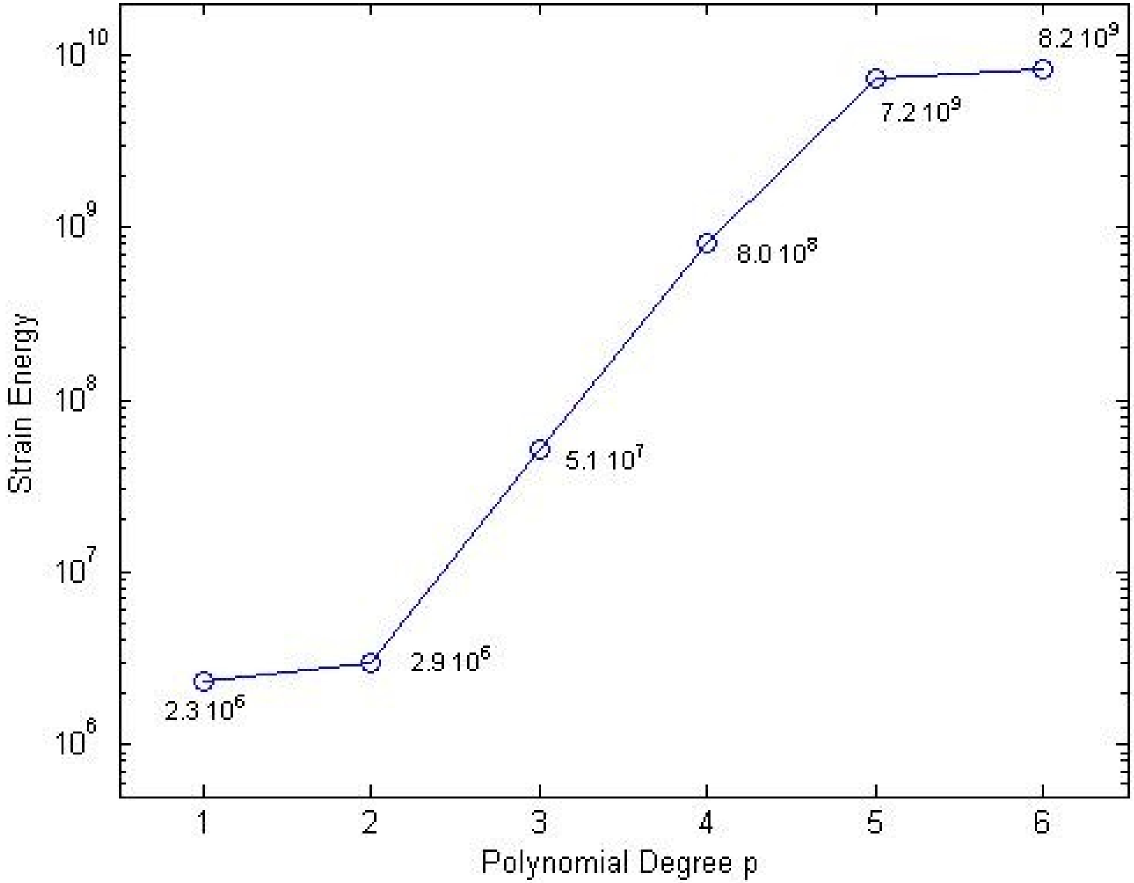
Uniaxial tension test



Structural analysis with FCM



Strain Energy as measure for Young's modulus



Computational effort for p=6:

- 770 sec on Intel quadcore-processor workstation
- No time for meshing

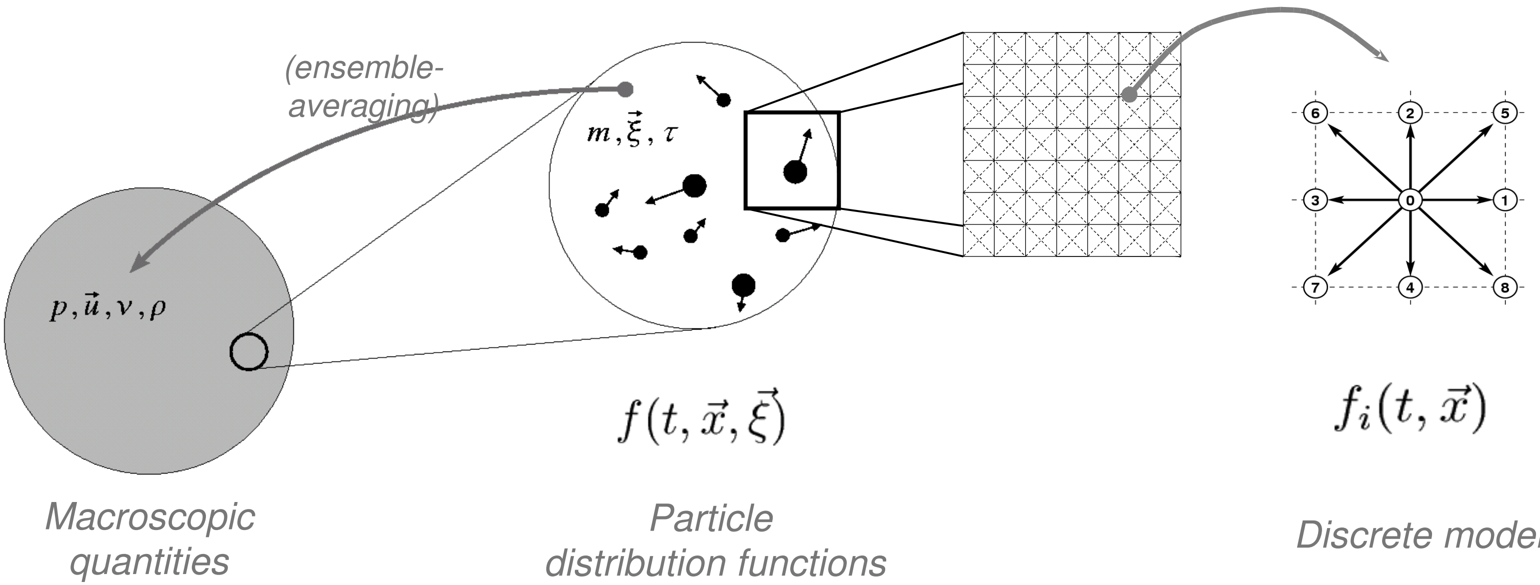
Summary

- Macroscopic mechanical properties of porous media, e.g. foam like structures are determined by their microscopic geometric structure
- The Lattice Boltzmann Method with free surface flow allows simulation of the formation of the microscopic structure of metal foams
- With the High order Finite Cell Method geometrically complex structures can be simulated without meshing

Future work

- More investigations to verification and validation necessary
- Extension of the method to obtain nonlinear material properties
- Combination with stochastic analysis

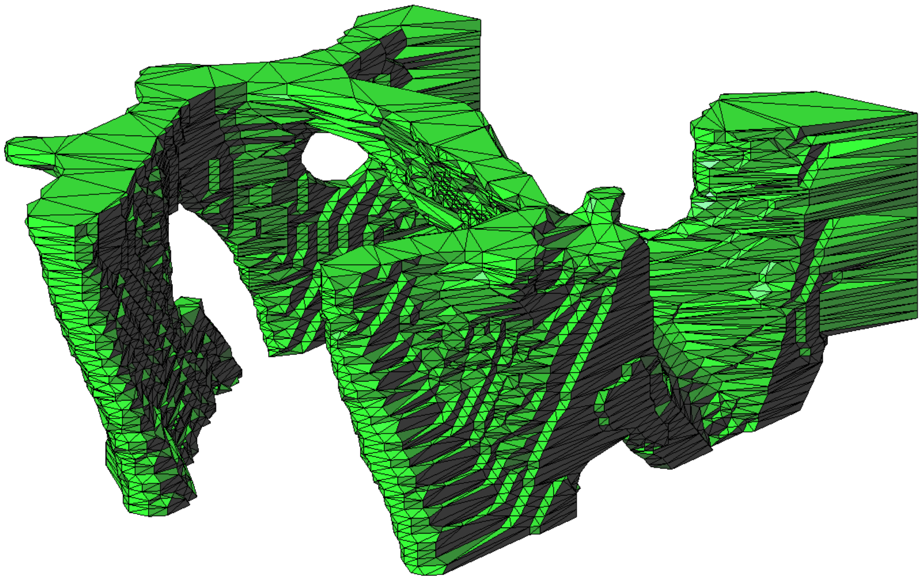
$$\frac{\partial f}{\partial t} + \vec{\xi}\,\frac{\partial f}{\partial \vec{x}} + \vec{F}\,\frac{\partial f}{\partial \vec{\xi}} = \Omega(f, f)$$



Multiple-relaxation-time lattice-Boltzmann equation

$$\underbrace{f_i(t+\Delta t, \vec{x}+\vec{\xi}_i\Delta t) - f_i(t,\vec{x})}_{\substack{\text{Propagation:}\\ \text{phase space } \mathcal{F}}} = -\,\mathbf{M}^{-1}\,\underbrace{\hat{\mathbf{S}}\left[m_i(t,\vec{x}) - m_i^{(0)}(t,\vec{x})\right]}_{\substack{\text{Collision:}\\ \text{moment space } \mathcal{M}}}$$

Finite Cells



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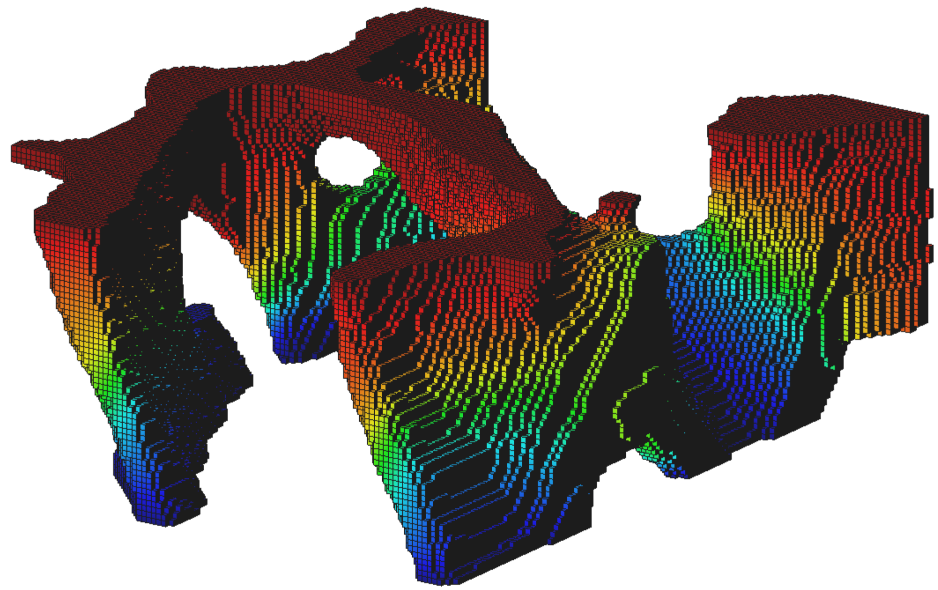

linear elastic material
behaviour assumed

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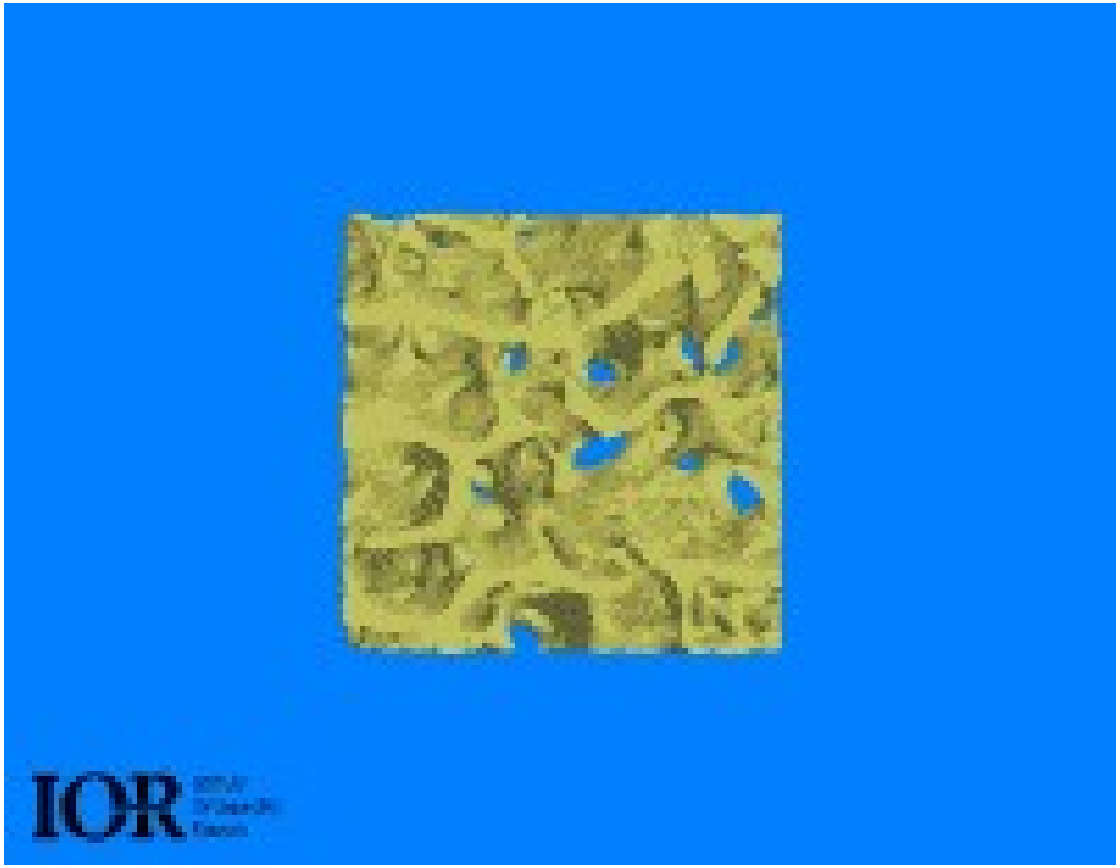

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Some related methods

- Embedded domain method
- Fictitious domain method
- Meshfree/meshless methods
- Level set methods

in general for low order approximations, only.

Also similarities to

- Extended/enhanced finite element methods
- Generalized finite element methods
- Unfitted discontinuous Galerkin method