INVERSE SCATTERING METHOD WITH VARIABLE SPECTRAL PARAMETER

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In the traditional scheme of the inverse scattering method, the spectral parameter of the auxiliary linear problem is assumed to be a constant. It is here proposed to regard the parameter as a variable quantity that satisfies an overdetermined system of differential equations which is uniquely determined by the auxiliary linear problem. The nonlinear equations that arise in such an approach contain, as a rule, an explicit dependence on the coordinates. This makes it possible to construct not only the well-known equations (gravitation equation, Heisenberg equation in axial geometry, etc.) but also a number of new integrable equations that have applied significance.

Introduction

The inverse scattering method, discovered in 1967, has now become an irreplaceable tool of mathematical physics, permitting the effective investigation of numerous nonlinear partial differential equations that occur in applications. A very general formulation of this method was proposed in [1] (see also [2]).

In the framework of this formulation, nonlinear integrable equations arise as conditions of compatibility of an overdetermined system of linear equations

$$\Phi_\xi=U\Phi, \quad \Phi_\eta=V\Phi. \tag{I.1}$$

Here, U, V, and Φ are complex-valued $N \times N$ matrix functions of ξ, η, and the spectral parameter λ, which is assumed to be an arbitrary complex constant. The dependence of U and V on λ is assumed to be rational. In the general case, we have

$$U(\lambda,\xi,\eta)=u_0(\xi,\eta)+\sum_{n=1}^{N_1}\frac{u_n(\xi,\eta)}{\lambda-\lambda_n(\xi)}, \quad V(\lambda,\xi,\eta)=v_0(\xi,\eta)+\sum_{n=1}^{N_2}\frac{v_n(\xi,\eta)}{\lambda-\mu_n(\eta)}, \tag{I.2}$$

where $\lambda_n \neq \mu_n$. The situations with coincident and multiple poles are obtained from (I.2) by limiting processes. An important role is played by the polynomial case

$$U=u_0+u_1\lambda+\ldots+u_i\lambda^i, \quad V=v_0+v_1\lambda+\ldots+v_j\lambda^j. \tag{I.3}$$

The condition of compatibility of Eqs. (I.1) has the form

$$U_\eta-V_\xi+[U,V]=0. \tag{I.4}$$

Equations for the functions u_n, v_n arise from the requirement that the condition (I.4) hold identically with respect to λ. To solve these equations, we use the "dressing method," based on the Riemann–Hilbert problem on the complex plane of λ. After appropriate reductions, the majority of systems that are integrable by the inverse scattering method can be made to fit the formulated scheme.

The dressing method has been most fully developed in the case when the poles λ_n (μ_n) do not depend on the variables ξ (η). The equations that then arise have constant coefficients. In particular, if we set

$$\Phi_\xi=\frac{u}{\lambda-1}\Phi, \quad \Phi_\eta=\frac{v}{\lambda+1}\Phi, \quad u=-g_\xi g^{-1}, \quad v=g_\eta g^{-1}, \quad g=\Phi|_{\lambda=0}, \tag{I.5}$$

then for g we obtain the equation of the "principal chiral field" (see [2])

$$(g_\xi g^{-1})_\eta+(g_\eta g^{-1})_\xi=0. \tag{I.6}$$

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An equation with nearly the same form as (I.6) is

$$(\alpha g_\xi g^{-1})_\eta+(\alpha g_\eta g^{-1})_\xi=0, \tag{I.7}$$

which has variable coefficients. Here, $\alpha = \alpha(\xi, \eta)$ is a scalar function that satisfies the equation

$$\alpha_{\xi\eta}=0. \tag{I.8}$$

Let g be a symmetric real 2×2 matrix and $\alpha^2 = \det g$; then Eq. (I.7) describes the gravitational field in Einstein's theory of gravitation under the condition that the metric tensor depends on only two variables.

In the general case, Eq. (I.7) does not fit in the scheme described above. However, it was shown in [3] that one can also apply the inverse scattering method to it for arbitrary matrix dimension N; in fact, the requirement for the matrix g to be real and symmetric is not necessary. It was found in [3] that Eq. (I.7) is the condition of compatibility of the overdetermined system

$$\mathbf{D}_1\Phi=\frac{u}{\lambda-\alpha}\Phi, \quad \mathbf{D}_2\Phi=\frac{v}{\lambda+\alpha}\Phi, \tag{I.9}$$

where

$$\mathbf{D}_1=\frac{\partial}{\partial\xi}-\frac{2\alpha_\xi}{\lambda-\alpha}\lambda\frac{\partial}{\partial\lambda}, \quad \mathbf{D}_2=\frac{\partial}{\partial\eta}+\frac{2\alpha_\eta}{\lambda+\alpha}\lambda\frac{\partial}{\partial\lambda}, \quad u=-\alpha g_\xi g^{-1}, \quad v=\alpha g_\eta g^{-1}, \quad g=\Phi|_{\lambda=0}. \tag{I.10}$$

The operators $\mathbf{D}_1, \mathbf{D}_2$ commute if the condition (I.8) is satisfied. In [3], the dressing method was developed for Eq. (I.7), and classes of exact solutions were constructed. A different approach to Eq. (I.7) is also possible. As was shown in [4,5], it can be extracted from the system (I.4) if one assumes that λ is a certain function of ξ and η and the additional complex constant z:

$$\lambda(z, \xi, \eta)=[h-f-z+\sqrt{(z-2h)(z+2f)}]/[f+h], \quad f=f(\xi), \quad h=h(\eta), \quad \alpha=f+h, \tag{I.11}$$

where f and h are arbitrary functions. In what follows, we shall call z the hidden spectral parameter. We note that in [5] the second approach was developed more deeply. In [4], only the Lax representation for the gravitational equations was found. In [5] for systems of the type (I.7) the dressing method was constructed, exact solutions found, and the conservation laws investigated.

In the present paper, we shall show that the device employed in [4,5] is equivalent to the device of [3] and can be developed to the level of a general method, which we shall call the inverse scattering method with variable spectral parameter. The traditional inverse scattering method [1,2] is a special case of this method.

Equation (I.7) is an example of an integrable equation having variable coefficients. This example is by no means unique. Many such equations are given, for example, in [6]. The simplest way of constructing such equations was already noted in [1]. In Eqs. (I.1) and (I.2) one can assume that the poles λ_n are arbitrary functions of the variable ξ and the poles μ_n are also arbitrary functions of the variable η. Then the conditions (I.4) give equations with variable coefficients. It was shown in [7] that in this way one can arrive at Eq. (I.7) in the special case $N = 2$, $g = g^{tr}$, which corresponds to the applications in the theory of gravitation.

It follows from [3,8] that the dressing method in the case of "moving poles" is significantly modified compared with the case of constant poles described in [1,2].

The proposed method provides a possibility of studying systematically integrable nonlinear partial differential equations with variable coefficients. One can show that to every equation with constant coefficients to which the scheme of the inverse scattering method [1] briefly described above applies there corresponds an entire class of variable coefficients that are amenable to the new method. We shall call equations of this class deformations of the original equation. In this class we shall also include equations integrable in the framework of the scheme of [1] with "moving poles" $\lambda_n(\xi)$, $\mu_n(\eta)$. Thus, Eq. (I.7) is a deformation of Eq. (I.6) of the principal chiral field. One further example of this kind was already known — the Heisenberg equation that models the evolution of cylindrically symmetric configurations of the magnetization of an isotropic magnet [9]:

$$S_t = S \times \left(S_{rr} + \frac{1}{r} S_r\right), \quad S^2 = 1, \quad S = (S_1, S_2, S_3). \tag{I.12}$$

This equation is a deformation of the Heisenberg equation used in the one-dimensional situation:

$$S_t = S \times S_{xx}, \quad S^2 = 1. \tag{I.13}$$

In the general case, as in the considered examples, the variable coefficients of the deformations satisfy a certain system of partial differential equations (for example, for the gravitational equation Eq. (I.8) plays the part of this system). In the general case, the system is nonlinear, and the question of its solutions is of independent interest. The point is that in general these systems do not have the Painlevé property, i.e., their exact reductions to ordinary differential equations admit moving critical points. Nevertheless, one can sometimes reduce the problem of constructing the general solution of these partial differential equations to the problem of integrating an ordinary differential equation. It could be that these systems are integrable, but, perhaps, in a quite new sense.

In some cases, substitutions of the variables and fields can reduce the deformation of an equation to the original undeformed equation. We shall say that such deformations are trivial. An example of this kind is the well-known Korteweg–de Vries equation with "cylindrical divergence":

$$u_t + 6uu_x + u_{xxx} + \frac{u}{2t} = 0. \tag{I.14}$$

The corresponding substitution was found for the first time in [10].

There exist special cases, of great interest, when the deformed equations have constant coefficients. Such, for example, is the system of equations

$$E_\eta = \rho, \quad N_\xi + {}^1/_2(\bar{\rho}E + \rho\bar{E}) = c, \quad \rho_\xi = NE, \tag{I.15}$$

(c is a constant), which is a nontrivial deformation of the well-known Maxwell–Bloch system (see below for physical applications of the system (I.15)).

The procedure for integrating the deformed equations by means of the dressing method differs appreciably from the method described in [1]. We intend to devote a separate paper to the description of this procedure.

1. General Case

The basic idea of the inverse scattering method with variable spectral parameter is as follows. Suppose the undeformed system is the condition of compatibility of (I.1), (I.2) in the case of "fixed poles" λ_n, μ_n. To obtain deformations of this system, we shall assume that in (I.2) the poles λ_n, μ_n are certain functions of ξ and η and that λ is a function of ξ and η and is locally an analytic function of the hidden spectral parameter z (in this case, the simple fractions $1/(\lambda - \lambda_n)$ and $1/(\lambda - \mu_n)$ are linearly independent). These functions are by no means arbitrary. They satisfy a system of equations that is uniquely fixed by the requirement that the condition (I.4) be satisfied identically with respect to z and the resulting system of nonlinear equations for the matrices u_n, v_n have precisely the gauge indeterminacy. By virtue of this, the equations for λ have the form

$$\frac{\partial}{\partial\eta}\frac{1}{\lambda-\lambda_n} = p_n + \sum_{m=1}^{N_1}\frac{a_{nm}}{\lambda-\lambda_m} + \sum_{m=1}^{N_2}\frac{b_{nm}}{\lambda-\mu_m}, \tag{1.1}$$

$$\frac{\partial}{\partial\xi}\frac{1}{\lambda-\mu_n} = q_n + \sum_{m=1}^{N_1}\frac{c_{nm}}{\lambda-\lambda_m} + \sum_{m=1}^{N_2}\frac{d_{nm}}{\lambda-\mu_m}, \tag{1.2}$$

where the coefficients $p_n, q_n, a_{nm}, b_{nm}, c_{nm}, d_{nm}$ are functions of ξ and η.

We consider the system (1.1). All the equations of this system must determine the same derivative λ_η. Hence, we have

$$p_1 = \ldots = p_{N_1} = p, \quad a_{nm} = a_n\delta_{nm}, \quad b_{nm} = b_m/(\lambda_n - \mu_m)^2. \tag{1.3}$$

At the same time

$$a_n = 2p\lambda_n - \tilde{p} - \sum_{m=1}^{N_2} \frac{b_m}{(\lambda_n - \mu_m)^2}, \tag{1.4}$$

$$\frac{\partial \lambda_n}{\partial \eta} = -p\lambda_n^2 + \tilde{p}\lambda_n + \hat{p} - \sum_{m=1}^{N_2} \frac{b_m}{\lambda_n - \mu_m}, \tag{1.5}$$

where $\hat{p}$ and $\tilde{p}$ are certain functions of ξ and η. Now

$$\frac{\partial \lambda}{\partial \eta} = -p\lambda^2 + \tilde{p}\lambda + \hat{p} - \sum_{m=1}^{N_2} \frac{b_m}{\lambda - \mu_m}. \tag{1.6}$$

Similarly, from (1.2)

$$q_1 = \ldots = q_{N_2} = q, \quad d_{nm} = d_n \delta_{nm}, \quad c_{nm} = c_m/(\mu_n - \lambda_m)^2, \tag{1.7}$$

with

$$d_n = 2q\mu_n - \tilde{q} - \sum_{m=1}^{N_1} \frac{c_m}{(\mu_n - \lambda_m)^2}, \quad \frac{\partial \mu_n}{\partial \xi} = -q\mu_n^2 + \tilde{q}\mu_n + \hat{q} - \sum_{m=1}^{N_1} \frac{c_m}{\mu_n - \lambda_m}, \tag{1.8}$$

where $\hat{q}$ and $\tilde{q}$ are as yet unknown functions and

$$\frac{\partial \lambda}{\partial \xi} = -q\lambda^2 + \tilde{q}\lambda + \hat{q} - \sum_{m=1}^{N_1} \frac{c_m}{\lambda - \lambda_m}. \tag{1.9}$$

Substituting (I.2) in (I.4) and taking into account (1.1)-(1.9), we obtain the matrix system

$$\begin{gathered}
\frac{\partial u_0}{\partial \eta} - \frac{\partial v_0}{\partial \xi} + [u_0, v_0] = -p \sum_{n=1}^{N_1} u_n + q \sum_{n=1}^{N_2} v_n, \\
\frac{\partial u_n}{\partial \eta} + \left[u_n, v_0 + \sum_{m=1}^{N_2} \frac{v_m}{\lambda_n - \mu_m}\right] = -a_n u_n + c_n \sum_{m=1}^{N_2} \frac{v_m}{(\mu_n - \lambda_m)^2}, \\
\frac{\partial v_n}{\partial \xi} + \left[v_n, u_0 + \sum_{m=1}^{N_1} \frac{u_m}{\mu_n - \lambda_m}\right] = -d_n v_n + b_n \sum_{m=1}^{N_1} \frac{u_m}{(\lambda_n - \mu_m)^2},
\end{gathered} \tag{1.10}$$

which contains the as yet unknown coefficient functions $\lambda_n, \mu_n, c_n, b_n, \hat{p}, \tilde{p}, p, \hat{q}, \tilde{q}, q$. For their determination, we note that Eqs. (1.6) and (1.9) must be compatible and determine λ as a function of ξ and η and the constant of integration z, the hidden spectral parameter. Calculating $\lambda_{\xi\eta}$ in two ways from (1.6) and (1.9), and making simple manipulations, we find

$$\begin{gathered}
\frac{\partial p}{\partial \xi} + p\tilde{q} = \frac{\partial q}{\partial \eta} + q\tilde{p}, \quad \frac{\partial \tilde{p}}{\partial \xi} - 2p\hat{q} = \frac{\partial \tilde{q}}{\partial \eta} - 2q\hat{p}, \quad \frac{\partial \hat{p}}{\partial \xi} - \hat{p}\tilde{q} + 3p\sum_{m=1}^{N_1} c_m = \frac{\partial \hat{q}}{\partial \eta} - \hat{q}\tilde{p} + 3q\sum_{m=1}^{N_2} b_m, \\
\frac{\partial c_n}{\partial \eta} + 2c_n\left(-\tilde{p} + 2p\lambda_n - \sum_{m=1}^{N_2} \frac{b_m}{(\lambda_n - \mu_m)^2}\right) = 0, \quad n = 1, \ldots, N_1, \\
\frac{\partial b_n}{\partial \xi} + 2b_n\left(-\tilde{q} + 2q\mu_n - \sum_{m=1}^{N_1} \frac{c_m}{(\mu_n - \lambda_m)^2}\right) = 0, \quad n = 1, \ldots, N_2.
\end{gathered} \tag{1.11}$$

Equations (1.11) in conjunction with Eqs. (1.5) and (1.8) form a system of compatibility conditions of Eqs. (1.6) and (1.9).

In the trivial special case when λ, λ_n, and μ_n are constants, the right-hand sides in (1.10) are also equal to zero. Then Eqs. (1.10) determine a general "undeformed" system that is integrable in the framework of the inverse scattering method (I.1), (I.2) with "fixed poles" λ_n, μ_n. In the general case, the system (1.10), augmented by Eqs. (1.5), (1.8), and (1.11), is a deformation of it. Note that the scalar system (1.5), (1.8), (1.11) is not dependent on the matrix system (1.10) and can be treated independently.

This system is a set of $2(N_1 + N_2) + 3$ equations for $2(N_1 + N_2 + 3)$ unknown functions. The underdetermination is here due to the possibility of making an arbitrary linear-fractional transformation

$$\lambda \to (\alpha\lambda+\beta)/(\gamma\lambda+\delta), \tag{1.12}$$

where α, β, γ, δ are certain functions of ξ and η.

PROPOSITION 1. By a linear-fractional transformation the system (1.6), (1.9) can be reduced to the form

$$\frac{\partial\lambda}{\partial\eta}+\sum_{m=1}^{N_2}\frac{b_m}{\lambda-\mu_m}=0,\quad \frac{\partial\lambda}{\partial\xi}+\sum_{m=1}^{N_1}\frac{c_m}{\lambda-\lambda_m}=0. \tag{1.13}$$

The conditions of compatibility of the system (1.13) have the form

$$\begin{aligned}&\frac{\partial\lambda_n}{\partial\eta}+\sum_{m=1}^{N_2}\frac{b_m}{\lambda_n-\mu_m}=0,\quad \frac{\partial c_n}{\partial\eta}-2c_n\sum_{m=1}^{N_2}\frac{b_m}{(\lambda_n-\mu_m)^2}=0,\\ &\frac{\partial\mu_n}{\partial\xi}+\sum_{m=1}^{N_1}\frac{c_m}{\mu_n-\lambda_m}=0,\quad \frac{\partial b_n}{\partial\xi}-2b_n\sum_{m=1}^{N_1}\frac{c_m}{(\mu_n-\lambda_m)^2}=0.\end{aligned} \tag{1.14}$$

Here, the number of equations and the number of unknown functions are the same and equal to $2(N_1 + N_2)$.

We omit the simple proof of Proposition 1. We merely note that the system (1.14) arises from (1.5), (1.8), and (1.11) if we set $\hat{p}=\tilde{p}=p=\hat{q}=\tilde{q}=q=0$. In addition, the matrix system (1.10) has gauge indeterminacy ($N_1 + N_2 + 1$ matrix equations for $N_1 + N_2 + 2$ matrices u_n, v_k, $n = 0, 1, \ldots, N_1$; $k = 0, 1, \ldots, N_2$). This is due to the possibility of making in the system (I.1) the gauge transformation

$$\Phi=g\tilde{\Phi},\quad \tilde{U}=-g^{-1}g_\xi+g^{-1}Ug,\quad \tilde{V}=-g^{-1}g_\eta+g^{-1}Vg, \tag{1.15}$$

where g is an arbitrary nondegenerate matrix function of ξ and η. Besides the transformation (1.15), in the system (I.1) we can make the change of coordinates

$$\xi'=\xi'(\xi),\quad \eta'=\eta'(\eta). \tag{1.16}$$

We shall say that two deformed systems differing by the transformations (1.12), (1.15), (1.16) are generalized gauge equivalent. We shall say that the deformations that can be reduced to the undeformed equations by means of point transformations and the transformations (1.15) are trivial, and we shall say that all the remainder are nontrivial. Suppose that the system of equations for λ has been reduced by a linear-fractional transformation to the form (1.13). Then $p = q = 0$, and in Eq. (1.10) we can make the functions u_0 and v_0 vanish simultaneously by means of a gauge transformation. Then the system (1.10) takes the simple form

$$\begin{aligned}&\frac{\partial u_n}{\partial\eta}+\left[u_n,\sum_{m=1}^{N_2}\frac{v_m}{\lambda_n-\mu_m}\right]=\sum_{m=1}^{N_2}\frac{b_m u_n+c_n v_m}{(\lambda_n-\mu_m)^2},\\ &\frac{\partial v_n}{\partial\xi}+\left[v_n,\sum_{m=1}^{N_1}\frac{u_m}{\mu_n-\lambda_m}\right]=\sum_{m=1}^{N_1}\frac{c_m v_n+b_n u_m}{(\mu_n-\lambda_m)^2}.\end{aligned} \tag{1.17}$$

One can arrive at the system (1.14), (1.17) in a quite different way. Let λ be a constant spectral parameter but suppose Eqs. (I.1) are replaced by the equations

$$\mathbf{D}_1\Phi=U\Phi,\qquad \mathbf{D}_2\Phi=V\Phi. \tag{1.18}$$

Here, the operators $\mathbf{D}_{1,2}$ have the form

$$\mathbf{D}_1=\frac{\partial}{\partial\xi}+F\frac{\partial}{\partial\lambda},\quad \mathbf{D}_2=\frac{\partial}{\partial\eta}+G\frac{\partial}{\partial\lambda},\quad F=-\sum_{m=1}^{N_1}\frac{c_m}{\lambda-\lambda_m},\quad G=-\sum_{m=1}^{N_2}\frac{b_m}{\lambda-\mu_m}, \tag{1.19}$$

while the matrices U and V are, as before, given by (I.2). We require that the operators $\mathbf{D}_1$ and $\mathbf{D}_2$ commute. We then obtain

$$F_\eta+GF_\lambda=G_\xi+FG_\lambda. \tag{1.20}$$

Equation (1.20) is the condition of vanishing of the commutator of the vector fields $\mathbf{D}_{1,2}$.

Substituting (1.19) in (1.20) and making simple transformations, we obtain Eqs. (1.14).

The condition of compatibility of the system (1.18) has the form

$$\mathbf{D}_1 U - \mathbf{D}_2 V + [U, V] = 0. \tag{1.21}$$

It is readily seen that Eq. (1.21) (in the case of the canonical gauge $u_0 = v_0 = 0$) is equivalent to the system (1.17).

In Eqs. (1.18) one can also make linear-fractional and gauge transformations and also stretch the coordinates. To conclude this section, we note that the scheme for obtaining deformed equations using differentiation with respect to the parameter λ follows directly from [3], whereas the technique associated with introducing the variable parameter λ arose from attempts to understand the result of Maison [4].

We note also that the system (1.13), (1.14) contains a simple special case. Suppose $c_n = 0$, $\partial\lambda/\partial\xi = 0$. Then $\mu_n = \mu_n(\eta)$ and $b_n = b_n(\eta)$ are arbitrary functions of one variable. The dependence $\lambda = \lambda(z, \eta)$ can be found by solving the ordinary differential equation (1.13) and the poles are $\lambda_n = \lambda|_{z=z_n}$.

2. Deformation of the Equations of the Principal Chiral Field

Turning to the consideration of specific examples, we note that in what follows we shall not necessarily reduce the considered system to the form (1.14), (1.17) but will exploit the freedom with respect to the gauge and linear-fractional transformations as we find convenient. Suppose the matrix functions U and V each have just one simple pole. By a linear-fractional transformation we can carry these poles to the points $\lambda = \pm 1$. Choosing the canonical gauge, we reduce Eqs. (I.1) to the form

$$\Phi_\xi = \frac{u}{\lambda - 1}\Phi, \quad \Phi_\eta = \frac{v}{\lambda + 1}\Phi. \tag{2.1}$$

In contrast to (I.6), λ here is variable and satisfies the equations

$$\frac{\partial}{\partial\eta}\frac{1}{\lambda - 1} = \frac{a}{\lambda - 1} + \frac{b}{\lambda + 1}, \quad \frac{\partial}{\partial\xi}\frac{1}{\lambda + 1} = \frac{c}{\lambda + 1} + \frac{d}{\lambda - 1}. \tag{2.2}$$

With allowance for (2.2), the conditions of compatibility of the system (2.1) have the form

$$u_\eta + {}^1/_2[u, v] = -au + dv, \qquad v_\xi - {}^1/_2[v, u] = -cv + bu. \tag{2.3}$$

The conditions of compatibility of the system (2.2) can be written in the form

$$b_\xi + 2bc = 0, \qquad d_\eta + 2da = 0, \tag{2.4}$$

$$(a+b)_\xi = (c+d)_\eta = -ac - 3bd. \tag{2.5}$$

It follows from (2.5) that we can introduce the function α in accordance with the formulas

$$\frac{\partial}{\partial\eta}\ln\alpha = a + b, \quad \frac{\partial}{\partial\xi}\ln\alpha = c + d. \tag{2.6}$$

With allowance for (2.6), it follows from (2.3) that

$$(\alpha u)_\eta = (\alpha v)_\xi. \tag{2.7}$$

Expressing u and v by means of (2.1), we obtain for Φ

$$(\alpha(\lambda-1)\Phi_\xi\Phi^{-1})_\eta = (\alpha(\lambda+1)\Phi_\eta\Phi^{-1})_\xi. \tag{2.8}$$

Equation (2.8) holds identically with respect to the "hidden parameter" z — the constant of integration of the system (2.2). In the trivial special case $a = b = c = d = 0$, $\lambda =$ const, $\alpha = 1$, (2.8) goes over for $\lambda = 0$ into the equation of the principal chiral field. The system (2.2)-(2.8) is the most general deformation of these equations. Equations (2.4)-(2.5) can be integrated simply in the special case

$$a = b = \frac{1}{2}\frac{\partial}{\partial\eta}\ln\alpha, \quad c = d = \frac{1}{2}\frac{\partial}{\partial\xi}\ln\alpha, \tag{2.9}$$

which simplifies to the single equation $\alpha_{\xi\eta} = 0$. The system (2.2) can then also be integrated and gives formula (I.11), which determines $\lambda = \lambda(z, \xi, \eta)$ as a function of the hidden spectral parameter z, which lies on a two-sheeted Riemann surface. Going to the limit $z \to \infty$ on the upper sheet, we obtain $\lambda \to 0$. At the same time, Eq. (2.8) goes over into the gravitation equation (I.7).

Other cases of integrability of the system (2.4)-(2.5) are known. Suppose $d = 0$, $b \neq 0$. Then the ansatz

$$b=\frac{1}{R^2}, \quad c=\frac{R_\xi}{R}, \quad a=-\frac{R_{\xi\eta}}{R_\xi}+\frac{R_\eta}{R} \tag{2.10}$$

makes it possible to integrate the equation for ξ twice, after which the system is reduced to an ordinary differential equation with moving critical points:

$$R_\eta=-\frac{1}{R}+\beta_{(\eta)}R+\gamma_{(\eta)}. \tag{2.11}$$

Here, $\beta_{(\eta)}$ and $\gamma_{(\eta)}$ are arbitrary functions. In the general case, this equation cannot be integrated by quadratures.

Simpler is the case $d = b = 0$. Then the ansatz $a=\frac{\partial}{\partial\eta}\ln\alpha$, $c=\frac{\partial}{\partial\xi}\ln\alpha$ again leads to the equation

$$\alpha_{\xi\eta}=0, \qquad \alpha=f(\xi)+h(\eta).$$

Equations (2.2) can now be integrated and give

$$\lambda=\frac{-f(\xi)+h(\eta)+z}{f(\xi)+h(\eta)}. \tag{2.12}$$

From Eq. (2.8) for $z = 0$ we have

$$(f\Phi_\xi\Phi^{-1})_\eta+(h\Phi_\eta\Phi^{-1})_\xi=0. \tag{2.13}$$

There is one further case of integrability: $c = d = 0$. Then a and b are arbitrary functions of η, and Eqs. (2.2) cannot be integrated in general form. It is interesting that we can now take a and b to be arbitrary constants. Then Eq. (2.8) takes the form

$$a(\Phi_\xi\Phi^{-1})_\eta+b(\Phi_\eta\Phi^{-1})_\xi+a(a+b)\Phi_\xi\Phi^{-1}=0. \tag{2.14}$$

Both (2.3) and (2.14) become equations with constant coefficients. In the general case the solution of the system (2.4), (2.5) is not known to us.

3. Deformations of the U–V System and the Maxwell–Bloch Equations

The equations of the principal chiral field are gauge equivalent to a certain system of equations for two matrix functions U and V known as the U–V system. In a special case, the sine–Gordon equation can be deduced from this system. It is convenient to consider the deformations of these equations independently. We consider the system (I.1), in which the functions U and V each have one simple pole. By a linear-fractional transformation we carry the poles to the points $\lambda = 0, \infty$. We set

$$U=\lambda u_1+u_0, \qquad V=v/\lambda. \tag{3.1}$$

The function λ satisfies the system of equations

$$\frac{\partial\lambda}{\partial\eta}=a\lambda+b+\frac{c}{\lambda}, \quad \frac{\partial}{\partial\xi}\frac{1}{\lambda}=\tilde{a}\lambda+\tilde{b}+\frac{\tilde{c}}{\lambda}. \tag{3.2}$$

Taking them into account, we extract from (I.4) the system of equations

$$\frac{\partial u_1}{\partial\eta}=-au_1+\tilde{a}v, \quad \frac{\partial u_0}{\partial\eta}+[u_1, v]=-bu_1+\tilde{b}v, \quad \frac{\partial v}{\partial\xi}+[v, u_0]=cu_1-\tilde{c}v. \tag{3.3}$$

For $a=\tilde{a}=b=\tilde{b}=c=\tilde{c}=0$, the system (3.3) goes over into the U–V system of [11]. (One usually takes $u_1 = J$, a constant diagonal matrix.)

In our case, the coefficients $a, \tilde{a}, b, \tilde{b}, c, \tilde{c}$ satisfy a system of equations that follows from (3.2):

$$\tilde{a}_\eta+2\tilde{a}a=0,\quad \tilde{b}_\eta+\tilde{b}a+3b\tilde{a}=0,\quad c_\xi+2c\tilde{c}=0,\quad b_\xi+b\tilde{c}+3\tilde{b}c=0,\quad a_\xi+\tilde{c}_\eta+2b\tilde{b}+4\tilde{a}c=0. \tag{3.4}$$

The choice of U and V in the form of (3.1) permits a stretching of the parameter λ ($\lambda \to e^\gamma\lambda$), and therefore the system (3.4) is underdetermined. By a stretching of λ we can make the function $a_{(\xi,\eta)}$ vanish and the function $\tilde{a}_{(\xi,\eta)}$ into a constant. Further, the following tree of variants is possible.

1. $\tilde{a}=1, c\neq 0$. By the substitution

$$c=e^{2\chi},\quad \tilde{c}=-\chi_\xi,\quad b=pe^\chi \tag{3.5}$$

the system (3.4) is reduced to the symmetric form

$$\tilde{b}_\eta+3pe^\chi=0,\quad p_\xi+3\tilde{b}e^\chi=0,\quad \chi_{\xi\eta}=2p\tilde{b}e^\chi+4e^{2\chi}. \tag{3.6}$$

The system (3.6) (like the system (2.4)-(2.5)) has not yet been solved in general form. In the special case $\tilde{b}$ = p = 0 it is equivalent to the Liouville equation

$$\chi_{\xi\eta}=4e^{2\chi}. \tag{3.7}$$

2. $\tilde{a}=1$, $c=0$, $b\neq 0$. In this case, the system (3.4) is reduced by the substitution

$$b=e^\chi,\quad \tilde{c}=-\chi_\xi,\quad \tilde{b}={}^1/_2\chi_{\xi\eta}e^{-\chi} \tag{3.8}$$

to the single equation

$$\chi_{\xi\eta\eta}-\chi_\eta\chi_{\xi\eta}+6e^{2\chi}=0. \tag{3.9}$$

We also make the substitution $Y_\eta = e^\chi$. Equation (3.9) can be integrated three times with respect to η, after which it becomes an ordinary differential equation of first order with moving critical points,

$$Y_\xi+Y^3+f_1(\xi)Y+f_0(\xi)=0, \tag{3.10}$$

where $f_{0,1}(\xi)$ are arbitrary functions. In the general case, it cannot be integrated by quadratures.

3. $\tilde{a}=1$, $b=c=0$. In this case, Eqs. (3.4) become trivial, and $\tilde{b}$ and $\tilde{c}$ can be made arbitrary functions of ξ. Then λ does not depend on η but it is not possible to find the explicit dependence of λ on ξ and z in the general case.

We consider the simplest case $\tilde{b} = \tilde{c} = 0$. Then $\lambda = 1/\sqrt{2(\xi + z)}$, and the system (3.3) takes the form

$$\frac{\partial u_1}{\partial\eta}=v,\quad \frac{\partial u_0}{\partial\eta}+[u_1,v]=0,\quad \frac{\partial v}{\partial\xi}+[v,u_0]=0. \tag{3.11}$$

We set

$$u_1=\begin{bmatrix}0 & \psi\\ \bar{\psi} & 0\end{bmatrix},\quad u_0=i\begin{bmatrix}s & 0\\ 0 & -s\end{bmatrix},\quad \bar{s}=s. \tag{3.12}$$

We have

$$s_\eta=i(\psi\bar{\psi}_\eta-\bar{\psi}\psi_\eta),\quad \psi_{\xi\eta}=2is\psi_\eta. \tag{3.13}$$

It would be of great interest to find an application of the system (3.13).

4. We now consider the case $\tilde{a}=0$. Then $\partial u_1/\partial\eta = 0$ and we can take $u_1 = J$, a constant matrix. In the general case a stretching of λ achieves $\tilde{b} = 1$. Then, if $c \neq 0$, the change of variables $c=e^{2\chi}$, $\tilde{c}=-\chi_\xi$, $b={}^1/_2\chi_{\xi\eta}$ leads to the equation

$$\chi_{\eta\xi\xi}-\chi_\xi\chi_{\eta\xi}+6e^{2\chi}=0, \tag{3.14}$$

which differs from (3.9) by the substitution $\xi\leftrightarrow\eta$. For c = 0, the substitution $b=e^\varphi$, $\tilde{c}=-\varphi_\xi$ leads to the Liouville equation $\varphi_{\xi\eta}=2e^\varphi$. The most interesting case is b = $\tilde{b}$ = 0. Then from (3.4) we have

$$\tilde{c}_\eta=0,\quad c_\xi+2c\tilde{c}=0. \tag{3.15}$$

Equation (3.3) simplifies to the form

$$\frac{\partial u_0}{\partial\eta}+[J,v]=0,\quad \frac{\partial v}{\partial\xi}+[v,u_0]=cJ-\tilde{c}v. \tag{3.16}$$

We set

$$J=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad u_0=\frac{1}{2}\begin{bmatrix}0 & E\\ -\bar{E} & 0\end{bmatrix},\quad v=\frac{1}{4}\begin{bmatrix}N & -\rho\\ -\bar{\rho} & -N\end{bmatrix}.$$

We have

$$E_\eta-\rho=0, \tag{3.17}$$

$$N_\xi+{}^1/_2(\rho\bar{E}+\bar{\rho}E)=-\tilde{c}N-4c, \tag{3.18}$$

$$\rho_\xi-NE=-\tilde{c}\rho. \tag{3.19}$$

The system of equations (3.17)-(3.19) for $c = \tilde{c} = 0$ goes over into the well-known Maxwell-Bloch system [12], which describes the propagation of a radiation pulse in a two-level medium. The case $\tilde{c} = 0$, c = const is interesting. The system (3.17)-(3.19) now takes the form (I.15). For the spectral parameter we have $\lambda = \sqrt{2c(\eta + x)}$, so that the deformation (I.15) of the Maxwell-Bloch equations is nontrivial. From the physical point of view this is the Maxwell-Bloch system in which there is pumping of atoms in an excited state. The value of $\sqrt{|\rho|^2 + N^2}$ varies in accordance with the law

$$\frac{\partial}{\partial\xi}\sqrt{|\rho|^2+N^2}=-4c\frac{N}{\sqrt{|\rho|^2+N^2}}.$$

Suppose $c = 0$, $\tilde{c}$ = const; then the deformation is trivial. For now $\lambda=z\psi(\xi)$, $\psi(\xi)=\exp[-\tilde{c}\xi]$. The substitution $v=\psi\tilde{v}$, $u_0=\psi\tilde{u}_0$, $\frac{1}{\psi}\frac{\partial}{\partial\xi}=\frac{\partial}{\partial\tilde{\xi}}$ reduces the system to the undeformed form

$$\Phi_{\tilde{\xi}}=(zJ+\tilde{u}_0)\Phi,\quad \Phi_\eta=\frac{\tilde{v}}{z}\Phi.$$

4. The Polynomial Case

A significant number of integrable systems belongs to the case when the matrices U and V in (I.1) are polynomials in the spectral parameter λ:

$$U=u_0+u_1\lambda+\ldots+u_m\lambda^m, \tag{4.1}$$

$$V=v_0+v_1\lambda+\ldots+v_n\lambda^n. \tag{4.2}$$

Suppose the spectral parameter is variable. Then the following system of equations for λ must hold:

$$\lambda_x=p_0+p_1\lambda+\ldots+p_k\lambda^k, \tag{4.3}$$

$$\lambda_t=q_0+q_1\lambda+\ldots+q_l\lambda^l, \tag{4.4}$$

where $k \le m$, $\ell \le n$. In addition, $p_k\neq0$, $q_l\neq0$. The conditions of compatibility of the system (4.3), (4.4) can be investigated in general form. To be specific, we take $k \le \ell$. Then we have

PROPOSITION 2. The following alternative holds: $k = 1$ or $k = \ell$.

Indeed, suppose $1 < k \le \ell$. Differentiating (4.3) with respect to t, and (4.4) with respect to x, and expressing the derivatives by means of (4.3) and (4.4), we find in the leading order in λ:

$$(l-k)p_kq_l\lambda^{k+l-1}=0. \tag{4.5}$$

Hence $k = \ell$.

We consider two possibilities.

1. Suppose $k = 1$. Equation (4.3) has the form

$$\lambda_x=p_0+p_1\lambda. \tag{4.6}$$

We make the transformation

$$\lambda=a+b\mu \tag{4.7}$$

and require fulfillment of the conditions $a_x=p_0+p_1a$, $b_x=p_1b$. Then the system (4.3), (4.4) takes the form

$$\mu_x=0,\qquad \mu_t=\tilde{q}_0+\tilde{q}_1\mu+\ldots+\tilde{q}_l\mu^l, \tag{4.8}$$

where $\tilde{q}_0, \ldots, \tilde{q}_\ell$ are arbitrary functions of t.

2. $k = \ell$. To find the general compatible solution of the system (4.3), (4.4), we note that it now admits an arbitrary curvilinear transformation of the coordinates: $x = x(x', t')$, $t = t(x', t')$. The transformation can be chosen to make $p_k \equiv 0$. By virtue of what was proved above, all p_i $(2 \leqq i \leqq k-1)$ then automatically vanish, and Eq. (4.3) is reduced to the form (4.6). Further, by a linear transformation of the parameter λ we can make p_0 and p_1 vanish. Thus, to obtain the general solution of the system (4.3), (4.4) for $k = \ell$ we must consider the general solution of the system (4.8) and then in this system make an arbitrary change of the coordinates and an arbitrary linear transformation of the function λ.

In the examples considered below, we shall encounter only the case $k < \ell$, when the system (4.3), (4.4) can be reduced to the form (4.8) by a linear transformation of λ. The system (4.8) can be simplified still further by using the linear transformation $\mu = c_{(t)} + d_{(t)}\hat{\lambda}$ and a change of the time t. We finally obtain

$$\hat{\lambda}_x = 0, \qquad \hat{\lambda}_t = \hat{q}_2\hat{\lambda}^2 + \hat{q}_3\hat{\lambda}^3 + \ldots + \hat{\lambda}^l.$$

This system arises from (4.8) if we set $\tilde{q}_0 = \tilde{q}_1 = 0$, $\tilde{q}_\ell = 1$. Examples of different deformations of systems of polynomial type are given in the Appendix.

5. The Case of the Spectral Parameter on a Curve of Finite Kind

In the system (I.1), the spectral parameter may be situated not only on a Riemann sphere but also on an algebraic curve of finite kind [13]. Instead of making an explicit uniformization of the algebraic curve, it is more convenient in this case to introduce analogs of simple fractions — linearly independent functions w_i that are rational on the curve and are connected by quadratic relations (see [13]). In general form,

$$U = \sum_{i=1}^{N} u_i w_i, \quad V = \sum_{i=1}^{N} v_i w_i. \tag{5.1}$$

Here, the functions w_i satisfy the set of quadratic relations

$$\Pi_{sij} w_i w_j + \Psi_{si} w_i = 0, \qquad s = 1, \ldots, N-\rho \text{ (}\rho\text{ is the kind of curve),} \tag{5.2}$$

which are taken into account when the conditions of compatibility of the system (I.1) are calculated.

Here, the spectral parameter — the uniformizing parameter of the system of quadrics (5.2) — is from the very beginning hidden, though it is constant. To make it variable, we assume that all the w_i are functions of x and t and satisfy the system of equations

$$\frac{\partial w_i}{\partial x} = A_{ijk} w_j w_k + P_{ij} w_j, \quad \frac{\partial w_i}{\partial t} = B_{ijk} w_j w_k + Q_{ij} w_j. \tag{5.3}$$

The overdetermined system (5.3) must be considered simultaneously with the algebraic equations (5.2). At the same time, not only the coefficients A_{ijk}, B_{ijk}, P_{ij}, Q_{ij} but also the parameters Π_{sij}, Ψ_{si} of the quadrics must be assumed to be functions of x and t.

The general problem of such kind has not been studied at all. However, we shall give a fairly interesting example. We consider the well-known Landau—Lifshitz equation [14,15]:

$$\mathbf{S}_t = \mathbf{S} \times (\mathbf{S}_{xx} + \hat{J}\mathbf{S}), \qquad \mathbf{S}^2 = 1, \tag{5.4}$$

where $\hat{J}$ is a constant diagonal 3×3 matrix, $\mathbf{S} = (S_1, S_2, S_3)$. In the case (5.4), the matrices U and V have the form

$$U = i \sum_{a=1}^{3} S_a \sigma_a w_a^{(1)}, \quad V = 2i \sum_{a=1}^{3} S_a \sigma_a w_a^{(2)} + i \sum_{a,b,c=1}^{3} \varepsilon_{abc} S_b \frac{\partial S_c}{\partial x} \sigma_a w_a^{(1)}. \tag{5.5}$$

Here, the functions $w^{(i)}$ satisfy the equations

$$(w_a^{(1)})^2 - (w_b^{(1)})^2 = 2A_a - 2A_b, \tag{5.6}$$

$$w_a^{(2)} = w_b^{(1)} w_c^{(1)},\ a, b \text{ and } c \text{ are not equal to each other.} \tag{5.7}$$

Here, σ_a are the Pauli matrices, and ε_{abc} is the absolutely antisymmetric tensor. The relation (5.6) determines an algebraic curve (torus). We calculate the condition of compatibility of the system (I.1) in the case (5.5)-(5.7):

$$\sum_{a=1}^{3} \sigma_a w_a^{(1)} \left[\frac{\partial S_a}{\partial t} - \sum_{b,\,c=1}^{3} \varepsilon_{abc} \left(S_b \frac{\partial^2 S_c}{\partial x^2} - 8 S_b A_c S_c \right) \right] + 2 \sum_{a=1}^{3} \sigma_a w_a^{(2)} \left[(S^2 - 1) \frac{\partial S_a}{\partial x} - \frac{1}{2} S_a \frac{\partial S^2}{\partial x} \right] =$$
$$\sum_{a=1}^{3} \sigma_a \left[2 S_a \frac{\partial w_a^{(2)}}{\partial x} - S_a \frac{\partial w_a^{(1)}}{\partial t} + \sum_{b,\,c=1}^{3} \varepsilon_{abc} S_b \frac{\partial S_c}{\partial x} \frac{\partial w_a^{(1)}}{\partial x} \right]. \tag{5.8}$$

In the calculation we used the relation

$$\sum_{a,\,b,\,c=1}^{3} \varepsilon_{abc} \sigma_a w_b^{(1)} w_c^{(2)} S_b S_c = -2 \sum_{a,\,b,\,c=1}^{3} \varepsilon_{abc} \sigma_a w_a^{(1)} S_b A_c S_c,$$

which is a consequence of the definitions (5.6) and (5.7) for $w_a^{(1)}$ and $w_a^{(2)}$. In the case of constant $w_a^{(1)}$, expanding (5.8) with respect to the basis of the functions $w_a^{(1)}$ and $w_a^{(2)}$, we obtain the Landau–Lifshitz equation (5.4), in which

$$\hat{J} = -8 \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}.$$

Now suppose the functions $w_a^{(1)}$ and A_a depend on x and t. In the general case, this dependence must be specified by the equations

$$\frac{\partial w_a^{(1)}}{\partial x} = f_a w_a^{(1)}, \quad \frac{\partial w_a^{(1)}}{\partial t} = g_a w_a^{(2)} + h_a w_a^{(1)}. \tag{5.9}$$

However, it can be shown that the functional arbitrariness that arises in (5.9) is imaginary: omitting the proof, we state

<u>PROPOSITION 3.</u> The requirement that the equations of the system of (5.9) be compatible with each other and with the definition (5.6), (5.7) and also fulfillment of the condition $S^2 = 1$ leave possible the variant

$$\frac{\partial w_a^{(1)}}{\partial x} = \frac{1}{x} w_a^{(1)}, \quad \frac{\partial w_a^{(1)}}{\partial t} = \frac{4}{x} w_a^{(2)}. \tag{5.10}$$

Substituting (5.10) in (5.8), we obtain a deformation of the Landau–Lifshitz equation:

$$S_t = S \times \left(S_{xx} + \frac{1}{x} S_x + x^2 \hat{J}_0 S \right), \quad S^2 = 1, \tag{5.11}$$

where

$$A_a(x) = x^2 A_a^0, \quad \hat{J}_0 = -8 \begin{bmatrix} A_1^0 & & \\ & A_2^0 & \\ & & A_3^0 \end{bmatrix}.$$

We note that in the cylindrical Landau–Lifshitz equation, in contrast to the deformation (5.11), the coefficient x^2 is not present in front of the term $S \times (\hat{J}_0 S)$. It remains to integrate the system (5.10). We seek the solution in the form

$$w_a^{(1)}(z, x, t) = x \hat{w}_a (\lambda = z - 4\rho t), \quad \rho = {}^1\!/_2 \sqrt{J_3^0 - J_1^0}$$

(the complex constant of integration z is a hidden spectral parameter), which automatically satisfies the first equation of the system (5.10). Substituting in the second equation, we obtain

$$\frac{d\hat{w}_a}{d\lambda} = -\rho^{-1} \hat{w}_b \hat{w}_c.$$

The solution of this last equation can be expressed in terms of elliptic Jacobi functions [15]:

$$\wp_1=\frac{\rho}{\operatorname{sn}(\lambda,k)},\quad \wp_2=\frac{\rho\operatorname{dn}(\lambda,k)}{\operatorname{sn}(\lambda,k)},\quad \wp_3=\frac{\rho\operatorname{cn}(\lambda,k)}{\operatorname{sn}(\lambda,k)},\quad k=\sqrt{\frac{J_2^0-J_1^0}{J_3^0-J_1^0}},\quad J_1^0\leqslant J_2^0\leqslant J_3^0,$$

these being defined in the rectangle $R=\{\lambda:|\operatorname{Re}\lambda|\leqslant 2k,\ |\operatorname{Im}\lambda|\leqslant 2k'\}$.

Note that Eq. (5.11) can be obtained by averaging over the reduction group of the operators $L_{1,2}$: $L_1 = -\partial_x + U$, $L_2 = -\partial_t + V$ for the cylindrical Heisenberg equation (I.12) [16].

We are grateful to I. R. Gabitov for a fruitful discussion of the paper.

Appendix

Omitting the calculations, we give the list of the deformations (and corresponding linear systems of polynomial type) of various well-known equations that we have found.

1. Deformations of the Korteweg–de Vries equation:

$$u_t=\frac{1}{2}v_{xxx}+2v_xu+vu_x,\quad v=-\frac{1}{\sqrt{x}}\int\sqrt{x}\left(2u_x+\frac{4u}{x}\right)dx. \tag{A.1}$$

To obtain (A.1), we used the "scalar" linear system

$$\Phi_{xx}+U\Phi=0,\quad \Phi_t=A\Phi_x+B\Phi, \tag{A.2}$$

$$U=\lambda+u,\quad A=4\lambda+v,\quad B=-\frac{1}{2}A_x, \tag{A.3}$$

where $\lambda_x = \lambda/x$, $\lambda_t = (12/x)\lambda^2$, $\lambda = -x/12(t + z)$. By the substitutions $u(x, t) = -5/16x^2 + \hat{u}(\hat{x}, \hat{t})$, $\hat{x} = x^{3/2}$, $\hat{t} = (27/8)t$ the system (A.1) is reduced to the form (in what follows, we omit the caret)

$$u_t+(xu)_{xxx}+u_{xx}=\left[-3xu^2-2u\int u\,dx\right]_x-3u^2. \tag{A.4}$$

The deformation (A.1) is nontrivial. To obtain Eq. (I.14) we must in (A.3) make the choice $v = -2u$, $\lambda_x = -1/12t$, $\lambda_t = -\lambda/t$.

2. Deformation of the modified Korteweg–de Vries equation:

$$u_t+(xu)_{xxx}=\pm2\left[\left(x\int u^2\,dx\right)_x u\right]_x. \tag{A.5}$$

To construct (A.5) it is necessary to apply the linear system (I.1):

$$\Phi_x=U\Phi,\quad \Phi_t=V\Phi, \tag{A.6}$$

where

$$U=i\lambda\begin{bmatrix}1 & 0\\ 1 & -1\end{bmatrix}+\begin{bmatrix}0 & u\\ \pm u & 0\end{bmatrix},$$

$$V=4i\lambda^3x\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}+4\lambda^2x\begin{bmatrix}0 & u\\ \pm u & 0\end{bmatrix}+2i\lambda\begin{bmatrix}\pm\left(x\int u^2\,dx\right)_x & -(xu)_x\\ \pm(xu)_x & \mp\left(x\int u^2\,dx\right)_x\end{bmatrix}+$$

$$\begin{bmatrix}0 & -(xu)_{xx}\pm2u\left(x\int u^2\,dx\right)_x\\ \mp(xu)_{xx}+2u\left(x\int u^2\,dx\right)_x & 0\end{bmatrix},$$

$$\lambda_x=0,\quad \lambda_t=4\lambda^3,\quad \lambda(z,t)=\frac{1}{\sqrt{-8(t+z)}}.$$

3. Deformation of Kaup's system [17]:

$$u_t+\frac{1}{2}(xu^2)_x+(x\eta)_x=-\eta,\quad \eta_t+(x\eta u)_x+(xu)_{xxx}=-\eta u. \tag{A.7}$$

For the system (A.7), the corresponding linear system has the form (A.2), where

$$U=\lambda^2+\frac{i\lambda u}{2}+\frac{\eta}{4}-\frac{u^2}{16},\quad A=-2ix\lambda-\frac{1}{2}xu,\quad B=-\frac{1}{2}A_x,\quad \lambda_x=0,\quad \lambda_t=-2i\lambda^2,\quad \lambda(z,t)=\frac{1}{2i(t+z)}.$$

The deformation (A.7) is nontrivial. We also give a trivial deformation of Kaup's system of interest from the point of view of applications in hydrodynamics:

$$u_t+uu_x+\eta_x=0,\qquad \eta_t+[(1-\alpha x+\eta)u]_x+u_{xxx}=0,\quad \alpha \text{ is a constant.} \tag{A.8}$$

For the system (A.8), the functions U, A, and B have the form

$$U=\lambda^2+\frac{i\lambda u}{2}+\frac{1}{4}(\eta+1-\alpha x)-\frac{u^2}{16},\quad A=-2i\lambda-\frac{u}{2},\quad B=-\frac{1}{2}A_x,$$

where $\lambda_x=0$, $\lambda_t=\frac{i}{4}\alpha$, $\lambda(z,t)=z+\frac{i}{4}\alpha t$. Note that the substitution

$$\eta(x,t)=\alpha x+\hat\eta(\hat x,\hat t),\quad \hat x=x+\frac{\alpha}{2}t^2,\quad u(x,t)=-\alpha t+\hat u(\hat x,\hat t),\quad \hat t=t, \tag{A.9}$$

reduces the system (A.8) to the original undeformed Kaup system [17] (we omit the caret):

$$u_t+uu_x+\eta_x=0,\qquad \eta_t+[(1+\eta)u]_x+u_{xxx}=0.$$

4. Deformations of the nonlinear Schrödinger equation:

$$i\psi_t+\psi_{xx}+\frac{1}{x}\psi_x\pm2|\psi|^2\psi=\frac{\psi}{x^2}\mp4\psi\int\frac{|\psi|^2}{x}\,dx. \tag{A.10$^\pm$}$$

Here and below, the linear systems have the form (A.6), and therefore we shall give only the corresponding matrix functions U and V. In the case (A.10$^\pm$), we have

$$U=i\lambda\begin{bmatrix}1&0\\0&-1\end{bmatrix}+i\begin{bmatrix}0&\psi\\\pm\bar\psi&0\end{bmatrix},\qquad V=-2i\lambda^2\begin{bmatrix}1&0\\0&-1\end{bmatrix}-2i\lambda\begin{bmatrix}0&\psi\\\pm\bar\psi&0\end{bmatrix}+$$

$$\begin{bmatrix}\pm i|\psi|^2\pm2i\int\frac{|\psi|^2}{x}\,dx & -\psi_x-\frac{1}{x}\psi\\ \pm\bar\psi_x\pm\frac{1}{x}\bar\psi & \mp i|\psi|^2\mp2i\int\frac{|\psi|^2}{x}\,dx\end{bmatrix},$$

$$\lambda_x=\frac{\lambda}{x},\quad \lambda_t=-\frac{4}{x}\lambda^2,\quad \lambda(z,x,t)=\frac{x}{4(z+t)}.$$

Equation (A.10$^\pm$), being a nontrivial deformation of the nonlinear Schrödinger equation, is remarkable in being gauge equivalent to the cylindrical Heisenberg equation (I.12). The substitutions

$$\psi(x,t)=\frac{x}{2}\hat\psi(\hat x,\hat t),\qquad \hat x=\frac{1}{4}x^2,\qquad \hat t=t$$

reduce Eqs. (A.10$^\pm$) to the form (we henceforth omit the caret)

$$i\psi_t+(x\psi)_{xx}\pm2\psi\left[x\int|\psi|^2\,dx\right]_x=0. \tag{A.11$^\pm$}$$

After we had completed our paper, we learnt that Eqs. (A.4), (A.5), and (A.11$^\pm$) had already been integrated in [6]:

$$i\left(\psi_t+\frac{\psi}{2t}\right)+\psi_{xx}\pm2|\psi|^2\psi=0,\qquad U=i\lambda\begin{bmatrix}1&0\\0&-1\end{bmatrix}+i\begin{bmatrix}0&\psi\\\pm\bar\psi&0\end{bmatrix},$$

$$V=-2i\lambda^2\begin{bmatrix}1&0\\0&-1\end{bmatrix}-2i\lambda\begin{bmatrix}0&\psi\\\pm\bar\psi&0\end{bmatrix}+\begin{bmatrix}\pm i|\psi|^2&-\psi_x\\\pm\bar\psi_x&\mp i|\psi|^2\end{bmatrix},\quad \lambda_x=\frac{1}{4t},\quad \lambda_t=-\frac{\lambda}{t},\quad \lambda(z,x,t)=\frac{z+x/4}{t}. \tag{A.12}$$

Equation (A.12), to us hitherto unknown, describes cylindrically diverging quasiplane envelope waves in a nonlinear medium. In addition, this equation is a trivial deformation of the nonlinear Schrödinger equation. The substitution

$$\psi(x,t)=\frac{1}{t}\exp\left[\frac{ix^2}{4t}\right]\hat{\psi}(\hat{x},\hat{t}),\quad \hat{x}=\frac{x}{t},\quad \hat{t}=-\frac{1}{t} \tag{A.13}$$

in Eq. (A.12) leads to the nonlinear Schrödinger equation (we omit the caret)

$$i\psi_t+\psi_{xx}\pm 2|\psi|^2\psi=0.$$

5. Deformation of the Schrödinger equation with differentiated nonlinearity [18]:

$$i\psi_t+(x\psi)_{xx}\pm i(x|\psi|^2\psi)_x-\frac{1}{2}\psi_x=0,\quad U=-i\lambda^2\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}-i\lambda\begin{bmatrix}0 & \psi\\ \pm\bar{\psi} & 0\end{bmatrix},$$

$$V=-2i\lambda^4x\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}-2i\lambda^3x\begin{bmatrix}0 & \psi\\ \pm\bar{\psi} & 0\end{bmatrix}+i\lambda^2x\begin{bmatrix}\pm|\psi|^2 & 0\\ 0 & \mp|\psi|^2\end{bmatrix}+ \tag{A.14}$$

$$\lambda x\begin{bmatrix}0 & \psi_x+\dfrac{\psi}{2x}\pm i|\psi|^2\psi\\ \mp\bar{\psi}_x\mp\dfrac{\bar{\psi}}{2x}+i|\psi|^2\bar{\psi} & \end{bmatrix},\quad \lambda_x=0,\quad \lambda_t=\lambda^3,\quad \lambda(z,t)=\frac{1}{\sqrt{-2(t+z)}}.$$

After the substitution $\psi(x,t)=x^{-1/4}\hat{\psi}(\hat{x},\hat{t})$, $\hat{x}=2x^{1/2}$, $\hat{t}=t$, the system (A.14) takes the form (we omit the caret)

$$i\psi_t+\psi_{xx}+\frac{1}{x}\psi_x\pm i(|\psi|^2\psi)_x=\frac{1}{4x^2}\psi\mp\frac{i}{2x}|\psi|^2\psi. \tag{A.15}$$

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