# A Sketch of the Rudiments of Set Theory

Modern mathematics is based on the notion of a *set*. We all know intuitively what a set is supposed to be: the set of people in the room, the set of positive real numbers, the set of rational numbers, the set of pages in our textbook, etc. But what sort of thing is a general, arbitrary set $A$? Informally, we tend to think of a set $A$ as being a collection of distinct objects $a \in A$. The last sentence is not a definition of the term *set*, because we don't know what a collection or an object is. In fact, we will not define the term *set*. Instead, the term *set* and the *belonging relation* $\in$ are the undefined notions upon which the towering edifice of modern mathematics sits. So at the beginning, all we know is that if $a$ and $A$ are these things called sets, then it may be that $a \in A$, in which case we say that $a$ belongs to $A$ or $a$ is an element of $A$. This symbolism is the formal analogue of our intuition about sets being collections of objects. Note that the elements of a set are themselves sets. In order to proceed, we need some *axioms* that tell us about sets and belonging. Our first axiom tells us something about the belonging relation $\in$ by relating it to the basic notion of equality.

**Axiom 1** (Axiom of Extension)**.** *Two sets are equal if and only if they have the same elements. In symbols:*

$$[A = B] \iff [x \in A \iff x \in B]$$

The Axiom of Extension tells us something that surely corresponds to our intuition about sets: a set is determined by its elements. We write $A \subset B$ if $x \in A \Rightarrow x \in B$, and we say that $A$ is a *subset* of $B$. The Axiom of Extension implies that $A = B$ if and only if $A \subset B$ and $B \subset A$. Hence, if you are trying to prove that two sets are equal, you should show that each is a subset of the other.

The remaining axioms of set theory that we will consider all allow us to make new sets out of old ones. The first of these is

**Axiom 2** (Axiom of Specification)**.** *If $A$ is a set and $S(x)$ is a logical condition, then there is a set $B$ whose elements are exactly those $x \in A$ such that $S(x)$ is true.*

Clearly the set $B$ furnished by the Axiom of Specification is a subset of $A$, and we write $B = \{x \in A \mid S(x)\}$. As an informal example, suppose that $A$ is the set of people in the room, and $S(x)$ is the sentence "$x$ is not a student".

Then $B = \{x \in A \mid S(x)\} = \{\text{Scott}\}$. Note that $B \neq \text{Scott}$, but rather $B$ is the set containing the single element Scott.

At this point we must face the unsettling possibility that there might not be any sets. At any rate, nothing we have said so far (besides our intuition) guarantees that even a single set exists. So to get things rolling, let's assume that at least one set $A$ exists.

**Claim.** *There exists a unique set with no elements. It is denoted $\emptyset$ and called the* empty set*.*

*Proof.* Indeed, if $A$ is a set (we just assumed that there *is* at least one set), then by the Axiom of Specification the set $\emptyset := \{x \in A \mid x \neq x\}$ exists, and this set clearly has no elements. Any other set with no elements must equal this set by the Axiom of Extension. □

In order to be assured of the existence of more sets than just the empty set, we need a new axiom.

**Axiom 3** (Axiom of Pairing)**.** *If $a$ and $b$ are sets, then there exists a set whose elements are exactly $a$ and $b$. This set is called the* unordered pair *of $a$ and $b$, and is denoted $\{a, b\}$.*

Note that if $a = b$, then the unordered pair $\{a, a\}$ has only one element, namely $a$. We write $\{a\}$ for this set and call it the *singleton* of $a$.

You might ask the following question: why can't we just use the Axiom of Specification to construct the unordered pair of $a$ and $b$? In other words, why not just define $\{a, b\} := \{x \mid x = a \text{ or } x = b\}$? The answer is that this would be OK provided we already knew the existence of a set $A$ containing both $a$ and $b$. Well, why not settle this once and for all by taking $A$ to be the "set of all sets"? It turns out that no such set exists, as the following famous paradox shows.

**Claim** (Russell's Paradox)**.** *There is no set of all sets.*

*Proof.* Suppose on the contrary that $A$ is the set of all sets. Then by the Axiom of Specification, the following set $B$ exists: $B = \{x \in A \mid x \notin x\}$. In words, $B$ consists of the sets that are not elements of themselves. Now let's ask ourselves whether $B \in B$? Well, if $B \in B$, then by the defining property of $B$ we see that $B \notin B$, a contradiction. On the other hand, if $B \notin B$, then since $B \in A$, we see that $B \in B$, another contradiction. The only way of avoiding these contradictions is to concede that $A$ must not be a set after all. □

Russell's Paradox demonstrates that we need to be careful when defining sets if we want to avoid logical contradictions. Even if a proposed set seems intuitively reasonable, we need to justify its existence by means of the axioms of set theory. The remaining axioms of set theory tell us under what circumstances we are justified in asserting the existence of a new set. Like the Axiom of Pairing, they would be consequences of the Axiom of Specification if there were a "set of all sets".

**Axiom 4** (Axiom of Union)**.** *If $A$ and $B$ are sets, then there exists a set $A \cup B$, called the* union *of $A$ and $B$, whose elements are exactly the elements of $A$ and the elements of $B$:*

$$A \cup B = \{x \mid x \in A \textit{ or } x \in B\}.$$

(DX) Prove that $A \cup \emptyset = A$, $A \cup B = B \cup A$, $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cup A = A$, and $[A \subset B \iff A \cup B = B]$.

We don't need a special axiom to get the *intersection* of two sets $A$ and $B$. Indeed, by the Axiom of Specification we can define $A \cap B := \{x \in A \mid x \in B\}$.

(DX) Prove that $A \cap \emptyset = \emptyset$, $A \cap B = B \cap A$, $A \cap (B \cap C) = (A \cap B) \cap C$, $A \cap A = A$, and $[A \subset B \iff A \cap B = A]$.

(DX) Prove the following *distributive laws*:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad \text{and} \qquad A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

We can also define the *difference* between two sets $A$ and $B$ without invoking a special axiom: $A - B := \{x \in A \mid x \notin B\}$. If all sets under discussion happen to be subsets of a single set $E$, then $E - A$ is often denoted $A'$ and called the *complement* of $A$.

(DX) Prove the De Morgan Laws: $(A \cup B)' = A' \cap B', (A \cap B)' = A' \cup B'$.

**Axiom 5** (Axiom of Powers)**.** *If $A$ is a set, then there exists a set $\mathcal{P}(A)$ whose elements are precisely the subsets of $A$. $\mathcal{P}(A)$ is called the* power set *of $A$. In symbols, $B \in \mathcal{P}(A) \iff B \subset A$.*

One of the most important and familiar operations on sets in the formation of ordered pairs. For instance, in calculus we are constantly considering ordered pairs of real numbers $(x, y)$. Intuitively, we know what this should be as a set: it should consist of the two elements $x$ and $y$, but somehow $x$ should be "first". Given $x$ and $y$, the Axiom of Pairing allows us to make the unordered pair $\{x, y\}$, but this is the same as the unordered pair $\{y, x\}$ by the Axiom of Extension. How can we use the axioms of set theory to capture the notion of order?

**Definition.** *If $a$ and $b$ are sets, then the* ordered pair $(a, b)$ *is defined to be the unordered pair* $\{\{a\}, \{a, b\}\}$.

**Claim.** *If $(a, b) = (x, y)$ then $a = x$ and $b = y$.*

*Proof.* You will do it on the first problem set. □

If $A$ and $B$ are sets, then we would like to consider the totality of all ordered pairs $(a, b)$ where $a \in A$ and $b \in B$. We now have enough axioms to show that such a set exists.

**Claim.** *If $A$ and $B$ are sets, then there exists a set $A \times B$ whose elements are exactly the ordered pairs $(a, b)$ for $a \in A$ and $b \in B$. The set $A \times B$ is called the* cartesian product *of $A$ and $B$.*

*Proof.* Suppose that $a \in A$ and $b \in B$. Then by the definition of union $a \in A \cup B$ and $b \in A \cup B$, so $\{a, b\} \subset A \cup B$. Also, $\{a\} \subset A \cup B$. By the definition of the power set, this means that both $\{a\}$ and $\{a, b\}$ are elements of $\mathcal{P}(A \cup B)$. But then $(a, b) = \{\{a\}, \{a, b\}\} \subset \mathcal{P}(A \cup B)$. Again by the definition of the power set, this means that $(a, b) \in \mathcal{P}(\mathcal{P}(A \cup B))$. Hence, we have found a single set $\mathcal{P}(\mathcal{P}(A \cup B))$ that contains all of the ordered pairs $(a, b)$ with $a \in A$ and $b \in B$. By the Axiom of Specification we can define

$$A \times B = \{x \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid x = (a, b) \text{ for some } a \in A, b \in B\}.$$

□

Note that the difficulty in this proof was in constructing a set large enough to contain all of the ordered pairs. We won't go further with these types of constructions and arguments, but I hope you are convinced that all of the mathematical constructs you are familiar with (and many you aren't) are really just sets that can be constructed through careful thought and a few well-chosen axioms.

Having axiomatically constructed the cartesian product, we are within striking distance of providing a rigorous definition of a *function*, which is perhaps the central concept of modern mathematics. But first we need to mention something more general: *relations*

**Definition.** *If $X$ and $Y$ are sets, then a* relation*, $R$, from $X$ to $Y$ is a subset of the cartesian product $X \times Y$.*

As an informal example, consider the dog-owning relationship $R \subset P \times D$, where $P$ is the set of people and $D$ is the set of dogs. Then $(p, d) \in R$ if and only if $p$ is a person, $d$ is a dog, and $p$ owns $d$.

**Definition.** *Suppose that $R \subset X \times X$ is a relation from a set $X$ to itself. Then we say that $R$ is*

*i)* reflexive *if $(a, a) \in R$ for every $a \in X$*

*ii)* transitive *if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$*

*iii)* symmetric *if $(a, b) \in R$ implies $(b, a) \in R$.*

*If $R$ is reflexive, transitive, and symmetric, then we say that $R$ is an* equivalence relation *on $X$.*

Equivalence relations will be extremely important to us later in the course, but for now we move on to the more familiar topic of functions.

**Definition.** *A* function *from $X$ to $Y$ is a relation $f \subset X \times Y$ such that for every $x \in X$ there is a unique $y \in Y$ such that $(x, y) \in f$.*

In practice, we don't generally think of a function $f$ as a set, but rather as a rule that assigns to every $x \in X$ an element $y \in Y$. In accordance with this dynamic conception of functions, we write $f : X \to Y$ instead of $f \subset X \times Y$ and $f(x) = y$ instead of $(x, y) \in f$.